A geometric approach to the classification of the equilibrium shapes of self-gravitating fluids

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Abstract

The classification of the equilibrium shapes that a self-gravitating fluid can take in a Riemannian manifold is a classical problem in Mathematical Physics. In this paper it is proved that the equilibrium shapes are isoparametric submanifolds. Some geometric properties of them are also obtained, e.g. classification and existence for some Riemannian spaces and relationship with the isoperimetric problem and the group of isometries of the manifold. Our approach to the problem is geometrical and allows to study the equilibrium shapes on general Riemannian spaces.

1 Introduction

Let (M, g) be an analytic, complete and connected (without boundary) Riemannian n-manifold and Ω an open connected subset of M (bounded or not) occupied by a mass of fluid. We say that a fluid is self-gravitating if the only significative forces are its interior pressure and its own gravitation. A fluid in these conditions represents a simplified stellar model of fluid-composed star. Depending on whether the gravitational field is modelled by Poisson or Einstein equations we say that the fluid is Newtonian or relativistic. An important problem in Fluid Mechanics consists in studying the shape that a self-gravitating fluid will take when it reaches the equilibrium state. By the term shape of a fluid it is meant the topological and geometrical properties of the boundary $\partial\Omega$. The mathematical description of this kind of fluids only involves three physical quantities, the gravitational potential, which is a function $f_1 : M \to \mathbb{R}$ (constant on $\partial\Omega$), and the density and pressure, which are two functions $f_2, f_3 : \Omega \to \mathbb{R}$. The set of partial differential equations fulfilled by the functions (f_1, f_2, f_3) is of free-boundary type because the domain Ω is an unknown of the problem. In the relativistic case the metric tensor g is also an unknown and it must satisfy the coupling condition

$$R_{ab} = f_1^{-1} f_{1;ab} + 4\pi(f_2 - f_3) g_{ab} \,, \tag{1}$$

R_{ab} standing for the Ricci tensor and ; standing for the covariant derivative.

The standard approaches to the problem of classifying the equilibrium shapes of $\partial\Omega$ generally employ analytical techniques. In the Newtonian case maximum principles for elliptic equations are used in order to prove the existence of symmetries of the solutions (f_1, f_2, f_3). Lichtenstein [1] and later on Lindblom [2] proved the existence of spherical symmetry (i.e. $\partial\Omega$ is a round sphere) when M is the Euclidean $\mathbb{R}^3$, Ω is bounded and the functions (f_1, f_2, f_3) satisfy some physical constraints. In the relativistic case arguments involving the positive mass theorem are used for obtaining the conformal flatness of the metric tensor. As a consequence of this technique Beig&Simon [3] and Lindblom&Masood-ul-Alam [4] proved again spherical symmetry when Ω is bounded and the solutions verify certain physical hypotheses. Despite of these important results many questions remain open: What about Newtonian fluids on Riemannian manifolds? What about unbounded domains Ω? Do the same results hold if we drop the physical assumptions?

In this paper we approach the problem in a different, more geometrical way. Indeed, since the gravitational potential f_1 is constant on $\partial\Omega$ then the study of the geometric properties of the level sets of f_1 is an effective procedure to classify the equilibrium shapes that the fluid can take, thus connecting our problem with the philosophy of the geometric theory of PDEs. The literature on this field is essentially focused on the study of general properties of the level sets of the solutions to differential equations, e.g. critical levels [5, 6], convexity or starshapedness of the levels [7, 8], order of vanishing and measure for level sets [9, 10], symmetries of the levels due to overdetermined boundary conditions [11, 12, 13, 14, 15], ... This work provides another contribution to the abundant literature on geometric theory of PDEs, and as far as we know, the techniques that we develop for classifying the shapes of $\partial\Omega$ (e.g. the analytic representation property) are new and independent of the other approaches to similar problems. It is interesting to observe that in Serrin's paper [11, 12] it is proved that solutions to certain boundary problems exist only when the domain Ω is a sphere (for related results see the literature on overdetermined boundary problems, e.g. [15] and references therein). However Serrin's method, somehow related to the classical approaches to Newtonian self-gravitating fluids by Lichtenstein and Lindblom (i.e. moving plane technique and maximum principles), bear no similarity with our methods (in fact the physical motivation of Serrin's problem is not related to static fluids).

Let me summarize the organization of this paper in three blocks.

- In sections 2 and 3 we establish the notation of the paper, prove some elementary lemmas and formulate the problem. The equations that we study include the Newtonian and the relativistic (without taking into account equation (1)) as particular cases. In section 4 we prove that the function f_1 is analytically representable across $\partial\Omega$ and as a consequence of this remarkable property f_1 is proved to satisfy the equilibrium condition. This result is particularly relevant from the physical viewpoint since it allows us to classify the equilibrium shapes of a self-gravitating fluid on any Riemannian manifold. Our techniques are different to the previous ones appeared in the literature on self-gravitating fluids, and could be of interest to PDE theorists working on (overdetermined) free-boundary problems.

- In the second block we study the geometrical consequences of the equilibrium condition. In section 5 we prove that the (regular) level sets of the function f_1 are isoparametric submanifolds, thus connecting the shapes of static fluids with these classical objects of differential geometry, and in section 6 we obtain classification theorems for certain spaces (M, g). In section 7 necessary and sufficient conditions for the existence of certain equilibrium shapes are obtained and the 3-manifolds admitting fluid-composed stars are classified. In section 8 the relationship between the equilibrium shapes, the isoperimetric problem and the Killing vector fields of M is studied. Apart from the physical relevance (most of our statements have not been obtained through the classical approaches) our results could be of interest to differential geometers.

- In sections 9 and 10 some open problems are discussed, being the most important the extension of our techniques to general relativistic fluids.

2 Notation and preliminary results

Let f be a smooth, real-valued function on M. For each $c \in f(M) \subset \mathbb{R}$ we have a pre-image $f^{-1}(c)$. Let V_c^i be the i-th connected component of $f^{-1}(c)$. The *partition* induced on M by f is defined as

$$\beta_M(f) = \bigcup_{i,c} V_c^i .$$

Analogously if we have an open set $U \subset M$ the partition induced on U by $f_{|U}$ is called $\beta_U(f)$. Each V_c^i is called a leaf of the partition. In general the dimension of the leaves of the partition is not constant because there can exist singular fibres ($\nabla f = 0$) and therefore $\beta_M(f)$ is a singular foliation.

We say that a function f represents the partition Γ of U if $\beta_U(f) = \Gamma$. If f is real analytic (C^ω) then Γ is said to admit an *analytic representation*. Analogously we call f analytically representable on $U \subset M$ if $\beta_U(f)$ admits an analytic representation. We will say that a family $\{f_i : M \to \mathbb{R}\}$ agrees fibrewise if $\beta_M(f_i) = \beta_M(f_j)$ for all i, j.

The following extension lemma will be useful in forthcoming sections.

Lemma 2.1. *Let f, g be real analytic functions on M. Let $U \subset M$ be an open set. Then $\beta_U(f) = \beta_U(g) \Longrightarrow \beta_M(f) = \beta_M(g)$.*

Proof. Since $\beta_U(f) = \beta_U(g)$ we have that $\mathrm{rank}(\mathrm{d}f, \mathrm{d}g) \leq 1$ in U. Since U is open and f, g are analytic functions this condition extends to the whole M, i.e. $\mathrm{rank}(\mathrm{d}f, \mathrm{d}g) \leq 1$ in M. This implies [16] that f and g are functionally dependent and hence there exists an analytic function $Q : \mathbb{R}^2 \to \mathbb{R}$ such that $Q(f, g) = 0$ in M, thus showing that the partitions induced by f and g agree. □

Let S be a codimension one orientable submanifold in M. The metric induced by g on S is given by $\beta_{ab} = g_{ab} - n_a n_b$ [17], where n^a is the unit normal vector field to S on M. The extrinsic curvature or second fundamental form of S is defined as $H_{ab} = \beta_a^c \beta_b^d n_{d;c} = \frac{1}{2} L_n(\beta_{ab})$ [17], L_n standing for the Lie derivative with respect to the unit normal vector field. If S is given as the zero-set of the function f, i.e. $S = \{p \in M : f(p) = 0\}$ and $\mathrm{d}f_{|S} \neq 0$ then the mean curvature H of S (the trace of the second fundamental form) is given by the following expression:

$$H = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right) \tag{2}$$

div and ∇ standing for the divergence and gradient operators on the Riemannian manifold.

If R'^a_{bcd} is the Riemann curvature tensor induced by R^a_{bcd} on S then Gauss theorem implies [17] that

$$R' = R - 2R_{ab} n^a n^b + (H_a^a)^2 - H_{ab} H^{ab} \,. \tag{3}$$

¿From this expression we deduce the relationship between the intrinsic sectional curvature of S (K'), the sectional curvature of M restricted to S (K) and the Gauss curvature of S ($\bar{K}$), namely $K' = K + \bar{K}$ [17].

Let us finish this section with the following lemma [18].

Lemma 2.2. *Let f be a smooth function on an open set $U \subseteq M$ saturated by level sets of f. If $|\mathrm{d}f| \geq m > 0$ on U then f is a (locally trivial) fibre bundle on U.*

Proof. The normal vector field $X = \frac{\nabla f}{(\nabla f)^2}$ is smooth on U and a symmetry of f because $L_X(f) = 1$ [19]. X is complete because (M, g) is complete and $|X| \leq \frac{1}{m}$, thus defining a global 1-parameter group of diffeomorphisms [20]. □

3 Statement of the problem

The problem (P) in which we are interested is a system of PDEs defined on M. Its form and additional regularity assumptions are inspired by the equations modelling static self-gravitating fluids, both Newtonian and relativistic [21]. The boundary of the fluid region, $\partial\Omega$, is assumed to be a codimension one analytic submanifold (connected or not). The functions f_2 and f_3 are required to be analytic in $\bar{\Omega}$, constant on $\partial\Omega$ and f_3 is not allowed to be constant in the whole Ω.

With these assumptions in mind we state now the equations of problem $(\mathbf{P})$ (Δ is the Laplace–Beltrami operator on the manifold). In this paper we consider only *classical solutions.*

$$\Delta f_1 = F(f_1, f_2, f_3) \text{ in } \Omega \tag{4}$$

$$H(f_1)\nabla f_3 + G(f_2, f_3)\nabla f_1 = 0 \text{ in } \Omega \tag{5}$$

$$f_1 = c,\ c \in \mathbb{R},\ \nabla f_1 \neq 0 \text{ and } f_1 \in C_t^2 \text{ on } \partial\Omega \tag{6}$$

$$\Delta f_1 = 0 \text{ in } M - \bar{\Omega} \tag{7}$$

where F, G and H are (not identically zero) analytic functions. We also impose that G is not a constant. The symbol C_t^2 in (6) means that f_1 is C^1 on the boundary and its tangential second derivatives $f_{1,ij}t^j$ are continuous for any (local) vector field $t = t^i\partial_i$ tangent to the boundary. This assumption is analogous to Synge's junction condition for the metric tensor in General Relativity [22]. The normal components of the second derivatives will not be continuous in general.

Note that the constant c in (6) is not a priori prescribed and that the domain Ω is an unknown of (P). For the sake of simplicity we have assumed that $\nabla f_1 \neq 0$ on $\partial\Omega$, but all the results of this paper hold only requiring that ∇f_1 is not identically zero on the boundary. It is interesting to observe that one only needs to assume that $\partial\Omega$ is smooth enough, its analyticity following from general properties of elliptic free-boundary equations [23, 24].

Remark 3.1. *The solutions to problem (P) are the functions (f_1, f_2, f_3) and the domains Ω satisfying all the required conditions. For certain values of F, G and H or certain manifolds (M, g) problem (P) could have no solutions at all. Since we are not interested in the existence problem we will suppose that solutions to (P) exist and characterize the structure of the level sets of these solutions. This is in strong contrast to the classical approaches where the problem of existence and uniqueness is first considered and then the geometrical restrictions arise.*

Let us prove three important (elementary) lemmas which will be used in the following sections. The first one is a consequence of [25, 26, 27], and is stated without proof.

Lemma 3.2. *The solutions f_1 to (P) are analytic on $M - \partial\Omega$.*

Lemma 3.3. *The solutions f_1, f_2 and f_3 of (P) agree fibrewise in Ω.*

Proof. ¿From (5) it follows that $\mathrm{d}\left(\frac{\mathrm{d}f_3}{G(f_2,f_3)}\right) = -\mathrm{d}\left(\frac{\mathrm{d}f_1}{H(f_1)}\right) = 0$ and hence $G_{,f_2} df_2 \wedge \mathrm{d}f_3 = 0$ (the subscript denotes, as usual, partial differentiation). Therefore ∇f_2 and ∇f_3 are linearly dependent. Also from (5) $\mathrm{d}f_3 = -\frac{G(f_2,f_3)}{H(f_1)}\mathrm{d}f_1$ and taking the exterior derivative in this equation we get $(G_{,f_2}\mathrm{d}f_2 + G_{,f_3}\mathrm{d}f_3) \wedge \mathrm{d}f_1 = 0$. The linear dependence of ∇f_2 and ∇f_3 implies the linear dependence of all ∇f_1, ∇f_2 and ∇f_3. This fact and the analyticity of the functions imply that they agree fibrewise in the whole Ω. □

For any point $p \in \partial\Omega$ consider a small enough open neighborhood $U \subset M$. Define the open sets $U_{\rm in} = U \cap \Omega \neq \emptyset$ and $U_{\rm out} = U \cap (M - \overline{\Omega}) \neq \emptyset$.

Lemma 3.4. *f_2 and f_3 are functions of f_1 in U_{in}.*

Proof. Suppose that ∇f_1 does not vanish in U (it is always possible by continuity and the fact that $(\nabla f_1)_{|\partial\Omega} \neq 0$). If we cover U with a local coordinate system $(x_1, \ldots, x_n)$ we can assume, without loss of generality, that $f_{1,x_1} \neq 0$. The implicit function theorem guarantees the following steps in $U_{\rm in}$:

$$x_1 = X_1(f_1, x_2, \ldots, x_n)$$
$$\Longrightarrow f_2 = f_2(X_1(f_1, x_2, \ldots, x_n), x_2, \ldots, x_n) \equiv F_2(f_1, x_2, \ldots, x_n)$$
$$\Longrightarrow f_3 = f_3(X_1(f_1, x_2, \ldots, x_n), x_2, \ldots, x_n) \equiv F_3(f_1, x_2, \ldots, x_n).$$

It is easy to check that $F_{i,x_2} = \ldots = F_{i,x_n} = 0$, $i = 2, 3$. One only has to take into account the implicit function theorem and lemma 3.3. Hence in $U_{\rm in}$ we get that $f_2 = F_2(f_1)$ and $f_3 = F_3(f_1)$, where F_2 and F_3 are analytic functions of their argument. □

Since the pressure, the density and the gravitational potential induce the same partition in $U_{\rm in}$ we can reduce our study to $\beta_M(f_1)$, which has the advantage of being defined on the whole M. The following sections are devoted to understand the topological and geometrical properties of this partition, which will only depend on the base manifold (M, g).

4 Analytic representation and proof of the classification theorem

When working with partitions of $U \subseteq M$ it is reasonable to try to represent them by functions as good as possible, e.g. analytic functions. The general programme would be to substitute the "pathological" function $g : U \to \mathbb{R}$, which represents the partition, by an analytic function f satisfying $\beta_U(g) = \beta_U(f)$.

In general a partition cannot be analytically represented. Even in the case in which the function g is analytic in the whole M except for the fibre $g^{-1}(0)$ (as happens with the potential f_1 solution to (P)), an analytic representation does not need to exist. The following examples illustrate the difficulties which may arise.

Example 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(x) = |x|^2 - 1$ ($|.|$ standing for the Euclidean norm) and let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(t) = t\exp(-1/t^2)$ if $t \neq 0$ and $h(0) = 0$. Then the function $g = f + h \circ f$ is C^∞ on $\mathbb{R}^n$, analytic on $\mathbb{R}^n - g^{-1}(0)$ and agrees fibrewise with f. So f is an analytic representation of g.*

Example 4.2. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined in coordinates $(x_1, \ldots, x_n)$ as*

$$g(x_1, \ldots, x_n) = \begin{cases} x_1(1 + x_2 \exp(-1/x_1^2)) & \text{if } x_1 > 0 \\ x_1 & \text{if } x_1 \leq 0 \end{cases}$$

It is not difficult to prove that g does not admit analytic representation in any neighborhood of the fibre $\{x_1 = 0\}$. Similar examples can be constructed when the "pathological" fibre is compact.

In spite of these results, the gravitational potential f_1, which is generally not analytic on the boundary, has a remarkable property: its partition can be represented by analytic functions in a neighborhood of $\partial\Omega$. Before proving this result let us comment on a physical reason supporting this property. Since $\partial\Omega$ is an analytic submanifold and $(\nabla f_1)_{|\partial\Omega} \neq 0$ then there exists an analytic function $I : M \to \mathbb{R}$, arbitrarily close to f_1 (in the C^∞ strong topology), such that $I^{-1}(0) = \partial\Omega$ [28]. This proves that a small perturbation of the gravitational potential makes its partition analytically representable. Since two arbitrarily close partitions are indistinguishable from the physical viewpoint, this suggests that $\beta_M(f_1)$ admits itself analytic representation.

The rigorous proof of this result is a modification of a technique developed by Lindblom in [29].

Theorem 4.3 (Analytic representation property)**.** *Let f_1 be any solution to (P), then $\beta_M(f_1) = \beta_M(I)$ where $I : M \to \mathbb{R}$ is a function analytic on the whole M.*

Proof. Let U be a small enough neighborhood of $p \in \partial\Omega$. On account of lemma 3.4 we can write the equations defining (P) in terms of just f_1:

$$\Delta f_1 = \hat{F}(f_1) \text{ in } U_{\text{in}} \tag{8}$$

$$f_1 = c,\ c \in \mathbb{R},\ \nabla f_1 \neq 0 \text{ and } f_1 \in C^2_t \text{ on } \partial\Omega \cap U \tag{9}$$

$$\Delta f_1 = 0 \text{ in } U_{\text{out}} \tag{10}$$

Consider a vector field $\xi = \xi^i \partial_i$ which is a symmetry of f_1 in U_{in}, i.e. $\xi(f_1) = 0$. Note that ξ can always be chosen analytic. The assumption that f_2 and f_3 are analytic in $\bar{\Omega}$ implies that $\hat{F}$ in equation (8) is an analytic function of f_1. It follows that the interior solution f_1, satisfying equation (8) and the boundary condition $f_1 = c$, is analytically continuable across the boundary [30], although its continuation does not generally coincide with the exterior solution. Consequently ξ^i and $\xi^i_{,j}$ can be extended to $\partial\Omega$ as analytic functions and in fact $\xi(f_1)_{|\partial\Omega} = 0$. Now let us extend the vector field ξ beyond the free-boundary in such a way that the extension is analytic in the whole U.

The components of the new vector field $\hat{\xi}$ in U are defined by the following boundary problem:

$$\Delta\hat{\xi}^k + \frac{2D_\mu D_k f_1}{f_{1,k}}(D^\mu \hat{\xi}^k) + \frac{R_{ik} D^i f_1}{f_{1,k}} \hat{\xi}^k = 0\,, \tag{11}$$

$k = 1, \ldots, n$, provided with the boundary conditions on $\partial\Omega$ given by the values $(\hat{\xi}^i)_{|\partial\Omega} = (\xi^i)_{|\partial\Omega}$ and $(\hat{\xi}^i_{,j})_{|\partial\Omega} = (\xi^i_{,j})_{|\partial\Omega}$ of the interior symmetry. Note that the symbol D stands for the covariant derivative and R_{ik} is the Ricci curvature. In order that (11) be well defined assume, without loss of generality, that in U all the components of ∇f_1 are non zero. Since all the terms of (11) are analytic, the boundary conditions and $\partial\Omega$ are also analytic and the equation is linear and elliptic there exists a (unique) extension which defines an analytic vector field in U [25, 26] (Cauchy-Kowalewsky theorem).

This analytic vector field has the property of being a symmetry of f_1 in U. Indeed, the first step consists in the following computation

$$\begin{aligned} \Delta(\hat{\xi}^k f_{1,k}) = D_\mu D^\mu (\hat{\xi}^k f_{1,k}) = \\ = f_{1,k}\Big(\Delta\hat{\xi}^k + \frac{2D_\mu D_k f_1}{f_{1,k}}(D^\mu \hat{\xi}^k) + \frac{D^\mu D_\mu D_k f_1}{f_{1,k}}\,\hat{\xi}^k\Big). \end{aligned} \tag{12}$$

Note now that $D_\mu D^\mu D_k f_1 = D_k(\Delta f_1) + R_{ik} D^i f_1$ and since f_1 is solution of (P) we get from equations (11) and (12) that

$$\Delta(\hat{\xi}^k f_{1,k}) = \hat{F}'(f_1)\hat{\xi}^k f_{1,k} \text{ in } U_{\text{in}} \tag{13}$$

$$\Delta(\hat{\xi}^k f_{1,k}) = 0 \text{ in } U_{\text{out}} \tag{14}$$

The boundary conditions are the following. First note that $f_{1,k}\hat{\xi}^k$ is C^1 on $\partial\Omega$. Indeed, recall that on $\partial\Omega$ f_1 is C^1 and its tangential second derivatives are continuous, therefore $f_{1,k}\hat{\xi}^k$ and $(f_{1,k}\hat{\xi}^k)_i = f_{1,ik}\hat{\xi}^k + f_{1,k}\hat{\xi}^k_{,i}$ are also continuous on the boundary. The same applies to $f_{1,k}\xi^k$. Since $f_{1,k}\xi^k = 0$ in U_{in} then $(f_{1,k}\xi^k)_{|\partial\Omega} = 0$ and $\partial_i(f_{1,k}\xi^k)_{|\partial\Omega} = 0$, and hence the boundary conditions are $(f_{1,k}\hat{\xi}^k)_{|\partial\Omega} = 0$ and $\partial_i(f_{1,k}\hat{\xi}^k)_{|\partial\Omega} = 0$. Note now that equation (13) is analytic in $\bar{U}_{\text{in}}$ because the interior solutions f_1, f_2, f_3 can be analytically continued across the boundary. Holmgren's theorem for linear analytic elliptic equations [31] implies that the (unique) C^1 solutions to (13) and (14) provided with the boundary conditions are $\hat{\xi}^k f_{1,k} = 0$ in U_{in} and $\hat{\xi}^k f_{1,k} = 0$ in U_{out}, thus showing that $\hat{\xi}^k f_{1,k} = 0$ in the whole U.

Summarizing we have a (local) Lie algebra of $n-1$ independent analytic symmetries of f_1 in U. From this Lie algebra we can reconstruct the partition $\beta_U(f_1)$ via Frobenius theorem, the analyticity of $\hat{\xi}$ implying that the partition is analytic [32]. This ensures the existence of an analytic function $\hat{I} : U \to \mathbb{R}$ such that $\beta_U(f_1) = \beta_U(\hat{I})$. By lemma 3.4 it follows that $\hat{I} = F(f_1)$, thus showing that $\hat{I}$ extends to a saturated neighborhood of $\partial\Omega$, $N(\partial\Omega)$, as analytic function $\hat{I} : N(\partial\Omega) \to \mathbb{R}$. This result and the analyticity of f_1 in $M - \partial\Omega$ prove that $\beta_M(f_1)$ is analytically representable across any leaf, and therefore there exists an analytic extension $I : M \to \mathbb{R}$ of $\hat{I}$ such that $\beta_M(f_1) = \beta_M(I)$, thus proving the claim.

□

Concerning the physical meaning of the analytic representation property (ARP) we must say the following. From the proof of theorem 4.3 it follows that the interior symmetries propagate across the free-boundary and remain symmetries of the exterior solution. This implies that a physical matching on a free-boundary (at least in static situations) does not only guarantee the continuity of the gravitational field but also the dependence between the external and internal properties of the fluid.

Definition 4.4. *An analytic function $I : M \to \mathbb{R}$ is of* equilibrium *on $U \subseteq M$ if I, $(\nabla I)^2$ and ΔI agree fibrewise on U. A partition induced by an equilibrium function is called an equilibrium partition.*

Let us now prove the classification theorem.

Theorem 4.5 (Classification theorem)**.** *If f_1 is a solution to the problem (P) then $\beta_M(f_1)$ is an equilibrium partition*

Proof. If $p \in \partial\Omega$ and U is a small neighborhood of p then f_2 and f_3 can be expressed as functions of f_1 in U_{in} (see lemma 3.4), thus implying that $\Delta f_1 = \hat{F}(f_1)$ in U_{in}. In U_{out} the potential satisfies $\Delta f_1 = 0$. Theorem 4.3 ensures the existence of an analytic function I agreeing fibrewise with f_1. The same technique as in lemma 3.4 can be applied in order to show that $f_1 = R(I)$ in U.

¿From equation (7) it follows $\Delta R(I) = 0$ in U_{out}, which is equivalent to

$$R''(I)(\nabla I)^2 + R'(I)\Delta I = 0, \tag{15}$$

and therefore $\frac{\Delta I}{(\nabla I)^2} = \Theta(I)$ in U_{out}. Since I is analytic so are ΔI and $(\nabla I)^2$ and hence $\frac{\Delta I}{(\nabla I)^2}$ is analytic in U. Let $\hat{\Theta}$ be the analytic continuation of Θ to U (which indeed exists because of the analyticity of I and $\frac{\Delta I}{(\nabla I)^2}$). Then $\frac{\Delta I}{(\nabla I)^2} = \hat{\Theta}(I)$ in U and in particular in U_{in}. On the other hand

$$R''(I)(\nabla I)^2 + R'(I)\Delta I = \hat{F}(I) \tag{16}$$

in U_{in} and together with (15) implies that ΔI and $(\nabla I)^2$ depend only on I. The argument applies to the whole of U by analyticity and in fact the property of agreeing fibrewise extends (lemma 2.1) to the whole M (although generally ΔI and $(\nabla I)^2$ can be written as functions of I only locally). Since I and f_1 agree fibrewise we have that $\beta_M(f_1)$ is an equilibrium partition. □

Theorem 4.5 provides a complete characterization of the level sets of the solutions to problem (P). In the following sections we will explore the geometrical and topological meaning of the equilibrium condition. This theorem applies to Newtonian self–gravitating fluids thus characterizing their equilibrium shapes on any Riemannian manifold. It also works for relativistic fluids without coupling between matter and geometry (see equation (1)). This kind of relativistic models, where the base space is predetermined, is used in some applications of interest [33]. In section 9 it will be discussed how to extend our techniques to general relativistic fluids.

Remark 4.6. *It is interesting to observe that we do not impose additional assumptions in order to characterize the structure of the level sets of the potential. In the literature additional hypotheses are usually considered: $|\nabla f_1|$ is a function of f_1 [34], existence of a "reference spherical model" [3] or physical constraints, e.g. positivity of the density and pressure, asymptotic structure of the potential and existence of state equation [1, 2, 4].*

5 General geometric properties of the equilibrium shapes

Theorem 4.5 reduces the original problem involving a difficult system of PDEs to a purely geometrical problem: the classification of equilibrium partitions on different spaces. For instance, in sections 7 and 8, we will show examples of manifolds which do not admit equilibrium functions of certain types, thus obtaining by geometrical arguments an existence result: (P) cannot have these types of solutions on these spaces.

Although the definition of equilibrium partition involves a particular function I the concept is mainly geometrical, as the following proposition shows.

Proposition 5.1. *Any analytic function representing an equilibrium partition Σ is an equilibrium function.*

Proof. By hypothesis there exists an equilibrium function I representing Σ. Consider another function $\hat{I}$ representing the same partition. By using the same argument as in lemma 3.4 it is immediate to prove that $\hat{I} = F(I)$ for certain open set U, F being an analytic function. Since I is of equilibrium we have that locally (by the same argument) $(\nabla I)^2$ and ΔI are functions of I. Now a straightforward computation yields that $(\nabla \hat{I})^2 = F'(I)^2(\nabla I)^2$ and $\Delta\hat{I} = F''(I)(\nabla I)^2 + F'(I)\Delta I$. This implies that locally $(\nabla\hat{I})^2$ and $\Delta\hat{I}$ are functions of $\hat{I}$, thus proving that $\hat{I}$ is a local equilibrium function. The globalization of this property follows from lemma 2.1. □

Let us now prove a remarkable result which relates the equilibrium condition to the well known *isoparametric* property. Recall that a smooth function $f : M \to \mathbb{R}$ is called isoparametric if $(\nabla f)^2 = F(f)$ and $\Delta f = G(f)$ in M, F and G being differentiable functions [35]. A regular level set of an isoparametric function is called isoparametric submanifold and the union of level sets is called isoparametric family. This concept was firstly introduced by Levi-Civita [36], Cartan [37, 38] and Segre [39] in a purely geometrical context. Two good surveys on this topic are the works of Nomizu [40] and Thorgbersson [41]. Note that in the literature it is sometimes considered another definition for isoparametric family [42], which is not equivalent to the one considered in this paper.

Before proving the theorem let us state some notation. Assume (without loss of generality) that the equilibrium function I has N different critical values $\{c_i\}_1^N$ (since I is analytic the set of critical values is discrete in $\mathbb{R}$) and that $I(M) = (-\infty, +\infty)$. The manifold M is therefore stratified as follows, $M = \bigcup_{i=1}^{N+1} M_i \cup C(f)$, $C(f)$ standing for the critical set of I and $M_i = I^{-1}(c_{i-1}, c_i)$ ($c_0 = -\infty$ and $c_{N+1} = +\infty$). Each set M_i is possibly made up by several connected components M_i^j.

Theorem 5.2. *An equilibrium function is isoparametric on each M_i^j and hence its regular level sets are isoparametric submanifolds.*

Proof. In the open regions M_i^j the equilibrium function I is submersive and by lemma 2.2 the partition is globally trivial. In proposition 5.1 it was proved that $(\nabla I)^2$ and ΔI are functions of I in certain open subset of M_i^j. The globalization of this property to the whole M_i^j stems from the existence of a global transversal (non-closed) curve to the fibres of I, which is a consequence of the triviality of the partition (note that I can be adapted to a coordinate system in M_i^j [18]). $\square$

It is interesting to observe that the isoparametric character of an equilibrium function is generally only local, the following example illustrating this fact.

Example 5.3. *The analytic function $I(x, y) = \cos\sqrt{x^2 + y^2}$ in $(\mathbb{R}^2, \delta)$ is of equilibrium type (its partition is formed by concentric circles). On the contrary it is not a global isoparametric function because $(\nabla I)^2 = 1 - I^2$ but ΔI cannot be globally expressed as a function of I due to the existence of the critical fibres $r = i\pi$, $i \in \mathbb{N} \cup \{0\}$. Anyway, as proved in theorem 5.2, ΔI is a well defined function of I in the domains $M_i = \{i\pi < r < (i+1)\pi\}$ and a straightforward computation yields $\Delta I = -I - \frac{(-1)^i\sqrt{1-I^2}}{i\pi + \arccos((-1)^i I)}$.*

The most remarkable feature of theorem 5.2 is that the idea of isoparametric submanifold, which was introduced in Differential Geometry several decades ago, naturally arises in the physical context of Fluid Mechanics. It is important to note that other authors have also employed the isoparametric condition in order to study the partitions induced by the solutions of certain PDEs [43, 44, 45], but the techniques that we use are different to these authors', specifically the analytic representation property (theorem 4.3). It worths mentioning the interesting paper of Shklover [46], where it is shown that the overdetermined Neumann and Dirichlet problems on certain manifolds admit solution if the boundaries are assumed to be isoparametric. The converse, i.e. the existence of solution implies that the boundary is isoparametric, is not proved.

The literature on the isoparametric property is extensive, theorems 4.5 and 5.2 connect it with the problem of classifying the shapes of static self-gravitating fluids. In the following sections, for its relevance in this context, we will state without proofs, some well known results about isoparametric submanifolds. Several other statements will be obtained for which we provide demonstrations because, to the best of our knowledge, they are new or at least not explicitly stated in any reference that we have consulted.

The following theorem characterizes the general properties that all the equilibrium partitions must satisfy on any Riemannian manifold. It is a well known result to experts in isoparametric families, but we provide a proof for the sake of completeness and because we believe that it is unknown to most people working on Mathematical Physics.

Theorem 5.4. *The partition induced by any equilibrium function I on M has a trivial fibre bundle structure on each M_i^j, each (regular) leaf has constant mean curvature and locally the (regular) leaves are geodesically parallel*

Proof. The first statement has been proved in theorem 5.2. In M_i^j we have that $(\nabla I)^2 = F(I)$ and $\Delta I = G(I)$. The expression of the mean curvature (equation (2)) is the following:

$$H = \operatorname{div}\Big(\frac{\nabla I}{|\nabla I|}\Big) = \frac{G(I)}{\sqrt{F(I)}} + \frac{-F'(I)}{2\sqrt{F(I)}},$$

thus implying that H is constant on each (regular) leaf of the partition. Consider now the following equalities

$$D_X g(\nabla I, \nabla I) = D_X F(I) = F'(I)(\nabla I)^j X_j \tag{17}$$

$$D_X g(\nabla I, \nabla I) = 2g(D_X \nabla I, \nabla I) = 2(\nabla I)^j_{;k}(\nabla I)^k X_j\,, \tag{18}$$

D_X standing for the covariant derivative with respect to the vector field X. Identifying (17) and (18) we obtain that

$$g(D_{\nabla I}\nabla I, X) = g(\frac{F'(I)}{2}\nabla I, X) \Longrightarrow D_{\nabla I}\nabla I = \frac{F'(I)}{2}\nabla I\,,$$

which is the condition on the integral curves of ∇I to be tangent to geodesics. Call λ the parameter of the flow induced by ∇I. Then

$$\frac{\mathrm{d}I}{\mathrm{d}\lambda} = (\nabla I)^2 = F(I) \Longrightarrow \int_{c_1}^{c_2} \frac{\mathrm{d}I}{F(I)} = \lambda_2 - \lambda_1\,,$$

which depends exclusively on the fibres of I and not on the path chosen. Since the arc length is related to λ by $\mathrm{d}s = \sqrt{F(I(\lambda))}\,\mathrm{d}\lambda$, we get that the (regular) leaves of the partition in M_i^j are geodesically parallel. □

It is interesting to observe that the geodesical parallelism of the equilibrium partitions implies that they are Riemannian (singular) foliations [47, 48], a property which will be important in forthcoming sections. In fact theorem 5.4 can be locally expressed as an equivalence.

Proposition 5.5. *A Riemannian codimension 1 (singular) foliation whose non-singular leaves have constant mean curvature is locally an equilibrium partition.*

Proof. Consider an open subset $U \subset M$ small enough so that the foliation in U (which we assume regular) can be represented by a function I. If M is compact, simply connected and the foliation has trivial holonomy then I is defined on the whole M [49]. Since the leaves are parallel let us prove that I must satisfy $(\nabla I)^2 = F(I)$ in U. Indeed it is immediate to see that $X((\nabla I)^2) = 2g(D_X\nabla I, \nabla I) = 2g(D_{\nabla I}\nabla I, X)$ for any vector field X on U. Since the foliation is geodesically parallel then the gradient lines of I are tangent to geodesics, that is $D_{\nabla I}\nabla I = \psi \nabla I$ for certain real-valued function ψ on U. Identifying we get that $X((\nabla I)^2) = 2\psi X(I)$ and therefore the symmetries of I are also symmetries of $(\nabla I)^2$, thus implying, via Frobenius theorem, that $(\nabla I)^2$ is (locally) a function of I. The constancy of the mean curvature H on the leaves is expressed as $H = \mathrm{div}(\frac{\nabla I}{\|\nabla I\|}) = G(I)$. After some computations, and taking into account that $(\nabla I)^2 = F(I)$, one readily gets that ΔI is also a function of I in U, thus proving the (local) equilibrium property. □

If we assume that the foliation is locally trivial (this is the case when M is compact and there are not dense leaves [47, 48]) then we can extend U to a saturated set Λ in such a way that the first integral I is well defined on the whole Λ. In general it will be defined on any globally trivial saturated set Λ.

In the following section we show further properties of the equilibrium shapes for certain particularly relevant spaces. The more we want to characterize an equilibrium partition the more we have to restrict the topology and geometry of the base space.

6 Classification of the equilibrium shapes on certain spaces

Isoparametric submanifolds in the Euclidean space $(\mathbb{R}^n, \delta)$ are classified: they possess constant principal curvatures and hence must be globally isometric to $\mathbb{R}^{n-1}$, S^{n-1} or $S^{n-1-k} \times \mathbb{R}^k$ $(1 \leq k \leq n-2)$ with their respective canonical metrics [39, 41]. From this result it is straightforward to obtain the classification of equilibrium partitions on $(\mathbb{R}^n, \delta)$.

Proposition 6.1. *The equilibrium partitions on the Euclidean space* $(\mathbb{R}^n, \delta)$ *are given by concentric spheres* S^{n-1}*, parallel hyperplanes* $\mathbb{R}^{n-1}$ *or parallel coaxial cylinders* $S^{n-1-k} \times \mathbb{R}^k$ $(1 \leq k \leq n-2)$*.*

Proof. Theorems 5.2 and 5.4 and Cartan's classification of isoparametric submanifolds in $(\mathbb{R}^n, \delta)$ prove the claim on each M_i^j. The globalization follows from lemma 2.1. □

Accordingly, a static self-gravitating fluid on the Euclidean space must take the shape of a round sphere, a cylinder or a region bounded by parallel hyperplanes, thus recovering the classical results of Lichtenstein and Lindblom on compact Newtonian fluids in $(\mathbb{R}^3, \delta)$ [1, 2].

Corollary 6.2. *If* $\mathbb{R}^n$ *is provided with a conformally flat metric, i.e.* $g = e^{2\phi}\delta$*, and* I *is an equilibrium function which agrees fibrewise with* ϕ*, then the equilibrium partitions* $\beta_{\mathbb{R}^n}(I)$ *are the same as in the Euclidean case.*

Proof. A straightforward computation yields that I is also an equilibrium function in $(\mathbb{R}^n, \delta)$, and hence proposition 6.1 applies. Since ϕ and I agree fibrewise then the partitions induced by I on $(\mathbb{R}^n, g)$ and $(\mathbb{R}^n, \delta)$ are globally isometric, thus proving the claim. □

Note that the assumption of I and ϕ agreeing fibrewise is usually considered in the literature on relativistic fluids [21], where the geometry of the base space is coupled with the matter.

Isoparametric submanifolds are also classified in the hyperbolic space $\mathbb{H}^n$ [37] and therefore a result analogous to proposition 6.1 can be obtained, i.e. a detailed classification of the equilibrium partitions of $\mathbb{H}^n$. Proceeding as in proposition 6.1 it is immediate to prove the following claim.

Proposition 6.3. *The equilibrium partitions on the hyperbolic space* $\mathbb{H}^n$ *are given by concentric spheres* S^{n-1}*, parallel hyperplanes* $\mathbb{H}^{n-1}$ *or parallel coaxial cylinders* $S^{n-1-k} \times \mathbb{H}^k$ $(1 \leq k \leq n-2)$*.*

A complete classification of the isoparametric submanifolds on the sphere S^n has not yet been accomplished, see e.g. [41]. The partition formed by concentric spheres S^{n-1} is of equilibrium, but this does not exhaust all the possibilities, although from the physical viewpoint this is indeed the most relevant situation (see the following section for details).

Apart from the canonical constant curvature manifolds, a geometric characterization of the equilibrium partitions can also be obtained for other Riemannian spaces. A particularly interesting case is when (M, g) is non-compact and has non-negative Ricci curvature.

Proposition 6.4. *If $I : M \to \mathbb{R}$ is a submersive equilibrium function on (M, g) then the leaves of $\beta_M(I)$ are totally geodesic submanifolds and ∇I is a Killing vector field.*

Proof. Since I is a submersion then it is a global isoparametric function and hence $(\nabla I)^2 = F(I) \neq 0$ and $\Delta I = G(I)$. Without loss of generality assume that $F(I) = 1$, this is equivalent to representing the partition $\beta_M(I)$ by the function $\hat{I} = \int \frac{\mathrm{d}I}{\sqrt{F(I)}}$. Note that $L_{\nabla I} I = \dot{I} = 1$ and therefore ∇I is complete, thus implying that $I(M) = \mathbb{R}$. Bochner's formula [50] for the function I reads as

$$\frac{1}{2}\Delta(\nabla I)^2 = \nabla I \cdot \nabla \Delta I + \mathrm{Ricci}(\nabla I, \nabla I) + ||D^2 I||^2 \, ,$$

and therefore $||D^2 I||^2 = -\mathrm{Ricci}(\nabla I, \nabla I) - G'(I)$. Taking into account that Ricci ≥ 0 and the well known inequality $||D^2 I||^2 \geq \frac{1}{n-1}(\Delta I)^2$ we get

$$G'(I) \leq -\frac{G(I)^2}{n-1} \, .$$

It is ready to see via integration that this differential inequality is satisfied by a function G verifying $G \leq \frac{n-1}{c_1+I} + c_2$, $c_1, c_2 \in \mathbb{R}$. Since I takes any real value we would have $\lim_{I \to -c_1^-} G(I) = -\infty$, which is a contradiction, and hence the only global solution to the inequality is $G(I) = 0$. Substituting this expression into Bochner's formula it is concluded that $||D^2 I||^2 = 0$, which is the condition for the fibres of I to be totally geodesic. Let us now prove that ∇I is a Killing of (M, g). Recalling the expression for the second fundamental form of the fibres of I (section 2) and taking into account that it is zero we get

$$L_{\nabla I}(g_{ab}) - L_{\nabla I}((\nabla I)_a (\nabla I)_b) = 0 \, .$$

A quite long, although not difficult, computation shows that the second term of this expression vanishes (just assume that $(\nabla I)^2 = 1$), thus proving the claim. □

It is interesting to observe that proposition 6.4 implies that M splits isometrically as $M = N \times \mathbb{R}$, where N is a fibre of I, thus recovering the *Cheeger-Gromoll splitting theorem* [51]. In fact this theorem was originally proved without assuming the existence of a submersive equilibrium function, but showing that it always exists (Busemann function), although it is not generally analytic.

In ending this section let us focus on 3-manifolds (M^3, g). The notation that we will use is explained in section 2. The following elementary lemma will be useful.

Lemma 6.5. *Let S be a codimension 1 submanifold of M^3 such that R, R' and $R_{ab}n^a n^b$ are constant on S. Then the Gauss curvature of S is also constant.*

Proof. Let u and v be two orthonormal vectors tangent to S at the point $p \in S$. The sectional curvature K of M^3 restricted to S is given by

$$K = R_{ab}(u^a u^b + v^a v^b) - \frac{R}{2} \, .$$

The expression $u^a u^b + v^a v^b$ is a projection tensor onto S and therefore $u^a u^b + v^a v^b = \beta^{ab}$, thus implying that $K = \frac{R}{2} - R_{ab}n^a n^b$. The intrinsic sectional curvature of S satisfies the relationship $K' = \frac{R'}{2}$. The assumptions of the lemma yield that K and K' are constant on S and hence, by Gauss theorem, the claim follows. □

When M^3 is flat, conformally flat or locally symmetric, and there exists some relationship between the geometry and the equilibrium function I, then further geometrical properties of the equilibrium partition $\beta_{M^3}(I)$ can be obtained.

Proposition 6.6. *Let $I : M^3 \to \mathbb{R}$ be an equilibrium function on a 3-manifold satisfying either of the following*

1. *M^3 is conformally flat, i.e. $g = e^{2\phi}\delta$, and $\beta_{M^3}(I) = \beta_{M^3}(\phi)$. If M^3 is flat this assumption is not necessary.*
2. *M^3 is locally symmetric and each fibre of I has parallel second fundamental form.*

Then the (regular) leaves of the equilibrium partition $\beta_{M^3}(I)$ have constant principal curvatures.

Proof. First consider the conformally flat case. Let S be a regular leaf of $\beta_{M^3}(I)$. The Ricci tensor and the scalar curvature of $(M^3, e^{2\phi}\delta)$ are given by

$$R_{ab} = \phi_{ab} - \phi_a\phi_b + \delta_{ab}(\Delta_E\phi + (\nabla_E\phi)^2_E) \tag{19}$$

$$R = e^{-2\phi}(4\Delta_E\phi + 2(\nabla_E\phi)^2_E)\,, \tag{20}$$

the subscript E meaning that the corresponding operator has the Euclidean form. It is immediate to check that

$$(\nabla\phi)^2 = e^{-2\phi}(\nabla_E\phi)^2_E \tag{21}$$

$$\Delta\phi = e^{-2\phi}((\nabla_E\phi)^2_E + \Delta_E\phi)\,. \tag{22}$$

Since the partitions of I and ϕ agree fibrewise we have that ϕ is an equilibrium function. Now from equations (20), (21) and (22) it is evident that R is constant on S. Note by looking at equation (3) that if $H_{ab}H^{ab}$ and $R_{ab}n^a n^b$ are both constant on S then R' is also constant. The following computation is immediate

$$H_{ab}H^{ab} = \frac{1}{4}(L_n\beta_{ab})(L_n\beta^{ab}) = \frac{1}{4}\Big[3(L_n e^{2\phi})^2 - (L_n e^{2\phi})(L_n n^a)n_a - (L_n e^{2\phi})(L_n n^b)n_b -$$
$$-(L_n e^{2\phi})(L_n n_a)n^a + (L_n n_a)(L_n n^a) + (L_n n_a)n^a(L_n n_b)n^b - (L_n e^{2\phi})(L_n n_b)n_a\delta^{ab} +$$
$$+(L_n n^a)n_a(L_n n_b)n^b + (L_n n_b)(L_n n^b)\Big]\,.$$

As $(L_n n^a)n_a = \frac{1}{2}L_n(n^a n_a) = 0$ the above expression simplifies to

$$H_{ab}H^{ab} = \frac{1}{4}\Big[3(L_n e^{2\phi})^2 + 2(L_n n_a)(L_n n^a)\Big]\,. \tag{23}$$

The vector field n normal to S is defined as $n = \frac{\nabla I}{|\nabla I|}$, and hence it is immediate to check that the first summand in equation (23) is constant on S. For the second summand notice the following computation

$$(L_n n_a)(L_n n^a) = |\nabla I|^2\Big(L_n\frac{1}{|\nabla I|}\Big)^2 + \frac{1}{|\nabla I|}\Big(L_n\frac{1}{|\nabla I|}\Big)(L_n|\nabla I|^2) +$$
$$+\frac{1}{|\nabla I|^2}(L_n(\nabla I)^a)(L_n(\nabla I)_a)\,.$$

One readily gets that the first and the second summands are constant on S. The third one requires more computations. Indeed

$$(L_n(\nabla I)^a)(L_n(\nabla I)_a) = \frac{1}{|\nabla I|^2}(\nabla I)^b(\nabla I)^c \frac{\partial(\nabla I)_a}{\partial x_b}\frac{\partial(\nabla I)^a}{\partial x_c} =$$
$$= \frac{1}{|\nabla I|^2}e^{-4\phi}\frac{\partial I}{\partial x_b}\frac{\partial I}{\partial x_c}\frac{\partial^2 I}{\partial x_a \partial x_b}\Big(\frac{\partial e^{-2\phi}}{\partial x_c}\frac{\partial I}{\partial x_a} + e^{-2\phi}\frac{\partial^2 I}{\partial x_c \partial x_a}\Big).$$

On the other hand $\frac{\partial^2 I}{\partial x_a \partial x_b}\frac{\partial I}{\partial x_a} = \frac{1}{2}\frac{\partial}{\partial x_b}(\nabla_E I)^2_E$ and ϕ agrees fibrewise with I. Taking these facts into account and after some computations it is obtained that $(L_n(\nabla I)^a)(L_n(\nabla I)_a)$ is constant on S and hence $H_{ab}H^{ab}$ is also constant. Similar computations show the constancy of $R_{ab}n^a n^b$ on S, thus proving that R' is also constant. Note now that lemma 6.5 can be applied to conclude that the Gauss curvature of S is constant. Since S has dimension 2 and its mean curvature is also constant the proposition for conformally flat manifolds follows. Now let us focus on the locally symmetric spaces, i.e. $R_{abcd;m} = 0$. It is immediate that R is constant on S. Denote by $||$ the induced covariant derivative on S. The following equation is readily obtained

$$0 = \beta^{ab}_{||e} = -(n^a n^b)_{||e}\,.$$

Therefore $(R_{ab}n^a n^b)_{||e} = R_{ab}(n^a n^b)_{||e} = 0$, which means that $R_{ab}n^a n^b$ is constant on S. Since the second fundamental form of S is parallel (i.e. $H_{ab||e} = 0$) we have that $(H_{ab}H^{ab})_{||e} = 0$, thus concluding that R', and hence the Gauss curvature, is constant on S. □

As we mentioned concerning corollary 6.2 the assumption of coupling between the equilibrium function and the underlying geometry (e.g. the conformal factor ϕ) is usual in the general relativistic setting. We are not aware whether equilibrium functions not fulfilling this hypothesis exist and if so, what geometrical properties they have.

Remark 6.7. *It would be interesting to obtain a full geometrical description of the equilibrium partitions for the 8 canonical 3-dimensional geometries [52]. The results on constant curvature and locally symmetric spaces obtained in this section apply to 5 of these manifolds, i.e.* $\mathbb{R}^3$, $\mathbb{H}^3$, S^3, $S^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$.

The geometrical properties of equilibrium partitions obtained in this section are relevant from the physical viewpoint because they give rise to the possible shapes of a static self-gravitating fluid on different Riemannian spaces.

7 Existence of equilibrium shapes and fluid-composed stars

In general it is a difficult task to know whether an equilibrium function exists on a Riemannian manifold. This problem is not only interesting from the mathematical viewpoint but also from the physical one. Indeed if certain space does not admit equilibrium partitions then a self-gravitating fluid would never reach static equilibrium, a non-existence result for the set (P) of PDEs. It would be desirable to classify the spaces admitting equilibrium functions, which would be the suitable spaces for doing relevant physics. In this section we will obtain some restrictions to the existence of equilibrium partitions. These restrictions are of geometrical or topological type

A mass of fluid generally encloses a contractible domain, e.g. think of a fluid-composed star, and hence the equipotential sets are contractible to an interior point. Let us prove that in this case the equilibrium partition in Ω has only one focal point (recall that the focal set is the set of points where the lines of ∇I intersect each other, i.e. the critical points). Since I is analytic in Ω and the critical set of an analytic function does not possess endpoints [53] then the only possibility for the (interior) focal set is that it is formed by a single point since otherwise the fluid domain would not be contractible (recall that the leaves of the distance function are tubes around the focal sets [35]).

Accordingly it is physically relevant to study the manifolds (M, g) which admit equilibrium partitions possessing an isolated focal point. Recall that the space (M, g) is harmonic with respect to $p \in M$ [54] if the determinant of the metric in normal Riemann coordinates is a function of the geodesic distance to p. If G is the determinant of the metric in polar Riemann coordinates then it is well known that the determinant of the metric in normal Riemann coordinates is $\tilde{G} = Gr^{2-2n}\Omega(\theta)^{-2}$, where $\Omega(\theta)^2$ is the determinant of the metric of the round sphere S^{n-1} in spherical coordinates. Therefore the condition of harmonicity with respect to p can be expressed as $\tilde{G} = F(r)^2$ or $G = F(r)^2 r^{2n-2}\Omega(\theta)^2$. Set $A(r) = F(r)r^{n-1}$.

Proposition 7.1. *A local equilibrium partition with an isolated focal point $p \in M$ exists on (M, g) if and only if the space is harmonic with respect to p. In this case the equilibrium partition is locally formed by geodesic spheres.*

Proof. Recall that in polar Riemann coordinates centered at $p \in M$ the metric tensor is locally expressed as $\mathrm{d}s^2 = \mathrm{d}r^2 + G_{ij}(r,\theta)\mathrm{d}\theta^i\mathrm{d}\theta^j$. The sufficiency condition stems from the fact that the function $I = \frac{1}{2}r^2$ is of equilibrium. Indeed $(\nabla I)^2 = r^2 = 2I$ and $\Delta I = \frac{\partial_r(rA(r))}{A(r)}$ which is an analytic function of r because $A(r) = r^{n-1} + O(r^n)$. Therefore r^2 induces a local equilibrium partition (the geodesic spheres) whose focal point is p. Conversely if one has an equilibrium partition with a focal point formed by the point p then the geodesical parallelism of the leaves implies that the partition must be formed by geodesic spheres centered at p. This stems from the fact that the focal varieties of Riemannian (singular) foliations are smooth submanifolds of M and the regular leaves of the partition are tubes (constant distance) over either of the focal varieties [35]. On account of proposition 5.1 the function $I = \frac{1}{2}r^2$ representing the same partition must be of equilibrium. The condition of ΔI being a function of r is expressed as $\partial_r \ln(r\sqrt{G}) = C(r)$, and a straightforward integration yields that $G = A(r)^2 B(\theta)^2$. Since $\tilde{G} = \frac{A(r)^2}{r^{2n-2}}\frac{B(\theta)^2}{\Omega(\theta)^2}$ and $\tilde{G} = 1$ at p $(r = 0)$ we obtain that $B(\theta) = \Omega(\theta)$ (note that $A(r) = r^{n-1}(1 + O(r))$) thus implying that $\tilde{G} = F(r)^2$. □

Since polar Riemann coordinates are only local then the partition formed by concentric geodesic spheres could not be globally defined (at least as an analytic partition). The globalization exists, for example, when the exponential map is a global diffeomorphism, e.g. if the space is simply connected and the sectional curvature is non-positive (Cartan-Hadamard's theorem).

Important examples of harmonic spaces with respect to p are provided by manifolds which are rotationally symmetric around p, in fact in dimension 2 this is always the case ($G = A(r)^2\Omega(\theta)^2 \iff$ rotational symmetry with respect to p). If (M, g) is harmonic with respect to any point then it is called *harmonic manifold*. Particular cases of harmonic manifolds are the canonical constant curvature spaces $S^n, \mathbb{R}^n$ and $\mathbb{H}^n$, where the local partitions can be globalized (in S^n there will appear a second focal point). A remarkable physical consequence is that in all these manifolds (static) fluid-composed stars can exist.

Another consequence is that spaces whose metric does not satisfy the assumption of harmonicity with respect to any point will not have (local) equilibrium partitions with isolated focal points. In these spaces static fluid-composed stars cannot exist and hence they are not physically admissible.

Remark 7.2. *A physically realistic 3-space (the universe) must allow the existence of static contractible fluid domains around any point (the positions of fluid-composed stars should not be not privileged!). This implies that M^3 is harmonic and hence, it can be proved (dimension 3) [54] that this is equivalent to be two-point homogeneous. Two-point homogeneous manifolds are classified [55], in dimension 3 they are (up to local isometry) $\mathbb{R}^3$, $\mathbb{H}^3$ and S^3. Therefore the existence of static fluid stars implies that the universe must be a constant curvature manifold, thus justifying the main hypothesis of the standard cosmological models.*

It is interesting to observe that for any Riemannian manifold (M, g) and any given point $p \in M$ there exists a smooth conformal factor Φ [56] such that in the new metric Φg the geodesic spheres centered at p form locally an equilibrium partition, thus allowing the existence of contractible fluid domains in equilibrium.

The following proposition establishes a topological constraint on M in order that a submersive equilibrium function exist.

Proposition 7.3. *Let I be an equilibrium function such that $\mathrm{d}I \neq 0$. Then $M \cong N \times \mathbb{R}$.*

Proof. Since $\mathrm{d}I \neq 0$ then $M_i^j = M$ and therefore theorem 5.4 implies that I is a globally trivial fibre bundle with a global transversal curve Σ. Since the lines of a gradient vector field are not closed then $\Sigma \cong \mathbb{R}$, thus proving the claim. Of course $N \cong I^{-1}(0)$. □

Proposition 7.3 shows that submersive equilibrium functions do not exist if M is not diffeomorphic to $N \times \mathbb{R}$. It would be interesting to get topological restrictions ensuring the non existence of any equilibrium function. In section 8 we will prove that surfaces without Killing vector fields do not admit equilibrium partitions (proposition 8.1). Note also that when M has non-negative Ricci curvature the claim of proposition 7.3 can be improved, as was shown in proposition 6.4.

8 Equilibrium shapes, Killing vector fields and isoperimetric domains

The results of section 7 suggest that equilibrium partitions are usually linked to certain geometric structures of the manifold. If these geometric structures fail to exist then equilibrium partitions do not exist. For example, consider the equilibrium partitions of the Euclidean space (proposition 6.1). These partitions have the remarkable property of being generated by isometries of $(\mathbb{R}^n, \delta)$. Furthermore, as a consequence of proposition 7.1, the equilibrium partitions with p as focal point are induced by isometric group actions on M whenever the space is rotationally symmetric around p. These facts indicate that the *isometries* of the manifold are somehow related to the equilibrium functions. The following proposition establishes the equivalence between both concepts for surfaces.

Proposition 8.1. *Let (M, g) be a 2-dimensional Riemannian space. Then the equilibrium partitions are 1-dimensional (singular) foliations generated by Killing vector fields of (M, g), and conversely, any Killing vector field whose orbits are closed in M, is tangent to the level sets of an equilibrium function.*

Proof. Let ξ be a Killing vector field of (M, g). Since its orbits are closed it follows that the action of ξ on M is proper. Theorem 8.3 proves that the foliation defined by ξ is of equilibrium.

Consider now an orthogonal (local) coordinate system on M_i^j defined by the functions (I, J), I being an equilibrium function on M. Recall that on M_i^j the function I is isoparametric and hence $(\nabla I)^2 = F(I)^2$ and $\Delta I = G(I)$. Since $\partial_I \cdot \partial_I = \frac{1}{F(I)^2}$ and $\partial_I \cdot \partial_J = 0$, the expression of the metric tensor in the new coordinates is $\mathrm{d}s^2 = F(I)^{-2}\mathrm{d}I^2 + N(I,J)^2\mathrm{d}J^2$. A straightforward computation yields that $G(I) = \Delta I = F(I)^2\partial_I(\log NF)$, and hence $N(I,J) = A(I)B(J)$. If we define $\mathrm{d}\hat{J} = \sqrt{B(J)}\mathrm{d}J$ we get that the metric in coordinates $(I, \hat{J})$ is a warped product $\mathrm{d}s^2 = F(I)^{-2}\mathrm{d}I^2 + A(I)\mathrm{d}\hat{J}^2$, thus implying that the vector field $\partial_{\hat{J}}$ (tangent to the level sets of I) is a local Killing. The analyticity of the metric implies that this Killing vector field (which is also analytic) globalizes [57]. □

If I is a submersive equilibrium function then (proposition 7.3) $M \cong \mathbb{R}^2$ or $S^1 \times \mathbb{R}$, and therefore the warped product expression in proposition 8.1 can be globalized to $\mathrm{d}s^2 = \mathrm{d}I^2 + A(I)\mathrm{d}\hat{J}^2$, where it has been assumed without loss of generality that $(\nabla I)^2 = 1$.

Remark 8.2. *As a consequence of proposition 8.1 equilibrium functions do not exist on surfaces not admitting Killing vector fields, e.g. negative curvature tori of genus $g \geq 2$ [50], this being the "generic" situation.*

Part of proposition 8.1 can be generalized to higher dimension, as we prove in the next theorem.

Theorem 8.3. *Let $\Xi = \{\xi_1, \ldots, \xi_p\}$, $p \geq n-1$, be a Lie algebra of Killing vector fields of (M,g). Ξ satisfies that rank$(\xi_1, \ldots, \xi_p) = n-1$ in M, up to a null measure set, and it generates a closed subgroup of the group of isometries. Then the (singular) foliation induced by Ξ is an equilibrium partition.*

Proof. Ξ generates an isometric group action G on M. G is connected, simply connected (take the universal covering) and closed in the group of isometries (by assumption). This defines a proper group action on the manifold and therefore M can be divided in two connected components [58], the principal part M^*, which is open and dense in M, and the singular part, which is formed by totally geodesic submanifolds. M^* is foliated by codimension 1 closed submanifolds of M, in fact this foliation is a Riemannian submersion from M^* to M^*/G [48]. Note that M^*/G is a differentiable Hausdorff 1-manifold, and therefore diffeomorphic to $\mathbb{R}$ or S^1. The submersion is analytic because we always assume in this paper that (M,g) is analytic, and therefore also the Killing vector fields. Call f the function representing the foliation in M^*; since it is a Riemannian submersion then f will satisfy that $(\nabla f)^2 = F(f)$ in the whole M^*, as proved in proposition 5.5. Since the action of G is transitive on each leaf (the leaves are extrinsically homogeneous, that is homogeneous by isometries of the ambient space) then the mean curvature must be constant at all points of the leaf. This follows from the fact that the second fundamental forms at two different points connected by an isometry correspond through this isometry. In terms of f this condition is expressed as $H = \mathrm{div}(\frac{\nabla f}{\|\nabla f\|}) = H(f)$. The following computation, $\mathrm{div}(\frac{\nabla f}{\|\nabla f\|}) = \frac{\Delta f}{\|\nabla f\|} - \frac{\nabla f \nabla(\|\nabla f\|)}{(\nabla f)^2}$, readily implies that $\Delta f = G(f)$. Since the non-principal set is nowhere dense the isoparametric condition extends to the whole M and therefore the foliation is of equilibrium. Note that the extended f is a function over $\mathbb{R}$ or S^1 and it could fail to be analytic in the singular set. □

Remark 8.4. *In general it is necessary to require that G is closed in the group of isometries. For example, take the flat 2-torus $S^1 \times S^1$ and consider the action by the real line which is given by an irrational translation. This induces a Killing vector field, but the group generated is not closed in the isometry group of the torus, which we know is compact (it is $O(2) \times O(2)$).*

In fact this action is not proper since the orbits are not (properly) embedded. Similar examples can be constructed in greater dimension.

Note that theorem 8.3 generalizes, in the Riemannian setting for arbitrary dimension, theorem 1 in [59]. In general the converse of this theorem is true only for 2-dimensional manifolds (proposition 8.1). Indeed consider a manifold which is not rotationally symmetric with respect to the point p but it is harmonic with respect to it. Then the geodesic spheres around p are equilibrium submanifolds but they are not induced by an isometric group action, thus showing that the converse theorem does not generally hold. It would be interesting to find conditions in order that the equilibrium partitions of a manifold be (singular) foliations induced by isometric group actions.

All these results show the deep relationship between isometries and equilibrium and suggest that physically relevant spaces should possess enough Killing vector fields. Consequently an effective procedure in order to obtain equilibrium partitions, and hence equilibrium configurations of self-gravitating fluids, is to compute the Killing vector fields of the space. It is probable that spaces which just admit a few isometries (or even no one) do not admit equilibrium functions either (as in dimension 2), lacking static configurations. Let us illustrate theorem 8.3 with an example.

Example 8.5. *Consider the space* $\mathbb{H}^2 \times \mathbb{R}$ *endowed with the metric* $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{F^2} + \mathrm{d}z^2$*, where* $F = \frac{2-x^2-y^2}{2}$ *and* $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 2\}$*. The Killing vector fields of this manifold are:* $X_1 = (F+y^2)\partial_x - xy\partial_y$*,* $X_2 = -xy\partial_x + (F+x^2)\partial_y$*,* $X_3 = -y\partial_x + x\partial_y$ *and* $X_4 = \partial_z$*. A straightforward computation yields that* $(\nabla f)^2 = F^2(f_x^2+f_y^2)+f_z^2$ *and* $\Delta f = F^2(f_{xx}+f_{yy})+f_{zz}$*. Some easy, although long, computations show that the codimension 1 partitions (up to null measure set) induced by the Killings vector fields are:*

- $\{X_3, X_4\} \implies f = x^2+y^2$*, which is an equilibrium function.*
- $\{X_i, X_j\}$*,* $i \neq j = 1,2,3 \implies f = z$*, which is an equilibrium function.*
- $\{X_1, X_4\} \implies f = \frac{x^2+y^2-2}{y}$*, which is an equilibrium function (with a singular set).*
- $\{X_2, X_4\} \implies f = \frac{x^2+y^2-2}{x}$*, which is an equilibrium function (with a singular set).*

¿From the physical viewpoint it is reasonable to compare the shapes of a compact self-gravitating fluid with the *isoperimetric* domains. By the term isoperimetric we mean the sets which minimize the area for variations which leave fixed the volume. In the Euclidean space the only compact equilibrium submanifold is the round sphere, which is exactly the solution to the isoperimetric problem. The physical meaning is clear: fluid-composed stars would minimize their surfaces in order to achieve equilibrium. For general Riemannian manifolds an equilibrium submanifold does not solve the isoperimetric problem. The most general result that can be proved is the following.

Proposition 8.6. *Let* S *be a compact equilibrium codimension 1 submanifold. Then* S *is a critical point of the* $(n-1)$*-area* $A(t)$ *for all variations* S_t *that leave constant the* n*-volume* $V(t)$ *enclosed by* S*.*

Proof. S is the level set of an analytic function and therefore it has no endpoints [53]. Since it is compact it encloses a finite volume. The equilibrium condition implies that the mean curvature is a constant H. Let S_t, $t \in (-\epsilon, \epsilon)$ and $S_0 = S$, be a variation of S. The first variation of the area at $t = 0$ is given by [60] $A'(0) = -(n-1)H \int_S f \mathrm{d}S$, where f is the normal component of the variation vector of S_t and $\mathrm{d}S$ is the $(n-1)$-area element of S.

Since the variation is volume preserving then $V'(0) = \int_S f \mathrm{d}S = 0$ and therefore we get that $A'(0) = 0$. □

This result cannot be improved in general. We can find manifolds for which equilibrium shapes are minimizers of the area and other manifolds for which they are maximizers or saddle points. Even the weaker condition of being stable, that is $A''(0) \geq 0$, is not generally verified. It would be interesting to classify all the spaces whose compact equilibrium submanifolds are stable. The following list gives some of them:

- Constant curvature simply connected manifolds. The geodesic spheres are the only stable submanifolds [60]. They are also of equilibrium on account of proposition 7.1.
- Rotationally symmetric planes with decreasing curvature from the origin. The geodesic circles are stable and enclose isoparametric domains [61], they are also of equilibrium.
- Rotationally symmetric spheres with curvature increasing from the equator and equatorial symmetry. The geodesic circles are stable and enclose isoperimetric domains [61], they are also of equilibrium.
- Rotationally symmetric cylinders with decreasing curvature from one end and finite area. The circles of revolution are stable, enclose isoperimetric domains [61] and a straightforward computation yields that they are also of equilibrium.

It is not difficult to construct examples of manifolds with equilibrium partitions whose leaves are not stable. For instance consider the plane with the following metric tensor in polar coordinates $\mathrm{d}s^2 = dr^2 + r^2(1+r^2)^2\mathrm{d}\theta^2$. The function $I = \frac{1}{2}r^2$ is of equilibrium, it induces the equilibrium partition given by the geodesic circles. Now, if you set $f(r) = r(1+r^2)$, the expression $f'^2 - ff'' = 1 + 3r^4$ is greater than 1 when $r > 0$. This implies [61] that no stable curves exist. Other similar examples in dimension 2 can be found in the work of Ritoré. Other interesting example is given by the symmetric spaces of rank 1. The geodesic spheres are transitivity hypersurfaces of the group of isometries and therefore they are equilibrium submanifolds (theorem 8.3). However not all the geodesic spheres are stable [60].

9 Equilibrium shapes of relativistic fluids

The free-boundary problem (P) includes, as particular cases, the equations ruling Newtonian and relativistic fluids on Riemannian manifolds. In the relativistic case Einstein's equations give rise to the additional constraint

$$R_{ab} = f_1^{-1} f_{1;ab} + 4\pi(f_2 - f_3)g_{ab}\,, \tag{24}$$

which expresses the coupling between the geometry of (M, g) and the potential f_1.

The metric tensor g can be proved to be analytic in $M - \partial\Omega$ [62] and C^2_t on $\partial\Omega$ (Synge's junction condition [22]). If equation (24) is not taken into account (relativistic fluid model on a fixed space) then all the results obtained in this paper apply. When equation (24) is considered, theorem 4.3 does not hold (its proof makes use of the analyticity of the metric on $\partial\Omega$), and therefore it is not possible to provide a full classification of the equilibrium shapes.

Even in the case in which ARP could be proved, the proof of theorem 4.5 would fail in general. This obstacle can be overcome when the metric is assumed to be conformally flat, i.e. $g = e^{2\phi}\delta$, and the conformal factor ϕ agrees fibrewise with f_1 (this assumption is common in the literature, see e.g. [21]). In this case it is not difficult to show, proceeding as in section 4, that ARP implies theorem 4.5. Under mild physical assumptions it can be proved that the manifold M is diffeomorphic to $\mathbb{R}^n$ [63], and therefore corollary 6.2 would imply (without imposing other physical constraints) that the equipotential sets are concentric spheres S^{n-1}, parallel hyperplanes $\mathbb{R}^{n-1}$ or parallel coaxial cylinders $S^{n-1-k} \times \mathbb{R}^k$ ($1 \leq k \leq n-2$). This would generalize, up to ARP, a theorem of Lindblom asserting that in dimension 3, and bounded domain Ω, conformal flatness implies spherical symmetry [64]. Note that the hypothesis of conformal flatness is proved in [3, 4] under several physical assumptions.

It would be interesting to prove ARP and theorem 4.5 for general relativistic fluids. This would yield a complete classification of the equilibrium shapes without taking into account physical restrictions. Furthermore it would allow to detect the spaces on which $\partial\Omega$ is a round sphere or not. Note that the approaches of Beig&Simon and Lindblom&Masood-ul-Alam are adapted to prove the spherical symmetry of the equipotential sets, thus failing in more general situations.

Our conjecture is that theorems 4.3 and 4.5 remain true in the relativistic setting, without additional hypotheses. A possible proof may involve the concept of analytic representation of a metric. This requirement is rather natural since g, as f_1, is an unknown of equations (P) extending across the free-boundary. The question is how to define the analytic representation of a metric and how to prove that the metrics which are solutions to equations (P) and (24) are analytically representable. If f_1 is shown to induce equilibrium partitions then Kunzle's work [34] (where his strong assumption $(\nabla f_1)^2 = F(f_1)$ would arise as a consequence of the equilibrium property) would imply the spherical symmetry of the fluid, just by taking into account some mild physical hypotheses.

10 Final remarks and open problems

This work has shown a connection between the level sets of the solutions to certain free-boundary problems and the equilibrium (isoparametric) condition. A remarkable consequence of this result has been the classification of the shapes of static fluids on manifolds. This suggests the interest, both mathematical and physical, of studying the structure of isoparametric submanifolds on Riemannian spaces. In this line many problems remain open, e.g. a complete classification in constant curvature or symmetric manifolds, and a better understanding of the interplay between geometry, topology and isoparametricity.

An interesting consequence of our work is that we have provided a technique for characterizing the shapes of fluids on different spaces which is independent of whether $\partial\Omega$ is a round sphere or not. Up to now all the techniques available in the literature are adapted to the spherical symmetry situation. Consequently a deeper relationship between the geometrical and topological properties of (M, g) and the shapes of the fluid region has been established.

We have shown that recovering the physical intuition that we have in the Euclidean space, e.g. the existence of contractible fluid domains, the connection with the isometries of (M, g) and the stability of the fluid regions, requires to restrict the base manifold.

It would be interesting to ascertain whether techniques similar to the ones developed in this paper could be useful in other situations where the most interesting properties are of geometrical type, e.g. shapes of self-gravitating fluids in rotation, propagating interfaces, burning flames or computer vision [65].

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