Complex structures on 4-manifolds with symplectic 2-torus actions

J.J. Duistermaat and A. Pelayo*

Abstract

We apply the general theory for symplectic torus actions with symplectic or coisotropic orbits to prove that a 4-manifold with a symplectic 2-torus action admits an invariant complex structure and give an identification of those that do not admit a Kähler structure with Kodaira's class of complex surfaces which admit a nowhere vanishing holomorphic $(2,\, 0)$-form, but are not a torus or a K3 surface.

1 Introduction

The existence of complex and Kähler structures on smooth manifolds is a classical problem studied in differential geometry. This paper concerns the existence of T-invariant complex structures, and of Kähler structures, on symplectic 4-manifolds equipped with 2-dimensional torus actions. A *symplectic form* on a compact manifold M is a closed, non-degenerate two-form $\sigma \in \Omega^2(M)$. In this case, $(M,\, \sigma)$ is called a *symplectic manifold*. If T is a torus, a T-action on M is *symplectic* if it preserves σ. We will prove:

Theorem 1.1. *Let T be a 2-torus and $(M,\, \sigma)$ be a compact, connected symplectic 4-manifold equipped a symplectic T-action. Then M admits a T-invariant complex structure, and M does not admit a Kähler structure if and only if it is isomorphic, as a symplectic T-manifold, to a Kodaira space $\mathbb{C}^2/\Gamma$ as described in Subsection 3.4. Moreover, if M admits a Kähler structure, then M admits a T-invariant complex structure and a corresponding Kähler form equal to σ.*

The proof of Theorem 1.1 may be considered an application of the the general theory of symplectic torus actions developed in [5, 13]. There are several works related to Theorem 1.1. Most significantly, Delzant [4] proved that a compact, connected symplectic manifold $(M,\, \sigma)$ equipped a Hamiltonian action of a torus T with $\dim T = \frac{1}{2} \dim M$ is isomorphic to a smooth toric variety with σ equal to a Kähler form on it, and the action of T extends to a holomorphic action of the complexification $T_{\mathbb{C}}$ (see for instance [10] for the definition of Hamiltonian action). Karshon [6] proved that if the circle S^1 acts Hamiltonianly on a compact, connected symplectic manifold then M admits a compatible, S^1-invariant Kähler structure.

2 Preliminaries

2.1 Symplectic torus actions

Let T be a torus. Let M be a manifold equipped with a symplectic form σ and a symplectic T-action. For every $x \in M$ the orbit $T \cdot x$ of the T-action containing x is a smooth manifold, and the mapping $T \to M : t \mapsto t \cdot x$ induces a diffeomorphism from T/T_x onto $T \cdot x$, where $T_x := \{t \in T \mid t \cdot x = x\}$

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denotes the stabilizer subgroup of x in T. The tangent mapping at $1\,T_x$ of this diffeomorphism is a linear isomorphism from $\mathfrak{t}/\mathfrak{t}_x$ onto the tangent space $\mathrm{T}_x(T\cdot x)$ at x of $T\cdot x$. Here $\mathfrak{t}$ and $\mathfrak{t}_x$ denote the respective Lie algebras of T and T_x. That is, if $X_M(x)$ denotes the infinitesimal action at x of an element X of the Lie algebra $\mathfrak{t}$ of T, then $\mathrm{T}_x(T\cdot x)=\{X_M(x)\mid X\in\mathfrak{t}\}$, $\mathfrak{t}_x=\{X\in\mathfrak{t}\mid X_M(x)=0\}$, and the aforementioned linear isomorphism $\mathfrak{t}/\mathfrak{t}_x\to\mathrm{T}_x(T\cdot x)$ is induced by the linear mapping $\mathfrak{t}\to\mathrm{T}_x\,M:X\mapsto X_M(x)$. For an effective torus action the minimal stabilizer subgroups are the trivial ones $T_x=\{1\}$, in which case the action of T is free at the point x, and the corresponding orbits are called the *principal orbits*. The set M^{reg} of all $x\in M$ such that $T_x=\{1\}$ is an open, dense, and T-invariant subset of M.

The orbit $T\cdot x$ is symplectic if, for every $y\in T\cdot x$, restriction of σ_y to the tangent space $\mathrm{T}_y(T\cdot x)$ of the orbit is a symplectic form. That is, if $(\mathrm{T}_y(T\cdot x))^{\sigma_y}$ denotes the orthogonal complement of $\mathrm{T}_y(T\cdot x)$ in $\mathrm{T}_y\,M$ with respect to the symplectic form σ_y, then $\mathrm{T}_y\,M$ is equal to the direct sum of $\mathrm{T}_y(T\cdot x)$ and its symplectic orthogonal complement.

The orbit $T\cdot x$ is coisotropic if, for every $y\in T\cdot x$, $\mathrm{T}_y(T\cdot x)$ is a coisotropic linear subspace of $\mathrm{T}_y\,M$. That is, $(\mathrm{T}_y(T\cdot x))^{\sigma_y}\subset\mathrm{T}_y(T\cdot x)$. Because the T-action preserves σ, the orbit $T\cdot x$ is coisotropic if and only if $(\mathrm{T}_y(T\cdot x))^{\sigma_y}\subset\mathrm{T}_y(T\cdot x)$. It follows from Benoist [2, Proposition 5.1] that if coisotropic orbits exist, then these are precisely the principal orbits. A special case is when, for $x\in M^{\mathrm{reg}}$, we have $(\mathrm{T}_y(T\cdot x))^{\sigma_y}=\mathrm{T}_y(T\cdot x)$, that is, the principal orbits are Lagrangian.

Benoist [2, Lemme 2.1] observed that if u and v are smooth vector fields on M which preserve σ, then their Lie bracket $[u,\,v]$ is Hamiltonian with Hamiltonian function equal to $\sigma(u,\,v)$, that is, $\mathrm{i}_{[u,\,v]}\,\sigma=-\mathrm{d}(\sigma(u,\,v))$. It therefore follows from the commutativity of T that if $X,\,Y\in\mathfrak{t}$, then $\mathrm{d}(\sigma(X_M,\,Y_M))=0$, which means that there is a unique antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ such that $\sigma^{\mathfrak{t}}(X,\,Y)=\sigma_x(X_M(x),\,Y_M(x))$ for every $x\in M$ and $X,\,Y\in\mathfrak{t}$. If $X\in\mathfrak{t}_x$, that is $X_M(x)=0$, then $\sigma^{\mathfrak{t}}(X,\,Y)=0$ for every $Y\in\mathfrak{t}$. It follows that $\mathfrak{t}_{\mathfrak{h}}\subset\mathfrak{l}\subset\mathfrak{t}$, if $\mathfrak{t}_{\mathfrak{h}}$ and $\mathfrak{l}$ denote the the sum of all $\mathfrak{t}_x$'s and the kernel of $\sigma^{\mathfrak{t}}$ in $\mathfrak{t}$, respectively.

2.2 Symplectic torus actions with coisotropic principal orbits

We assume in the sequel that there exist coisotropic orbits, or equivalently, that all principal orbits are coisotropic. We provide the necessary background from [5] to understand the proof of Theorem 1.1. The section serves also as a concise review of [5].

2.2.1 The orbit space as an $\mathfrak{l}^*$-parallel space

Because T acts freely on M^{reg}, the orbit space M^{reg}/T is a smooth manofld of dimension $\dim M-\dim T$, and the restriction to M^{reg} of the canonical projection $\pi:M\to M/T:x\mapsto T\cdot x$ exhibits M^{reg} as a principal T-bundle over M^{reg}/T. If $x\in M^{\mathrm{reg}}$ then $\mathfrak{t}_x=\{0\}$, $\mathfrak{l}\simeq(\mathrm{T}_x(T\cdot x))^{\sigma_x}\subset\mathrm{T}_x(T\cdot x)\simeq\mathfrak{t}$, and therefore $\dim M=\dim\mathfrak{t}+\dim\mathfrak{l}$, or equivalently $\dim(M^{\mathrm{reg}}/T)=\dim\mathfrak{l}$.

For each $X\in\mathfrak{t}$, the one-form $\theta_X:=-\mathrm{i}_{X_M}\,\sigma$ is T-invariant and closed. It is horizontal, that is, satisfies $\mathrm{i}_{Y_M}\,\theta_X=\sigma(Y_M,\,X_M)=\sigma^{\mathfrak{t}}(Y,\,X)=0$ for every $Y\in\mathfrak{t}$, if and only if $X\in\mathfrak{l}$. It follows that, for every $X\in\mathfrak{l}$, there is a unique closed one-form $\widehat{\sigma}_X$ on M^{reg}/T such that $\theta_X=\pi^*(\widehat{\sigma}_X)$. Let $\widehat{\sigma}$ denote the closed $\mathfrak{l}^*$-valued one-form on M^{reg}/T which assigns to each tangent vector v at each point p the element $X\mapsto(\widehat{\sigma}_X)_p(v)$ of $\mathfrak{l}^*$. For each $p\in M^{\mathrm{reg}}/T$ the $\mathfrak{l}^*$-valued linear form $\widehat{\sigma}_p$ on $\mathrm{T}_p(M^{\mathrm{reg}}/T)$ is a linear isomorphism from $\mathrm{T}_p(M^{\mathrm{reg}}/T)$ onto $\mathfrak{l}^*$. It follows from the closedness of $\widehat{\sigma}$ that we can provide M^{reg}/T with an atlas of local charts in $\mathfrak{l}^*$ with tangent map equal to $\widehat{\sigma}$, and any two such local charts differ by a translation in $\mathfrak{l}^*$. In this way M^{reg}/T has the structure of an *$\mathfrak{l}^*$-parallel space*. Using the local model for the symplectic T-action of Benoist [2, Proposition 1.9], the whole orbit space M/T is a $\mathfrak{l}^*$-parallel space. For each singular point $p=T\cdot x$ there is a $\mathbb{Z}$-basis X_j, $j\in J$, of the integral lattice of T_x in $\mathfrak{t}_x$, such that in the local chart around p the orbit space M/T is identified with the corner

$$\{\xi \in \mathfrak{l}^* \mid \xi(X_j) \geq 0 \;\forall j \in J\}. \tag{1}$$

Here the *integral lattice of a torus S in its Lie algebra $\mathfrak{s}$* is defined as the kernel of the exponential mapping $\exp : \mathfrak{s} \to S$.

If $\zeta \in N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, a linear form on $\mathfrak{l}$ which is zero on every $\mathfrak{t}_x$, then the straight line solution curves of the constant vector field ζ on M/T do not meet the singular points of M/T, and it follows that we have a global translational action of N on M/T. The period group $P \subset N$ group of this action is a cocompact discrete subgroup of N, and N/P is a $\dim N$-dimensional torus.

A polytope Δ in the dual of the Lie algebra $\mathfrak{s}$ of a torus S is called a *Delzant polytope* if, for each vertex v of Δ there is a $\mathbb{Z}$-basis X_j of the integral lattice of S in $\mathfrak{s}$, such that near v the polytope Δ is equal to the set $\{\xi \in \mathfrak{s}^* \mid \xi(X_j) \geq v(X_j) \;\forall j \in J\}$. The restriction of linear forms on $\mathfrak{l}$ to $\mathfrak{t}_{\mathrm{h}}$ defines a linear projection from $\mathfrak{l}^*$ onto ${\mathfrak{t}_{\mathrm{h}}}^*$, with kernel equal to N. Therefore the N-orbit space $(M/T)/N$ has the structure of a ${\mathfrak{t}_{\mathrm{h}}}^*$-parallel space. The description (1) of the corners of M/T now leads to the following conclusions.

Proposition 2.1. $T_{\mathrm{h}} := \exp \mathfrak{t}_{\mathrm{h}}$ *is closed in T, hence a subtorus of T, and the ${\mathfrak{t}_{\mathrm{h}}}^*$-parallel space $(M/T)/N$ is isomorphic to a Delzant polytope Δ in ${\mathfrak{t}_{\mathrm{h}}}^*$.*

Let $C \simeq {\mathfrak{t}_{\mathrm{h}}}^*$ be a linear complement of N in $\mathfrak{l}^*$, and let $p \in M/T$. Then the mapping $\Phi_p : (\eta,\, \zeta) \mapsto p + (\eta + \zeta)$ defines an isomorphism of $\mathfrak{l}^*$-parallel spaces from $\Delta \times (N/P)$ onto M/T.

Because Δ is contractible, the projection from M/T onto the torus N/P along Δ induces an isomorphism from the cohomology of N/P onto the cohomology of M/T, where the latter cohomology, according to the theorem of Koszul [12], is isomorphic to the cohomology defined by the closed basic differential forms on M. If $X \in \mathfrak{l}$, then the closed basic one-form $-\mathrm{i}_{X_M}\,\sigma$ is exact, that is, there is a T-invariant smooth function μ_X on M such that $-\mathrm{i}_{X_M}\,\sigma = \mathrm{d}\mu_X$, if and only if $X \in \mathfrak{t}_{\mathrm{h}}$. In other words, *$X_M$ is a Hamiltonian vector field if and only if $X \in \mathfrak{t}_{\mathrm{h}}$*, where the Hamiltonian function μ_X can be chosen to depend linearly on $X \in \mathfrak{t}_{\mathrm{h}}$. The *momentum mapping* $\mu : M \to {\mathfrak{t}_{\mathrm{h}}}^*$ assigns to each $x \in M$ the linear form $\mu(x) : X \mapsto \mu_X(x)$ on $\mathfrak{t}_{\mathrm{h}}$. It follows from our characterization of $(M/T)/N$ as a ${\mathfrak{t}_{\mathrm{h}}}^*$-parallel space that μ can be identified with the composition of the canonical projection from M onto M/T, followed by the canonical projection from M/T onto $(M/T)/N \simeq \Delta$, when $\mu(M)$ is identified with the Delzant polytope Δ in Proposition 2.1. In this way we recover the theorem of Atiyah [1] and Guillemin and Sternberg [9] that the image of the momentum mapping of every Hamiltonian torus action on a compact connected symplectic manifold is a convex polytope, with the additional information that the polytope is a Delzant one. The canonical projection from M onto $M/T \simeq \Delta \times (N/P)$ can be viewed as a non-Hamiltonian momentum mapping, with image equal to the Cartesian product of the Delzant polytope Δ with the torus N/P.

For later purposes, we choose a complementary torus T_{f} to T_{h} in T, of which the Lie algebra $\mathfrak{t}_{\mathrm{f}}$ is a complementary linear subspace of $\mathfrak{t}_{\mathrm{h}}$ in $\mathfrak{t}$. Because the local models for symplectic T-actions with coisotropic orbits show that all stabilizer subgroups are connected, T_{f} is a maximal subtorus of T which acts freely on M. For the complementary linear subspace C of $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^0$ in $\mathfrak{l}^*$, which appears in Proposition 2.1, we can take $C = (\mathfrak{l}/(\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}))^*$. We will also choose a complementary linear subspace $\mathfrak{c}$ of $\mathfrak{t}_{\mathrm{f}} \cap \mathfrak{l}$ in $\mathfrak{t}_{\mathrm{f}}$, which is a complementary linear subspace of $\mathfrak{l}$ in $\mathfrak{t}$, and denote the linear projection from $\mathfrak{t}$ onto $\mathfrak{l}$ along $\mathfrak{c}$ by $X \mapsto X_{\mathfrak{l}}$.

2.2.2 Lifts of the constant vector fields on M/T

The next step in [5] is the construction of T-invariant vector fields L_ξ on M, depending linearly on $\xi \in \mathfrak{l}^*$, which are interwined by the canonical projection $\pi : M \to M/T$ with the constant vector fields ξ on the

$\mathfrak{l}^*$-parallel space. Furthermore, it is arranged that, for any $\xi,\, \xi' \in \mathfrak{l}^*$, the Lie brackets $[L_\xi,\, L_{\xi'}]$ and the symplectic products $\sigma(L_\xi,\, L_{\xi'})$ are as simple as possible.

The condition that π intertwines L_ξ with ξ is equivalent to the condition that $\sigma(L_\xi,\, X_M) = \xi(X)$ for every $X \in \mathfrak{l}$. Note that this condition implies that if L_ξ has a finite value at $x \in M$, then $\xi = 0$ on $\mathfrak{t}_x$. In other words, if $\xi \notin N = \mathfrak{l}^* \cap (\mathfrak{t}_\mathfrak{h})^0$, then the vector field L_ξ necessarily has singularities at the points x such that ξ does not vanish identically on $\mathfrak{t}_x$. We will conversely arrange that, for every $\zeta \in N$, L_ζ is a smooth vector field on the whole manifold M. When $\eta \in C$, we use the local models of the T-action in order to obtain that L_η has a mild singular behaviour near the points x where $\eta|_{\mathfrak{t}_x} \neq 0$, which we call *admissible*.

Proposition 2.2. *There exists an antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$, and vector fields L_ξ, $\xi \in \mathfrak{l}^*$, on M, such that the following conditions i) – xii) hold.*

i) *For every $\xi \in \mathfrak{l}^*$, L_ξ is a smooth vector field on M^{reg}, with an admissible singular behaviour when approaching points of $M \setminus M^{\text{reg}}$. If $\zeta \in N$, then L_ζ has a smooth extension to M.*

ii) *L_ξ depends linearly on $\xi \in \mathfrak{l}^*$.*

iii) *$[L_\xi,\, X_M] = 0$ for all $\xi \in \mathfrak{l}^*$ and $X \in \mathfrak{t}$.*

iv) *$[L_\eta,\, L_{\eta'}] = 0$ for all $\eta,\, \eta' \in C$.*

v) *$[L_\eta,\, L_\zeta] = 0$ for all $\eta \in C$ and $\zeta \in N$.*

vi) *$[L_\zeta,\, L_{\zeta'}] = c(\zeta,\, \zeta')_M$ for all $\zeta, \zeta' \in N$.*

vii) *$\sigma(L_\xi,\, X_M) = \xi(X_\mathfrak{l})$ for all $\xi \in \mathfrak{l}^*$ and $X \in \mathfrak{t}$.*

viii) *$\sigma(L_\eta,\, L_{\eta'}) = 0$ for all $\eta,\, \eta' \in C$.*

ix) *$\sigma(L_\eta,\, L_\zeta) = 0$ for all $\eta \in C$ and $\zeta \in N$.*

x) *$\sigma_x(L_\zeta(x),\, L_{\zeta'}(x)) = -\mu(x)(c_\mathfrak{h}(\zeta,\, \zeta'))$ for all $\zeta,\, \zeta' \in N$ and $x \in M$. Here $c_\mathfrak{h}(\zeta,\, \zeta')$ denotes the $\mathfrak{t}_\mathfrak{h}$-component of $c(\zeta,\, \zeta')$ in the direct sum decomposition $\mathfrak{l} = \mathfrak{t}_\mathfrak{h} \oplus (\mathfrak{l} \cap \mathfrak{t}_\mathfrak{f})$.*

xi) *$\zeta(c(\zeta',\, \zeta'')) + \zeta'(c(\zeta'',\, \zeta)) + \zeta''(c(\zeta,\, \zeta')) = 0$ for all $\zeta,\, \zeta',\, \zeta'' \in N$.*

xii) *$c(P \times P)$ is contained in the integral lattice of T in $\mathfrak{t}$.*

Note that iii) means that each vector field L_ξ is T-invariant, whereas vii) is a strengthening of the condition that $\sigma(L_\xi,\, X_M) = \xi(X)$ for every $X \in \mathfrak{l}$, the condition that L_ξ is a lift to M of the constant vector field ξ on M/T.

It follows from iv), v), and vi) in Proposition 2.2 that $c : N \times N \to \mathfrak{l}$ represents the Chern class of the principal T-bundle $\pi : M^{\text{reg}} \to M^{\text{reg}}/T \simeq \Delta^{\text{int}} \times (N/P)$. This implies xii) in Proposition 2.2.

2.2.3 The fibration into Delzant spaces

Let $\pi_{N/P} : M \to N/P$ be the mapping $\pi : M \to M/T \simeq \Delta \times (N/P)$ followed by the projection $\Delta \times (N/P) \to N/P$ onto the second factor. This defines a locally trivial smooth fiber bundle over the torus N/P, of which each fiber F is a compact and connected T-invariant smooth submanifold of M, where moreover $F \cap M^{\text{reg}}$ is a dense open subset of F.

For each $x \in F \cap M^{\mathrm{reg}}$, the linear span D_x of the vectors $Y_M(x)$, $Y \in \mathfrak{t}_{\mathrm{h}}$, and the lifts $L_\eta(x)$ of the $\eta \in C \simeq {\mathfrak{t}_{\mathrm{h}}}^*$ is a linear subspace of $\mathrm{T}_x F$, complementary to the tangent space at x of the T_{f}-orbit through x. It follows from the commutativity of the infinitesimal T_{h}-action, and iii) and iv) in Proposition 2.2, that the smooth distribution D in $F \cap M^{\mathrm{reg}}$ is T_{f}-invariant and integrable. Although the vector fields L_η can have singularities when approaching points of $F \backslash M^{\mathrm{reg}}$, where the Y_M can have zero limits, the lifts in Proposition 2.2 can be arranged in such a way that D has a smooth extension to a flat infinitesimal connection for the principal T_{f}-bundle $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$, also denoted by D.

There is a unique symplectic form $\sigma_{F/T_{\mathrm{f}}}$ on F/T_{f} such that, for every integral manifold I of D in F, we have $(\pi_{F/T_{\mathrm{f}}} \circ \iota_I)^* \, (\sigma_{F/T_{\mathrm{f}}}) = {\iota_I}^*(\sigma)$, where $\iota_I : I \to M$ denotes the inclusion mapping of I into M. The projection $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$ intertwines the T_{h}-action on F with a Hamiltonian T_{h}-action on the symplectic manifold $(F/T_{\mathrm{f}},\, \sigma_{F/T_{\mathrm{f}}})$, where $\dim(F/T_{\mathrm{f}}) = 2 \dim T_{\mathrm{h}}$. Furthermore, the image of the momentum map is equal to the Delzant polytope Δ. Delzant [4] proved that all compact and connected Hamiltonian T_{h}-spaces $(M_{\mathrm{h}},\, \sigma_{\mathrm{h}})$ such that $\dim M_{\mathrm{h}} = 2 \dim T_{\mathrm{h}}$ and the image of the momentum map is equal to Δ are isomorphic to each other. Following Guillemin [10], we call such a Hamiltonian T_{h}-space *the Delzant space defined by the Delzant polytope* Δ.

Delzant [4] also proved that every Delzant space can be provided with the structure of a toric variety, and Danilov [3, Theorem 9.1] observed that every toric variety is simply connected, because it has an open cell of which the complement is a complex algebraic subvariety of complex codimension one. It follows that, for every integral manifold I of D, the covering map $\pi_{F/T_{\mathrm{f}}}|_I : I \to F/T_{\mathrm{f}}$ is a diffeomorphism, and therefore I*, provided with the symplectic form* ${\iota_I}^*(\sigma)$ *and the Hamiltonian* T_{h}*-action, is a Delzant space*. As F is fibered by the compact integral manifolds of D, and M is fibered by the F's, M is fibered by these Delzant spaces.

2.2.4 The model

Recall that the L_ζ, $\zeta \in N$, in Proposition 2.2 are smooth vector fields on the whole manifold M. Together with the infinitesimal actions X_M, $X \in \mathfrak{t}$, of T, the L_ζ, $\zeta \in N$, form a Lie algebra of smooth vector fields on M, see iii) and vi) in Proposition 2.2. More precisely, let $\mathfrak{g} = \mathfrak{t} \times N$ be provided with the structure of a two-step nilpotent Lie algebra defined by

$$[(X,\, \zeta),\, (X',\, \zeta')] = -(c(\zeta,\, \zeta'),\, 0)$$

for all $(X,\, \zeta),\, (X', \zeta') \in \mathfrak{t} \times N$. Then the mapping $(X,\, \zeta) \mapsto X_M + L_\zeta$ is an injective anti-homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of all smooth vector fields on M. If we provide the vector space $G = T \times N$ with the product

$$(t,\, \zeta)\, (t',\, \zeta') = (t\, t'\, \mathrm{e}^{-c(\zeta,\, \zeta')/2},\, \zeta + \zeta'), \tag{2}$$

then G is a two-step nilpotent Lie group G with Lie algebra equal to $\mathfrak{g}$ and the mapping $(X,\, \zeta) \mapsto (\exp(X),\, \zeta)$ as the exponential mapping. The mapping $(t,\, \zeta) \mapsto t_M \circ \mathrm{e}^{L_\zeta}$ is a (left) action of G on M, that is, a homomorphism from G to the group of all diffeomorphism of M, with the map $(X,\, \zeta) \mapsto X_M + L_\zeta$ as its infinitesimal action.

Let $M_{\mathrm{h}} = I$ be a Delzant submanifold of M as in Subsection 2.2.3. For any $x \in M$ there exists $\zeta \in N$ such that $y = \mathrm{e}^{L_\zeta}(x)$ is an element of the fiber F of $\pi_{N/P} : M \to N/P$ which contains I, and then there is an element $t \in T_{\mathrm{f}}$ such that $t \cdot y \in I$. It follows that the mapping

$$A : G \times M_{\mathrm{h}} \to M : (g,\, z) \mapsto g \cdot z \tag{3}$$

is surjective.

Conversely, if $t \in T$, $\zeta \in N$, $x \in M_{\mathrm{h}}$, and $x' := t \cdot \mathrm{e}^{L_\zeta} \cdot x \in M_{\mathrm{h}}$, then $\pi_{N/P}(x) = \pi_{N/P}(x') = \pi_{N/P}(x) + \zeta$ implies that $\zeta \in P$. Here $P \subset N$ is the period subgroup of the action of N on M/T as discussed in front of Proposition 2.1.

For every $\zeta \in P$ and $p = T \cdot x \in M/T$, the curve $[0,\, 1] \ni s \mapsto p + s\,\zeta$ is a loop in $M/T \simeq \Delta \times (N/P)$. The curve $\gamma : s \mapsto \mathrm{e}^{s\,L_\zeta}(x)$ in M is the horizontal lift of δ, starting at x, with respect to the distribution D of Subsection 2.2.3, and there is an element $t \in T$ such that $\gamma(1) = t \cdot \gamma(0)$, called the *holonomy of the loop* δ. If $\gamma(0) = x \in M^{\mathrm{reg}}$, then the element $t = \tilde{\tau}_\zeta(x)$ of T is unique. The mapping $\tilde{\tau}_\zeta : M^{\mathrm{reg}} \to T$ has a unique extension to a smooth mapping $\tilde{\tau}_\zeta : M \to T$, which satisfies

$$\tilde{\tau}_\zeta(t \cdot \mathrm{e}^{L_{\zeta'}}(x)) = e^{c(\zeta,\, \zeta')} \cdot \tilde{\tau}_\zeta(x) \tag{4}$$

for every $(t,\, \zeta') \in G = T \times N$ and every $x \in M$. Furthermore, the mapping $\tilde{\tau}_\zeta$ is constant on every Delzant submanifold of M as defined in Subsection 2.2.3. In view of the surjectivity of the mapping A in (3) it follows that the mapping $\tilde{\tau}_\zeta : M \to T$ is determined by (4) and its value on M_{h}, which we denote by τ_ζ. Finally we have the product formula

$$\tau_{\zeta'}\ \tau_\zeta = \tau_{\zeta+\zeta'}\ \mathrm{e}^{c(\zeta',\, \zeta)/2} \quad \text{for all} \quad \zeta,\, \zeta' \in P. \tag{5}$$

With these preparations, we now present our global model for the symplectic T-action with coisotropic principal orbits.

Theorem 2.3. *Let $H := \{(t,\, \zeta) \in G \mid \zeta \in P \text{ and } t\,\tau_\zeta \in T_{\mathrm{h}}\}$. Then H is a closed and commutative Lie subgroup of G, and $((t,\, \zeta),\, x) \mapsto (t\,\tau_\zeta) \cdot x$ defines a smooth action of H on M_{h}. If we let $h \in H$ act on $G \times M_{\mathrm{h}}$ by sending $(g,\, x)$ to $(g\,h^{-1},\, h \cdot x)$, and ψ denotes the canonical projection from $G \times M_{\mathrm{h}}$ onto the H-orbit space $G \times_H M_{\mathrm{h}}$, then there is a unique diffeomorphism α from $G \times_H M_{\mathrm{h}}$ onto M such that $A = \alpha \circ \psi$.*

On $G \times_H M_{\mathrm{h}}$ we have the action of G induced by the action $(g,\, (g',\, x)) \mapsto (g\,g',\, x)$ of G on $G \times M_{\mathrm{h}}$, and α intertwines this G-action on $G \times_H M_{\mathrm{h}}$ with the G-action on M, and therefore in particular intertwines the left T-action on $G \times_H M_{\mathrm{h}}$ with the T-action on M. The projection from $G \times M_{\mathrm{h}}$ onto the second factor induces a locally trivial smooth fiber bundle $M \simeq G \times_H M_{\mathrm{h}} \to G/H$, with the Delzant manifolds of Subsection 2.2.3 as its fibers.

The two-form $\omega := \psi^*(\alpha^*(\sigma)) = A^*(\sigma)$ on $G \times M_{\mathrm{h}}$ is given by the explicit formula

$$\begin{aligned} \omega_a(\delta a,\, \delta' a) \;\; = \;\; & \sigma^{\mathfrak{t}}(\delta t,\, \delta' t) + \delta\zeta({X'}_{\mathfrak{l}}) - \delta'\zeta(X_{\mathfrak{l}}) - \mu(x)(c_{\mathfrak{h}}(\delta\zeta,\, \delta'\zeta) \\ & + (\sigma_{\mathrm{h}})_x(\delta x,\, ({X'}_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) - (\sigma_{\mathrm{h}})_x(\delta' x,\, (X_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) + \sigma_{\mathrm{h}\,x}(\delta x,\, \delta' x). \end{aligned}$$

Here $\delta a = ((\delta t,\, \delta\zeta)$ and $\delta' a = ((\delta' t,\, \delta'\zeta)$ are tangent vectors to $G \times M_{\mathrm{h}}$ at $a = ((t,\, \zeta),\, x)$, $X = \delta t + c(\delta\zeta,\, \zeta)/2$, and $X' = \delta' t + c(\delta'\zeta,\, \zeta)/2$.

2.2.5 The classification

Suppose that we have the following ingredients 1) – 7).

1) A torus T.

2) An antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on the Lie algebra $\mathfrak{t}$ of T.

3) A subtorus T_{h} of T of which the Lie algebra $\mathfrak{t}_{\mathrm{h}}$ is contained in $\mathfrak{l} := \ker \sigma^{\mathfrak{t}}$.

4) A Delzant polytope Δ in $\mathfrak{t_h}^*$.

5) A discrete cocompact subgroup P of the additive group $N := (\mathfrak{l}/\mathfrak{t_h})^*$ of $\mathfrak{l}^*$.

6) An antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ satisfying xi) and xii) in Proposition 2.2.

7) A mapping $\tau : \zeta \mapsto \tau_\zeta : P \to T$ which satisfies (5).

Then we have the following existence theorem:

Let $(M_\mathrm{h}, \sigma_\mathrm{h})$ be the Delzant space defined by the Delzant polytope Δ in 4). Let $G = T \times N$ be the group defined in terms of the c in 6) as in (2). Then the form ω in Theorem 2.3 defines a symplectic form on the manifold $G \times_H M_\mathrm{h}$. Furthermore the action of T on $G \times_H M_\mathrm{h}$ is effective, preserves the symplectic form, and has coisotropic principal orbits.

We also have that two such symplectic T-spaces are isomorphic if and only if the ingredients in 1) – 6) are the same, and the mappings τ, τ' in 7) of the two are related to each other by the condition that there is an element $\zeta' \in N$ and a symmetric bilinear form α on $\mathfrak{l}^*$, such that $\tau'_\zeta = \tau_\zeta \exp(c(\zeta, \zeta') + \alpha(\zeta))$ for every $\zeta \in P$. Here $\alpha(\zeta)$ denotes the element of $\mathfrak{l} \subset \mathfrak{t}$ such that $\xi(\alpha(\zeta)) = \alpha(\xi, \zeta)$ for every $\xi \in \mathfrak{l}^*$. This completes the classification of compact connected symplectic T-spaces with coisotropic principal orbits.

3 Proof of Theorem 1.1

In this section we present a proof of Theorem 1.1. Before this, we briefly recall the classification of effective symplectic actions of 2-dimensional tori T on 4-dimensional compact connected symplectic manifolds (M, σ), which we need for the proof.

3.1 The classification of symplectic 2-torus actions on 4-manifolds

The following classification of symplectic 4-manifolds with 2-torus action was given in [13, Theorem. 8.2.1].

Theorem 3.1. *Let (M, σ) be a compact connected symplectic 4-dimensional manifold equipped with an effective symplectic action of a 2-torus T. Then (M, σ) is equivariantly symplectomorphic to one of the following.*

(1) A Delzant space, also called a symplectic toric manifold.

(2) A product $T \times S^2$ with a product symplectic form and a split[1] T-action.

(3) A 4-dimensional torus with an invariant symplectic form and T as a Lagrangian subtorus.

(4) A nontrivial T-bundle over a 2-torus with Lagrangian fibers as in Subsection 3.3.

(5) A symplectic T-manifold with symplectic orbits as in Subsection 3.2, with $\dim T = 2$, which we may view as an orbifold T-bundle over an orbisurface with symplectic fibers.

[1] one subcircle of T acting by translations on T, and a complementary subcircle acting on S^2 by rotations about the vertical axis of $\mathbb{R}^3$

In Theorem 3.1, the cases (1) – (5) are mutually exclusive. Indeed, in Case (1) the action of T is Hamiltonian and therefore has fixed points, and the principal orbits are Lagrangian. In Case (2) the action of T has 1-dimensional stablilizer subgroups but no fixed points, and the principal orbits are Lagrangian. In Case (3) the action of T is free, the orbits are Lagrangian submanifolds of M, and the manifold is not diffeomorphic to a torus. In Case (4) the action is free, the orbits are Lagrangian submanifolds of M, and the manifold is diffeomorphic to a torus; Finally, in Case (5) the orbits are symplectic submanifolds of M.

Remark 3.1 We conclude by observing that in the case of M in Section 2.2 being 4-dimensional and T being 2-dimensional, with free Lagrangian orbits, and c is a non-zero antisymmetric bilinear mapping from $\mathfrak{t}^* \times \mathfrak{t}^*$ to $\mathfrak{t}$, then we are in Case (4) of Theorem 3.1. In this case $\sigma^{\mathfrak{t}} = 0$, $\mathfrak{l} = \mathfrak{t}$, $T_{\mathfrak{h}} = \{1\}$, $\mathfrak{t}_{\mathfrak{h}} = \{0\}$, $N = \mathfrak{l}^* = \mathfrak{t}^*$, and P is a cocompact subgroup of $N = \mathfrak{t}^*$. We only need to check that all holonomiy mappings τ are equivalent. Choose a $\zeta' \in \mathfrak{t}^*$ which does not vanish on the one-dimensional image I of c. If the bilinear form $(\zeta,\, \zeta'') \mapsto \zeta''(c(\zeta,\, \zeta'))$ is symmetric, then it follows from the antisymmetry of c that $\zeta'(c(\zeta, \zeta')) = \zeta(c(\zeta',\, \zeta')) = 0$ for all $\zeta \in \mathfrak{t}^*$, which leads to a contradiction because $\zeta' \neq 0$ implies that I is equal to the image of $\zeta \mapsto c(\zeta,\, \zeta')$. It follows that all mappings τ as in 7) in the list of ingredients in Subsection 2.2.5 are equivalent to each other, and therefore $T_{\mathbb{Z}}$, P, and c are the only invariants of the symplectic T-spaces in Case (4) of Theorem 3.1. $\oslash$

3.2 Symplectic orbits: torus bundle over orbisurface with symplectic fibers

In this section we allow T to be a torus of any even dimension. From the following ingredients (i) – (iv) we will cook up a symplectic T-space $(M,\, \sigma,\, T)$ with $\dim M = \dim T + 2$ and symplectic orbits.

(i) An integer $g \geq 0$, an integer $n \geq 0$, and, for each $1 \leq i \leq n$, an integer $o_i > 1$.

(ii) A positive real number $\lambda > 0$.

(iii) A nondegenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$.

(iv) Elements a_j, b_j, $1 \leq j \leq g$ and c_i, $1 \leq i \leq n$, of T such that for each $1 \leq i \leq n$ the order of c_i in T is equal to o_i and the product $c_1 \, \ldots c_n$ of the c_i's is equal to 1.

Let $\mathcal{O}$ be a compact connected oriented orbisurface which as a topological space is a surface of genus g, and has n singular points s_i with singularity order equal to o_i for each $1 \leq i \leq n$. Let $\sigma^{\mathcal{O}}$ be an orbifold smooth area form on $\mathcal{O}$, positive with respect to the orientation of $\mathcal{O}$ and with total area equal to λ. Let p_0 be a base point in $\mathcal{O}$, not equal to one of the singular points. Let α_j, β_j, $1 \leq j \leq g$, and γ_i, $1 \leq i \leq n$, be generators of the orbifold fundamental group $\Gamma := \pi_1^{\text{orb}}(\mathcal{O},\, p_0)$ of $\mathcal{O}$ subject to the relations that the product of the product of the commutators $[\alpha_j,\, \beta_j]$ over all $1 \leq j \leq g$ is equal to the product of the γ_i, $1 \leq i \leq i \leq n$, and ${\gamma_i}^{o_i} = 1$ for every i. Here the α_j, β_j correspond to loops in $\mathcal{O}$ which generate the fundamental group $\pi_1(\mathcal{O},\, p_0)$ of the topological space $\mathcal{O}$, whereas each γ_i is defined by a loop consisting of a path δ_i from p_0 to a point in an orbifold chart around the singular point s_i, followed by a loop in the orbifold chart around s_i, and concluded by running backward along δ_i to p_0.

Let $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}}^{\text{orb}} \to \mathcal{O}$ denote the orbifold universal covering of $\mathcal{O}$, which is a simply connected smooth surface provided with the smooth area form ${\pi_{\mathcal{O}}}^*(\sigma^{\mathcal{O}})$, on which the discrete group Γ acts properly by means of symplectomorphisms, with orbit space isomorphic to $\mathcal{O}$. Let μ be the unique homomorphism from Γ to T such that $\mu(\alpha_j) = a_j$, $\mu(\beta_j) = b_j$, and $\mu(\gamma_i) = c_i$ for all $1 \leq j \leq g$ and $1 \leq i \leq n$. Let $\gamma \in \Gamma$ act on $T \times \widetilde{\mathcal{O}}^{\text{orb}}$ by sending $(t,\, \widetilde{p})$ to $(t\, \mu(\gamma)^{-1},\, \gamma \cdot \widetilde{p})$. This action is proper and free, and the orbit space $M_{\text{model}} = T \times_{\Gamma} \widetilde{\mathcal{O}}^{\text{orb}}$ is a compact connected smooth manifold with $\dim M_{\text{model}} = \dim T + 2$.

The canonical projection $\psi : T \times \widetilde{\mathcal{O}}^{\text{orb}} \to M_{\text{model}}$ is a smooth covering and there is a unique symplectic form σ_{model} on M_{model} such that $\psi^*(\sigma_{\text{model}}) = \sigma^T \oplus \pi_{\mathcal{O}}{}^*(\sigma^{\mathcal{O}})$, where σ^T denotes the invariant symplectic form on T which on the Lie algebra $\mathfrak{t}$ is equal to $\sigma^{\mathfrak{t}}$. Let T act on $M_{\text{model}} = T \times_\Gamma \widetilde{\mathcal{O}}^{\text{orb}}$ by translations on the first factor. We now have the following conclusion, which is [13, Proposition 7.3.6].

Proposition 3.2. $(M_{\text{model}}, \sigma_{\text{model}}, T)$ *is a symplectic T-space with symplectic orbits. Conversely, every* $(\dim T + 2)$*-dimensional symplectic T-space with symplectic orbits is isomorphic to one of these models.*

3.3 Nontrivial torus bundle over a torus with Lagrangian fibers

Let ϵ_1, ϵ_2 be an $\mathbb{R}$-basis of $\mathfrak{t}^*$, and let γ be a nonzero element of $\mathfrak{t}$ such that $\exp(\gamma) = 1$. From these ingredients we will cook up a symplectic T-space (M, σ, T) with $\dim T = 2$, $\dim M = 4$, the T-action is free and the orbits of T are Lagrangian submanifolds of (M, σ). This exhibits M as a topologically nontrivial T-bundle over a torus M/T.

Let $c\colon \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$ be the antisymmetric bilinear mapping such that $c(\epsilon_1, \epsilon_2) = \gamma$. In $G := T \times \mathfrak{t}^*$, define the product

$$(t, \zeta)\,(t', \zeta') = (t\,t'\,\mathrm{e}^{-c(\zeta,\,\zeta')/2}, \zeta + \zeta').$$

With this product, G is a non-commutative two-step nilpotent Lie group. For any integers ζ_1 and ζ_2, assign to $\zeta = \zeta_1\,\epsilon_1 + \zeta_2\,\epsilon_2$ the element τ_ζ in T of order two by means of the formula

$$\tau_\zeta = \mathrm{e}^{\zeta_1\,\zeta_2\,\gamma/2}\,. \tag{6}$$

With the notation $P := \mathbb{Z}\,\epsilon_1 + \mathbb{Z}\,\epsilon_2$, the set $H := \{({\tau_\zeta}^{-1}, \zeta) \mid \zeta \in P\}$ is a cocompact discrete commutative subgroup of G, and $M = G/H$ is a compact and connected smooth manifold on which G acts transitively from the left.

Equip $G = T \times \mathfrak{t}^*$ with the canonical cotangent bundle symplectic form σ^G. Then σ^G is invariant under the right H-action, and therefore equal to the pull-back of a unique symplectic form σ on M by means of the canonical covering map $G \to G/H = M$. The left action of T on $M = G/H$ is free, leaves σ invariant, and has Lagrangian orbits. The projection from $T \times \mathfrak{t}^*$ onto the second factor $\mathfrak{t}^*$ induces a projection from M onto the torus $\mathfrak{t}^*/P$ with fibers equal to the T-orbits. That is, M is a principal T-bundle over the torus $\mathfrak{t}^*/P$, which is topologically nontrivial because its Chern class is equal to the nonzero c.

The product $(X, \zeta)\,(X', \zeta') = (X + X' - c(\zeta, \zeta')/2, \zeta + \zeta')$ turns $\mathfrak{t} \times \mathfrak{t}^*$ into a group such that $\epsilon : (X, \zeta) \mapsto (\mathrm{e}^X, \zeta)$ is a surjective homomorphism from $\mathfrak{t} \times \mathfrak{t}^*$ onto G, inducing an isomorphism from $(\mathfrak{t} \times \mathfrak{t}^*)/\Gamma$ onto $G/H = M$. Here $\Gamma = \epsilon^{-1}(H)$ is a non-commutative cocompact discrete subgroup of $\mathfrak{t} \times \mathfrak{t}^*$. This agrees with the identification of M with a Kodaira space $\mathbb{C}^2/\Gamma$ as in Subsection 3.4.

3.4 Kodaira's spaces $\mathbb{C}^2/\Gamma$

Kodaira has given a classification in [11, Theorem 19] of the compact connected complex analytic surfaces M which carry a holomorphic $(2, 0)$-form ω without zeros. Such an M is real 4-dimensional, and the real part σ of ω is a symplectic form on M, with the additional property that every complex analytic curve in M is a real two-dimensional Lagrangian submanifold of (M, σ).

The first and second cases in Kodaira [11, Theorem 19] consist of the complex tori and the K3 surfaces, respectively. Kodaira proved that the surfaces which are not isomorphic to a complex torus or a K3 surface have the following description. Let $\alpha_3, \alpha_4 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and let $\beta_j \in \mathbb{C}$, $1 \leq j \leq 4$ be such that β_1 and β_2 also are linearly independent over $\mathbb{R}$. Furthermore assume that

$$\overline{\alpha_3}\,\alpha_4 - \overline{\alpha_4}\,\alpha_3 = m\,\beta_2, \tag{7}$$

where m is a non-zero integer. Note that this implies that β_2 is purely imaginary. Let Γ be the group of affine transformations in $\mathbb{C}^2$ generated by

$$\gamma_j : (w_1,\, w_2) \mapsto (w_1 + \alpha_j,\, \overline{\alpha_j}\, w_1 + w_2 + \beta_j),$$

where $\alpha_1 = \alpha_2 = 0$. The condition (7) implies the relation $\gamma_3\,\gamma_4 = {\gamma_2}^m\,\gamma_4\,\gamma_3$ in Γ, where γ_1 and γ_2 belong to the center of Γ. This allows one to write any element of Γ as ${\gamma_1}^{k_1}\,{\gamma_2}^{k_2}\,{\gamma_3}^{k_3}\,{\gamma_4}^{k_4}$ for unique integers k_j, $1 \leq j \leq 4$. If the element leaves a point of $\mathbb{C}^2$ fixed, the fact that its first coordinate remains fixed implies that $k_3 = k_4 = 0$, and then the fact that its second coordinate remains fixed implies that $k_1 = k_2 = 0$. Therefore the action of Γ is free and a similar argument shows that it is proper. It follows that the orbit space $M := \mathbb{C}^2/\Gamma$ is a connected complex two-dimensional complex analytic surface. The standard $(2,\, 0)$-form $\mathrm{d}w_1 \wedge \mathrm{d}w_2$ on $\mathbb{C}^2$ is Γ-invariant, and therefore defines a holomorphic $(2,\, 0)$-form on $M = \mathbb{C}^2/\Gamma$ without zeros. Because $\mathbb{C}^2$ is simply connected, the fundamental group of $M = \mathbb{C}^2/\Gamma$ is isomorphic to Γ. Therefore the homology group $\mathrm{H}_1(M,\, \mathbb{Z})$ of M, isomorphic to the abelianization of the fundamental group, is isomorphic to $\mathbb{Z}^3 \times (\mathbb{Z}/m\,\mathbb{Z})$, and so M is not diffeomorphic to a torus. The translations $(w_1,\, w_2) \mapsto (w_1,\, w_2 + c)$, $c \in \mathbb{C}$, commute with the γ_j, hence with all elements of Γ, and therefore induce an action of $(\mathbb{C},\, +)$ on M. At each point the stabilizer subgroup of this action is $\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2$, and we obtain a free complex analytic action of the elliptic curve $T := \mathbb{C}/(\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2)$ on M, which exhibits M as a complex analytic T-fiber bundle over the elliptic curve $\mathbb{C}/(\mathbb{Z}\,\alpha_3 + \mathbb{Z}\,\alpha_4)$. In particular M is compact. This completes the description of Case 3 in Kodaira [11, Theorem 19].

Theorem 3.3. *With $M = \mathbb{C}/\Gamma$, $\sigma = \mathrm{Re}(\omega)$, and $T = \mathbb{C}/(\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2)$, $(M,\, \sigma,\, T)$ is a symplectic T-space as in Subsection 3.3 = Case (4) in Theorem 3.1. Conversely every symplectic T-space as in Subsection 3.3 is isomorphic to a Kodaira space* $(\mathbb{C}/\Gamma,\, \mathrm{Re}(\omega),\, \mathbb{C}/(\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2))$.

Proof. We identify both $\mathfrak{t}$ and $\mathfrak{t}^*$ with $\mathbb{C}$, when $P = \mathbb{Z}\,\alpha_3 + \mathbb{Z}\,\alpha_4$. It suffices to determine c for $\mathbb{C}^2/\Gamma$ and check c and P are as general as in Subsection 3.3. Consider the vector fields

$$L_j := (\alpha_j,\, \alpha_j\,\overline{w_1}) = (\mathrm{Re}(\alpha_j),\ \mathrm{Im}(\alpha_j),\, \mathrm{Re}(\alpha_j)\,u_1 + \mathrm{Im}(\alpha_j)\,v_1,\, -\mathrm{Re}(\alpha_j)\,v_1 + \mathrm{Im}(\alpha_j)\,u_1)$$

on $\mathbb{C}^2 \simeq \mathbb{R}^4$, where $w_k = u_k + \mathrm{i}\,v_k$ with $u_k,\, v_k \in \mathbb{R}$, and $L_1 = L_2 = 0$. The vector fields L_3, L_4 are lifts as in Proposition 2.2 of the $\mathbb{Z}$-basis α_3, α_4 of the lattice P in $\mathfrak{t}^*$. Furthermore

$$[L_3,\, L_4] = \Big(0,\, 0,\, \mathrm{Re}(\alpha_3\,\overline{\alpha_4}) - \mathrm{Re}(\alpha_4\,\overline{\alpha_3}),\, \mathrm{Im}(\alpha_3\,\overline{\alpha_4}) - \mathrm{Im}(\alpha_4\,\overline{\alpha_3})\Big) = (0,\, m\,\beta_2),$$

where the second identity follows from (7). That is, $c(\alpha_3,\, \alpha_4)$ is equal to the non-zero element $m\,\beta_2$ of the integral lattice $T_{\mathbb{Z}}$ of T in $\mathfrak{t}$.

Assume conversely that we are in the situation of Subsection 2.2.1. Then for any $\mathbb{R}$-basis θ_1, θ_2 of $\mathfrak{t}^*$, $Z = c(\theta_1,\, \theta_2)$ is a non-zero element of $\mathfrak{t}$, which implies that $\theta_1(Z)$ and $\theta_2(Z)$ are not both equal to zero. By interchanging θ_1 and θ_2 if necessary, we can arrange that $r = \theta_2(Z) \neq 0$. Let ρ be any non-zero real number. The linear forms $\epsilon_1 := \rho\,(r\,\theta_1 - \theta_1(Z)\,\theta_2)$ and $\epsilon_2 = r^{-1}\,\theta_2$ on $\mathfrak{t}$ form an $\mathbb{R}$-basis of $\mathfrak{t}^*$ such that $\epsilon_2(Z) = 1$, $\epsilon_1(Z) = 0$, and $c(\epsilon_1,\, \epsilon_2) = \rho\,Z$. Therefore, if e_1, e_2 is the $\mathbb{R}$-basis of $\mathfrak{t}$ dual to ϵ_1 and ϵ_2, then $c(\epsilon_1,\, \epsilon_2) = \rho\,e_2$.

If we provide $\mathfrak{t}$ with the complex structure such that e_1 and e_2 correspond to 1 and i, respectively, and we identify the complex number $\alpha \in \mathbb{C} \simeq \mathfrak{t}$ with the real-linear form $\beta \mapsto \mathrm{Re}(\alpha\,\beta)$ on $\mathbb{C} \simeq \mathfrak{t}$, then

$$c(\alpha,\, \alpha') = (\rho/2)\,(\alpha\,\overline{\alpha'} - \alpha'\,\overline{\alpha}).$$

This leads to the conclusion that we can arrange that the integral lattice $T_{\mathbb{Z}}$ of T in $\mathfrak{t} \simeq \mathbb{C}$ is of the form $\mathbb{Z}\,\beta_1 + \mathbb{Z}\,\beta_2$, the lattice P in $\mathfrak{t}^* \simeq \mathbb{C}$ is of the form $\mathbb{Z}\,\alpha_3 + \mathbb{Z}\,\alpha_4$, and $c(\alpha_3,\, \alpha_4) = \overline{\alpha_3}\,\alpha_4 - \overline{\alpha_4}\,\alpha_3 = m\,\beta_2$, where β_1, β_2 form an $\mathbb{R}$-basis of $\mathbb{C}$, α_3, α_4 form an $\mathbb{R}$-basis of $\mathbb{C}$, and m is a non-zero integer. □

Remark 3.2 It follows from $\mathrm{H}_1(M,\,\mathbb{Z}) \simeq \mathbb{Z}^3 \times (\mathbb{Z}/m\,\mathbb{Z})$ that the first Betti number $b_1(M) = 3$. The first example of a symplectic manifold that does not admit a Kähler structure by Thurston [14], which did not refer to Kodaira [11, Theorem 19], is one of the above examples with $m = 1$. Thurston observed that as all odd-dimensional Betti numbers of a Kähler manifold are even, see for instance Griffiths and Harris [8, p. 117], his example of a compact connected 4-dimensional symplectic manifold cannot be provided with a Kähler structure. We see that the same conclusion holds for all examples in Kodaira [11, Theorem 19, Case 3]. $\oslash$

3.5 Proof of Theorem 1.1

We first prove that in the cases (1), (2), (3), and (5) of Theorem 3.1 the manifold M admits a T-invariant complex structure with a corresponding Kähler form equal to σ.

Case (1). As mentioned in Section 1, Delzant [4] proved that every Delzant space is isomorphic to a smooth toric variety with σ equal to a Kähler form on it, where the action of T extends to a holomorphic action of the complexification $T_{\mathbb{C}}$ of T. See Guillemin [10, App. 2] for a more explicit description of the Kähler structure.

Case (2). Take an invariant complex structure on T and the standard one on the Riemann sphere S^2. Then the Hermitian forms on T and S^2 of which the respective symplectic forms are the imaginary parts are positive or negative definite. By changing the signs of the complex structures if necessary, we obtain a T-invariant complex structure on M with σ equal to a corresponding Kähler form.

Case (3). Every symplectic vector space admits a complex structure and corresponding Hermitian form such that the symplectic form is equal to the imaginary part of the Hermitian form, see for instance [7, Section 15.5]. If we apply this to the Lie algebra of the torus M in Case (3) of Theorem 3.1, we obtain a T-invariant complex structure on M with σ equal to a corresponding Kähler form.

In Case (4) of Theorem 3.1, which according to Theorem 3.3 is Kodaira's space $\mathbb{C}^2/\Gamma$, we have a T-invariant complex structure on M, whereas Remark 3.2 implies that M does not admit any Kähler structure.

Case (5). The orbifold universal covering $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ of the orbisurface $\mathcal{O}$ in Subsection 3.2 can be identified with the Riemann sphere, the Euclidean plane, or the hyperbolic plane, on which the orbifold fundamental group Γ of $\mathcal{O}$ acts by means of orientation preserving isometries, see Thurston [15, Section 5.5]. If we provide this $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ with the standard complex structure, then for a suitable strictly positive multiple of the Hermitian form the imaginary part is a Γ-invariant area form such that the area of $\mathcal{O} = \widetilde{\mathcal{O}}^{\mathrm{orb}}/\Gamma$ is equal to the number λ in (ii) of Subsection 3.2. As we can provide the 2-dimensional torus T with a T-invariant complex structure such that σ^T is equal to a Kähler form, we obtain a T-invariant complex structure on $M = T \times_{\Gamma} \widetilde{\mathcal{O}}^{\mathrm{orb}}$ with σ equal to a corresponding Kähler form. This completes the proof of Theorem 1.1.

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J.J. Duistermaat
Mathematisch Instituut, Universiteit Utrecht
P.O. Box 80 010, 3508 TA Utrecht, The Netherlands
e-mail: J.J.Duistermaat@uu.nl

A. Pelayo
University of California–Berkeley
Mathematics Department, 970 Evans Hall 3840
Berkeley, CA 94720-3840, USA.
E-mail: apelayo@math.berkeley.edu