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CURRENT RESEARCH & FUTURE PLANS

1. G1: Hamiltonian Dynamics and Symplectic Geometry

http://www.math.wustl.edu/~apelayo/GR1-Publications.html

In 1985 Gromov revolutionatized symplectic geometry by introducing pseudoholomorphic curves and using them to prove the celebrated Symplectic Nonsqueezing theorem: you cannot squeeze a big ball into an infinite thin cylinder by means of a symplectic map, or a Hamiltonian flow (even though the cylinder has infinite volume).

Gromov proved this result by studying properties of moduli spaces of pseudoholomorphic curves. Gromov's Nonsqueezing result may also be interpreted as a strong instability result for Hamiltonian differential equations. Coming from the variational theory of Hamiltonian dynamics, Ekeland and Hofer gave a proof of Gromov's Nonsqueezing Theorem by studying periodic solutions of Hamiltonian systems. Hofer and Ekeland-Hofer introduced new invariants of symplectic manifolds, the so called symplectic capacities.

The unifying theme of G1 is the use of analytic and dynamical techniques to construct symplectic invariants of manifolds endowed with some interesting additional structure on it such as an integrable system, a Lagrangian foliation, a family of random maps, a circular symmetry, etc. The invariants I have constructed come primarily from studying the dynamics of Hamiltonian flows, singular affine structures and their limits, and normal forms of singularities; I have also used pseudoholomorphic curves and ergodic theory.

Specific topics I have worked on:

1.1. **Symplectic Capacities & Embeddings.**

Studying the existence or inexistence of symplectic embeddings of balls and cylinders into balls and cylinders, following Gromov, Polterovich, Guth, Hind & Kerman. These results have consequences in terms of symplectic capacities. They may also be interpreted terms of stability (or instability) of Hamiltonian differential equations, following the works of Kuksin and Bourgain in the 1990s. I gave talks on this viewpoint which is perhaps less known in geometry at the PDE Seminar in Austin, and at the conference honoring Alan Weinstein in July 2013 (I'm preparing notes post in `http://www.math.wustl.edu/~apelayo/Summ-G1.html`);

1.2. **Random Dynamical Systems.**

The interaction between random and deterministic ideas in dynamics is a highly active topic of research, see for instance the recent work of Bourgain-Sarnak-Ziegler (2013), and Sarnak's Lectures (2011) on Mobius function Randomness and Dynamics. I have studied the generalization of the Poincare-Birkhoff Fixed Point Theorem to twist maps that are random with respect to a given probability measure. While random dynamics has been explored quite throughly, eg. Brownian motions, the implications of the area-preservation assumption remain relatively unknown. Currently I continue working on random versions of theorems in symplectic geometry using a theory of random generating functions introduced jointly with Rezakhanlou, following Chaperon and Viterbo;

1.3. **Semitoric Systems.**

Using symplectic methods to construct invariants of finite-dimensional integrable systems of so called semitoric type: these are systems with 2 degrees of freedom on 4-dimensional manifolds and for which one of the Hamiltonians generates a periodic motion. Semitoric systems retain some of the rigidity of toric systems but may have in addition nodal singularities (corresponding to singular fibers which are multipinched tori). Semitoric systems model some simple physical systems such as the spherical pendulum or the coupled spin-oscillator (known as Jaynes-Cummings model in quantum optics);

1.4. **Singular Affine Structures.**

Studying singular affine structures induced by singular Lagrangian fibrations into the plane, and connecting them to the works of Gross and Siebert on mirror symmetry, the work of Symington on the symplectic topology of almost toric manifolds, and the work of the Fomenko school on topological classifications of integrable systems;

1.5. **Moduli Spaces.**

Endowing certain spaces from symplectic and algebraic geometry with geometric structures, such as symplectic or metric structures. With the aid of these structures, one can raise new questions about maps or other objects defined on these spaces (for example symplectic capacities), for which regularity questions may now be posed. In some cases these structures lead us to constructing new invariants.

2. G2: Spectral Theory and Semiclassical Analysis

HTTP://WWW.MATH.WUSTL.EDU/~APELAYO/GR2-PUBLICATIONS.HTML

In the 1970s and 1980s Colin de Verdière, Duistermaat, Guillemin, and Sjöstrand, proved a series of major results in spectral theory of (differential, pseudodifferential) operators. Among the most striking ones are Colin de Verdière's inverse results for collections of commuting pseudodifferential operators on cotangent bundles. The problems in G2 belong to the realm of classical questions in inverse spectral theory going back to these pioneer works.

I am primarily concerned with the study of semiclassical spectral theory of quantum integrable systems using microlocal analysis for pseudodifferential operators and Berezin-Toeplitz operators. Even though quantum integrable systems date back to the early days of quantum mechanics, such as the work of Bohr, Sommerfeld and Einstein, the theory did not blossom at the time. The development of semiclassical analysis with microlocal techniques in the last forty years now permits a constant interplay between spectral theory and symplectic geometry. The goal of my research in G2 is to use this interplay to prove inverse results about quantum systems.

I have been particularly concerned with isospectral problems for quantum integrable systems, that is, studying when the semiclassical joint spectrum of the system, given by a sequence of commuting Toeplitz operators on a sequence of Hilbert spaces, determines the classical integrable system given by the symplectic manifold and commuting Hamiltonians. Recent results at the intersection of symplectic and spectral geometry suggest that symplectic invariants are better encoded in spectral information than Riemannian invariants (Charles, myself, and Vũ Ngọc proved this recently for quantum toric systems).

The question of isospectrality in Riemannian geometry may be traced back to Weyl (1910) and is most well known thanks to an article by Kac (1966), who himself attributes the question to Bochner. Kac popularized the sentence: "can one hear the shape of a drum?", to refer to this type of isospectral problem. Bochner and Kac's question has a negative answer in this case, even for planar domains with Dirichlet boundary conditions; major results in this direction were proven Osgood-Phillips-Sarnak, and Zelditch, among others. An important observation of Sarnak is that in this context a much better question to ask is whether the set of isospectral domains is finite or compact.

Specific topics I have worked on:

2.1. **Semiclassical Inverse Spectral theory for Singularities.**

Studying the semiclassical spectral theory of singularities of quantum integrable systems, combining analytic methods developed in the past fifteen years in semiclassical analysis with results from symplectic geometry going back to the foundational works of Weinstein in the 1970s;

2.2. **Global Semiclassical Spectral Theory.**

Studying and describing the semiclassical spectral theory in detail for large classes of quantum integrable systems. Then using this theory to study isospectrality questions for these systems, in collaboration with Charles, Polterovich, and Vũ Ngọc. In some cases one can prove that the semiclassical spectrum of a quantum system determines the classical system given by the principal symbols. One of the main results in this group is an analogue of the Colin de Verdière inverse spectral results on pseudodifferential operators, for the case of any prequantizable compact manifold and Berezin-Toeplitz operators (it uses microlocal methods developed since around 2003);

2.3. **Quantization and Quantum Integrable Systems.**

Studying the existence of different types of quantizations and for different types of classical systems. Currently studying the relations to important works on the more algebraic side by Reshetikhin, Yakimov, and others. In the past five years I worked with Bertram Kostant in writing a treatment on geometric quantization from the angle of Lie theory and representation theory;

2.4. **Spectral Theory for Integrable Systems from Physics.**

I have worked on understanding quantum versions of physical systems typical in mechanics (see Marsden and Ratiu book for several examples). This includes the Jaynes-Cummings model from quantum optics.

3. G3: Group Actions

HTTP://WWW.MATH.WUSTL.EDU/~APELAYO/GR3-PUBLICATIONS.HTML

In 1982 Atiyah and independently Guillemin & Sternberg proved one of the most influential results in the equivariant symplectic theory of group actions: the momentum map (introduced by Lie, Kostant, and Souriau) image of the action of a compact connected abelian Lie group (i.e. a torus) equals the convex hull of the images of the action fixed points, in particular it is convex. A school was created around this topic which flourished, and the area of Hamiltonian group actions is now a mainstream topic in equivariant geometry.

Several classification results have been achieved since, which involve some form of the momentum map image as an invariant. If one drops the condition on the action to be Hamiltonian (that is we no longer have a momentum map) the convexity result does not hold. However, it is still possibly to classify such actions in some cases which are of interest both in complex algebraic geometry (for instance the Kodaira variety falls into this class), and in symplectic topology. I worked on several of these cases.

Shortly after the work of Atiyah and Guillemin & Sternberg, Duistermaat & Heckman (1982) proved the DH Theorem which has extensive applications in geometry and analysis. This influential result expresses the Fourier transform of the push-forward of the Liouville measure by the momentum map (DH measure) of a Hamiltonian torus action in terms of the behavior around the fixed points (assumed isolated).

In 1984 Atiyah & Bott generalized the DH Theorem (to include non isolated points) and presented it using the far reaching setting of equivariant cohomology. Several of the problems I have worked on G3 concern the so called logarithmic concavity conjecture, which asks when the logarithm of the DH measure is a concave function. Although this is not in general true (proved by Karshon), it holds in important cases.

Specific topics I have worked on:

3.1. **Lagrangian Actions.**

Studying symplectic nonnecessarily Hamiltonian actions with Lagrangian orbits (which includes all Hamiltonian torus actions with maximal dimensional orbits); more generally study (and classify) symplectic actions whose orbits are coisotropic manifolds; such manifolds include the Kodaira variety (also known as Kodaira-Thurston manifold); the study of such actions involves classical geometry going back to foundational results of Cartan, Haefliger, Koszul, and Weinstein as well as recent work of Benoist and others;

3.2. **Logarithmic Concavity Conjecture.**

Studying in which settings the logarithmic concavity conjecture holds; the main tool is Hodge theory;

3.3. **Kosnowski Conjecture.**

This conjecture (1979) relates the existence of fixed points to the Hamiltonian character of a circle action on a compact manifold, when the fixed points are isolated. Full solutions are due to Frankel for compact Kahler manifolds and to McDuff for compact symplectic 4-manifolds. I have worked on the remaining cases of this conjecture from several angles with different collaborators: using Hodge theory, Atiyah-Bott localization in equivariant cohomology, and most recently methods originating in equivariant K-theory;

3.4. **Toric Poisson Geometry.**
Going back to the pioneer work of Weinstein in Poisson geometry, I have been working with my collaborators on extending to the Poisson setting the theory of symplectic toric manifolds;

3.5. **Symplectic Fiber Bundles.**
Proving that symplectic non-Hamiltonian actions whose orbits are symplectic can be viewed as symplectic orbifold bundles over an orbifold. The base of the orbifold and the monodromy of the flat connection over it (given by the symplectic orthogonal complements to the tangent spaces to the orbits), essentially classify them;

3.6. **Existence of invariant Kahler and complex structures.**
Delzant classified in 1988 Hamiltonian actions of n-tori on compact $2n$-dimensional manifolds (for any natural number n); I extended this result to symplectic, non-necessarily Hamiltonian actions of tori when $2n = 4$. As a consequence, one can determine which compact symplectic 4-manifolds endowed with torus actions admit complex and/or Kähler structures which are also invariant under the action; this is done by using a dictionary betwen the symplectic and complex analytic theory, using Kodaira's classification of complex analytic surfaces.

4. G4: Homotopy type theory

HTTP://WWW.MATH.WUSTL.EDU/~APELAYO/GR4-PUBLICATIONS.HTML

Homotopy type theory is a new field of mathematics which interprets type theory from a homotopical perspective. Voevodsky's univalent foundations program is a research theme based on homotopy type theoretic ideas, and which can be carried out in a computer proof assistant. The goal of Voevodsky's program is developing mathematics in the setting of type theory with an additional axiom which he introduced, the Univalence Axiom. Homotopy type theory is connected to several topics of interest in modern algebraic topology, such as infinity-groupoids and Quillen model structures.

Type theory was invented by Russell (1908), but it was first developed as a rigorous formal system by Church (1930s, 1940s). It now has numerous applications in computer science, especially in the theory of programming languages. In type theory objects are classified using a primitive notion of "type", similar to the datatypes used in programming languages. And as in programming languages, these elaborately structured types can be used to express detailed specifications of the objects classified, giving rise to principles of reasoning about them.

In homotopy type theory, one regards the types as spaces, or homotopy types, and the logical constructions as homotopy-invariant constructions on spaces. In this way, one is able to manipulate spaces directly, without first having to develop point-set topology or even define the real numbers.

We hope that in the future, because this development of mathematics can be carried out in a proof assistant such as Coq so that the proofs carry some algorithmic content, it will be possible to extract good algorithms from the proofs.

Specific topics I have worked on:

4.1. **Univalent Study of Basic p-adic Structures.**

One of my motivations for the construction of the aforementioned algorithms is to help solve some problems concerning operator theory and integrable systems which are of particular interest in applications. For instance, one outstanding problem is: given numerical spectral data about a p-adic quantum system (coming from an experiment), extract an algorithm to reconstruct the classical integrable system (this is connected to groups G1 and G2). Voevodsky, myself, and Warren have taken first steps in this direction by giving a homotopy type theoretic construction of the p-adic numbers in the setting of Voevodsky's univalent foundations (we also implemented the construction in the Coq proof assistant). The construction involves a univalent description of basic results in number theory and group theory;

4.2. **First Steps in Univalent p-adic Systems.**

A p-adic symplectic form may be defined as in the real case. The closedness condition makes sense in the p-adic setting, and so does the non-degeneracy condition (in fact, over any field). In the real setting, a theorem of Darboux says that all symplectic forms are locally equivalent, so real symplectic manifolds have no local invariants. It is natural to wonder whether this result holds in the p-adic setting "as is". One should restrict to the analytic setting since the closedness condition is a system of differential equations. We are working on the symplectic p-adic setting further, with an eye towards the development and univalent implementation of a theory of p-adic integrable systems;

4.3. **Revisiting the Basic Literature on Homotopy Type Theory.**

Awodey, myself, and Warren have worked on explaining in non-specialized, non-technical, terms some of the basic literature on homotopy type theory and Voevodsky's univalent foundations program (which requires knowledge of homotopy theory and mathematical logic) making en emphasis on the connections to other branches of mathematics and theoretical computer science.

5. Research plans for 2013-2016

Of the following, my main projects are 5.1, 5.3, and 5.5, which concern primarily:

- spectral geometry and symplectic geometry (and going back and forth between them);
- probabilistic methods in symplectic geometry.

5.1. **Semiclassical Symplectic Topology.**

A major result in symplectic topology is Gromov's Nonsqueezing theorem (1985), which shows the existence of 2-dimensional symplectic capacities. Symplectic capacities were introduced by Ekeland and Hofer in 1989, who came from the angle of Hamiltonian dynamics. The goal of this project, which is joint with Charles, Polterovich, and Vũ Ngọc, is investigating a semiclassical notion of symplectic capacity.

We understand some preliminary particular situations, following new insights of Polterovich. I believe that this topic will be a major one in the years to come for both analysts and geometers, nonetheless it is is a long term difficult project.

5.2. **Semiclassical Spectral Geometry of Semitoric Systems.**
One of my long term interests is to use microlocal analysis and symplectic geometry to solve questions in inverse spectral theory concerning quantum integrable systems. This is a long term project with Vũ Ngọc concerning inverse geometry for collections of commuting semiclassical operators, as initiated in the pioneer work of Colin de Verdière, Duistermaat, Guillemin, and Sjöstrand. We have worked on this project for approximately six years. I estimate that we have completed at least one half of the project.

In my NSF CAREER award application I proposed a strategy to show that the semiclassical joint spectrum of a quantum semitoric system determines the classical system of principal symbols (all the symbols but possibly one generate periodic Hamiltonian flows). I need to write this up but I have already all pieces of the puzzle (local results for singularities and a global strategy to glue local results), but one (recovering the so called twisting index invariant).

I think that we are quite close to a result. The strategy consists of two parts. First one computes the symplectic invariants from semiclassical joint spectrum (there are five invariants). The second part comes down to using our article in *Acta Mathematica* to reconstruct the system from the symplectic invariants. Even though the second part is already done in this article, the work needed to complete the first part is substantial. In particular in requires adapting microlocal results proved by Laurent Charles recently in the context of Berezin-Toeplitz quantization.

A detailed sketch of proof is given in my article in *DCDS-A* in 2012. This project is continuation of the work described in Section 1.3 and Section 2.

5.3. L^∞ **Bounds for Eigenfunctions of Quantum Integrable Systems.**
Two years ago I began to study major contributions to the isospectral problem for Riemannian manifolds that Sarnak had made with Osgood and Phillips. They showed in dimension two the isospectral set, while not finite, is compact in C^∞. I had considered analogue inverse problems for a collection of commuting operators. Jointly with Charles and Vũ Ngọc I proved that an important class of integrable systems (*quantum toric systems*) given by r independendent commuting operators (whose principal symbols generate periodic Hamiltonian flows) are determined by their semiclassical joint spectrum. This solves isospectrality for quantum toric systems (this is in an article in *Annales Sci. Ec. Norm. Sup.* dedicated to Sarnak).

In view of this work, Sarnak asked me (in October 2011) a question concerning r independent commuting operators: *how much can one improve the* L^∞ *bounds for the joint eigenfunctions he obtained?* Sarnak had shown that in the case of the commutative ring of differential operators of rank r, one gets an optimal improvement depending on r. I am working on this question now. Recently Simon Marshall obtained the corresponding L^p bounds.

5.4. **Construction of Symplectic Invariants of Integrable Systems.**
Building on the seminal contributions of Arnol'd, Eliasson and Duistermaat on symplectic theory of integrable systems, the goal here is to prove that large classes of finite dimensional integrable Hamiltonian systems $F = (f_1, \ldots, f_n) \colon M \to \mathbb{R}$ are characterized, up to symplectic isomorphisms, by an explicit list of invariants. This is a long term program with Ratiu and Vũ Ngọc. It is continuation of the work described in Sections 1.3 and 1.4.

The time is ripe for advances in this subject. The reason is that several new effective tools and ideas are now understood, which was not the case, say, ten years ago. There are three key ingredients that are understood today much better: singular affine structures, symplectic singularity theory, and the microlocal analysis of Berezin-Toeplitz operators. The technology and ideas developed by many since the 1970 and 1980s, but particularly in the past ten years, could be greatly developed with a view towards a better understanding of the symplectic and spectral aspects of integrable systems.

5.5. **Arnol'd Conjecture for Random Maps.**
Arnol'd formulated the higher dimensional analogue of the Poincaré-Birkhoff Theorem: the Arnol'd Conjecture (1978). The first breakthrough on the conjecture was by Conley and Zehnder (1983) who proved it for the $2n$-torus. A proof for the $2n$-torus using generating functions was later given by Chaperon (1984). The second breakthrough was by Floer (1988). Related results were proven eg. by Hofer-Salamon, Liu-Tian, Ono, Weinstein.

I am currently interested in probability and in applying it to problems in symplectic geometry (as continuation of the work in Section 1.2). The goal of this project is proving a random Arnol'd Conjecture, as continuation of our work on the random Poincaré-Birkhoff theorem, which we proved using the notion of *random* generating function which we introduced. This is a project with Rezakhanlou.

5.6. **Homotopy Type Theory of Integrable Systems.**
This is a joint project with Warren and Voevodsky, continuing on the steps taken in Section 4. It continues our program to build a p-adic theory of integrable systems in homotopy type theory, as part of the univalent foundations program. Out future plans are explained in details towards the end of our article arXiv:1302.1207.

5.7. C^k**-Regularity of Symplectic Capacities.**
This is a recent project with Figalli, in which we are studying properties of density functions for symplectic ball packings.

5.8. **Log Geometry and Poisson Geometry.**
This is a two year old project with Gualtieri and Ratiu continuing the work of Section 3.4 and to which recently my new postdoc Songhao Li has joined. A toric Poisson manifold is a $2n$-dimensional Poisson manifold endowed with an action of an n-dimensional torus. This paper proposes what we believe is the correct notion of momentum map J on toric Poisson manifolds, and constructs several examples to illustrate the definition.

In this construction the momentum polytope Δ is a manifold with corners, and the momentum map blows up in a logarithmic fashion along a distinguished set of facets of Δ. The convexity of J, which in the classical symplectic theory of Atiyah, Guillemin-Sternberg follows is a difficult theorem, is part of our definition. A paper is almost finished on this. Songhao Li is working on this. I also would like to ask Li to explore the theory of integrable systems in the Poisson setting.

Other reports at `http://www.math.wustl.edu/~apelayo/Projects_v2.html`

- Outline of Research Plans 2013-2016 (1 page);
- Report of work at IAS 12/2010-08/2013 (5 pages).

St Louis, September 2013.