Álvaro Pelayo
Washington University in St. Louis

REPORT FOR WORK AT IAS, 12/2010 - 08/2013

1. Hamiltonian Dynamics

I had studied for years Hofer's deep contributions to symplectic geometry and Hamiltonian dynamics and, as an analysis minded geometer, his book with Eduard Zehnder was a primary reference for my research. It was my privilege to have Helmut Hofer as my mentor during my stay. Member B. Bramham and I used to joke that after talking to Hofer we felt we could prove anything (there was never a shortage of conjectures of Hofer to investigate).

During my stay I learned in detail new theories (e.g., Polyfolds) and investigated problems outside of my core expertise (e.g., type theory and K-theory). I found Hofer's Lectures on his theory of Polyfolds (at the Institute) particularly inspiring. From these lectures I learned not only about a new subject, but also about Hofer's way of thinking. I could see how his approach was conceptual, and he was concerned about setting the right definitions before doing computations.

During my stay I learned that former Member Barney Bramham has used methods in symplectic dynamics (going back to pioneer work of Hofer and collaborators) to partially answer long standing questions in dynamical systems posed by A. Katok. I was able to incorporate part of these ideas into my toolbox, and Bramham and I started to work on a joint project to encode foliation invariants of integrable systems. This originated in a suggestion that Hofer made after one of my talks at the Institute in 2011, where I explained that the global symplectic dynamics of integrable systems is understood in dimension four provided that there are no hyperbolic singularities, but not otherwise.

I thought we could not incorporate hyperbolic singularities because one of the invariants (a polygonal set encoding the affine structure) seems impossible to define when there are hyperbolic singularities. But Hofer convinced me that a less concrete foliation invariant may be suited for this case.

From a lecture of Hofer at Princeton University I learned about the problem of existence of intermediate capacities. This is a question posed by Hofer in 1989. Intermediate symplectic capacities are the symplectic invariants in symplectic topology which might exist between volume and 2-dimensional invariants. Larry Guth had proven around 2008 that no such capacities existed that were also exhaustive. I spent several months thinking about this question and with Vũ Ngọc we removed the "exhaustiveness" condition.

This led to a complete solution of Hofer's question in the negative. I wrote a paper with this result in joint work with Vũ Ngọc. In the same paper we solved a question posed by Hofer that same year about the existence of symplectic embeddings of cylinders. Using the same idea, we wrote a paper answering questions of Hind-Kerman (posed in an *Invent. Math.* paper).

2. Spectral Geometry

One of my long term interests is to use microlocal analysis and symplectic geometry to solve questions in inverse spectral theory concerning quantum integrable systems. A large part of my stay at IAS I spent it working on isospectrality questions in symplectic geometry going back to influential works of Colin de Verdière in 1979, 1980 on inverse spectral geometry of pseudodifferential operators.

I was fortunate that Peter Sarnak pointed out to me what I consider major projects for me to work on in the next five years. Former Member Nalini Anatharaman (a major contributor to positive entropy for quantum limits of chaotic quantizations and a friend for several years) and I had several discussions influenced by joint conversations with Sarnak, and comments Sarnak had made to us. These discussions were on different topics, but a recurring one was a collection of major results of Sarnak on the isospectral problem for Riemannian manifolds.

During my stay I proved (with Charles and Vũ Ngọc) that quantum toric integrable systems given by r independent commuting operators on $2r$-dimensional phase space are determined by their semiclassical joint spectrum. This solves the isospectrality problem for quantum toric systems (this is published in a paper in *Annales Sci. Ec. Norm. Sup.* dedicated to Sarnak's 60th Birthday). In fact, the results we proved contained a complete description of the joint spectrum of such systems, and it is as a consequence of this result that we proved isospectrality. I gave several talks on this topic at the Institute.

After my first talk, around October 2011, Sarnak and I started discussing how this problem relates to some fundamental questions in a letter that he wrote to C. Morawetz (Courant), and some of the follow up correspondence (all in Peter Sarnak's website). Sarnak agreed with me that the toric isospectrality question I had worked on was quite natural, which I found really encouraging. His opinion was very influential in the choices I made of problems to work on during the remaining time of my stay at the Institute.

After speaking with Sarnak in the early Fall of 2011, I began to study his major contributions to the isospectral problem for Riemannian manifolds with Osgood and Phillips. They showed, in dimension two, that the isospectral set while not finite is compact in C^∞. The flavor of their spectacular isospectral results was similar to that of my aforementioned work on quantum integrable systems with Charles and Vũ Ngọc (in the context of symplectic geometry and semiclassical microlocal analysis).

Later, in collaboration with Polterovich we generalized these results to cover finite general collections of operators (non-necessarily commuting, or forming an integrable system); concretely, we proved an $\hbar$-limit theorem saying that the spectral limit as $\hbar$ approaches 0 of the convex hull of the joint spectrum of the operators converges to convex hull of the image of the joint map of principal symbols of the operators. These results apply to a general notion of semiclassical quantization introduced by myself, Polterovich, and Vũ Ngọc, which in particular includes $\hbar$-pseudodifferential quantization and Berezin-Toeplitz quantization.

In view of this work, Sarnak posed a question to me in October 2011 about r independent commuting operators: *how much can one improve the* L^∞ *bounds for the joint eigenfunctions he obtained?* Sarnak had shown that in the case of the commutative ring of differential operators of rank r, one gets an optimal improvement depending on r. I am working on this question now. Recently Simon Marshall obtained the corresponding L^p bounds.

3. Semiclassics and Integrabiliy

During many informal discussions with Thomas Spencer during tea and lunch hours, I learned from him about many important aspects of infinite dimensional integrable systems which I was not aware of. He also convinced me that integrable systems are of major importance currently in physics. Spencer pointed out to me a very interesting and I think really promising topic, which I am currently investigating: the relation between Fukaya categories and semiclassical analysis. This is related to work of Tamarkin and Witten on asymptotic analysis of quantum systems. It is connected with semiclassical aspects of symplectic topology, which is an ongoing project with Charles, Polterovich, and Vũ Ngọc.

A highlight of my stay was when Helmut Hofer and Thomas Spencer asked me to give a presentation about symplectic dynamics to the Board of Trustees. It was a 15 minute presentation to the Board, in which I explained the goals of the Hofer-Mather Symplectic Dynamics Program, and my work on semiclassical analysis.

During my stay I met with Edward Witten several times at his office to discuss questions on spectral theory of integrable systems and scattering theory. From him I got a good intuition for a certain trick that physicists use which I understood but from the perspective of semiclassical analysis. I also understood from him what type of inverse question was natural to ask from a physical view point.

4. Homotopy type theory and Voevodsky's Program

In early 2011 I began to work on homotopy type theory and Voevodsky's Univalent Foundations Program after I heard a talk by Vladimir Voevodsky. At that time few people were involved (I believe no one outside of logic and homotopy theory). This project represented a shift of my focus and required considerable time investment. I was fortunate to learn and contribute to the field at its flourishing stage. Helmut Hofer had expressed faith in this program and encouraged me to pursue it, which I did. It paid off because this research theme is now regarded as an important new development in mathematics. I wrote three articles on the subject and was an official participant in the Univalent Foundations Program book.

I had many discussions with Voevodsky and was fortunate to learn about his point of view. What I found most striking is how carefully he set up the the foundations of his new program. At times it appeared to me we spent too much time debating seemingly unimportant choices of notation, and preliminary definitions. They turned out to have a great impact down the road.

5. Grant support and Publications

I was supported by two grants held by IAS faculty, and also during the summer by my personal grants. I wrote 18 papers spanning about 600 pages, primarily on the areas of semiclassical analysis, spectral geometry, symplectic dynamics, and homotopy type theory. I also was official participant of the Univalent Foundations Program Book and cowrote a book with Bertram Kostant. I also have 3 papers almost finished (with Figalli, Gualtieri, Reshetikhin, Ratiu, and Vũ Ngọc).

6. Closing

My research program expanded and flourished at the Institute thanks in part to the excellent infrastructure (eg. offices, housing nearby), cultural activities, the obviously world class faculty, and the faculty's willingness to let the Members learn from them. Hofer, Sarnak, Spencer and Voevodsky all spent significant time discussing with me. No amount of study would make up for the privilege of talking to them in person. I also enjoyed interacting with the other IAS members. In particular, I was delighted to overlap with the stays of old friends Nalini Anantharaman and Barney Bramham. We had many discussions on topics of common interest. The atmosphere at IAS was that of a friendly interaction between members and faculty.

7. Publications

Available at http://www.math.wustl.edu/~apelayo/:

Papers by area

Spectral Geometry:

(1) *Semiclassical inverse spectral theory for singularities of focus-focus type*
Á. Pelayo, S.Vũ Ngọc
arXiv:1302.2260

(2) *Semiclassical quantization and spectral limits of h-pseudodifferential and Berezin-Toeplitz operators*
Á. Pelayo, L. Polterovich, S.Vũ Ngọc
arXiv:1302.0424

(3) *Symplectic spectral geometry of semiclassical operators*
Á. Pelayo
Bull. Belgian Math. Soc. 20 (2013) 405-415

(4) *Isospectrality for quantum toric integrable systems*
(dedicated to Peter Sarnak on his 60th Birthday)
L. Charles, Á. Pelayo, S.Vũ Ngọc
Annales Sci. Ec. Norm. Sup. 43 (2013) 815-849

(5) *First steps in symplectic and spectral theory of integrable systems*
Á. Pelayo, S.Vũ Ngọc
Discrete and Cont. Dyn. Syst. A 32 (2012) 3325-3377

Integrable Systems:

(1) *The affine invariant of generalized semitoric systems*
Á. Pelayo, T. Ratiu, S.Vũ Ngọc
arXiv:1307.7516

(2) *Fiber connectivity and bifurcation diagrams for almost toric systems*
Á. Pelayo, T. Ratiu, S.Vũ Ngọc
arXiv:1108.0328

Random Dynamics:

(1) *Poincare-Birkhoff theorems in random dynamics*
(dedicated to Alan Weinstein on his 70th Birthday)
Á. Pelayo, F. Rezakhanlou
arXiv:1306.0821

Kosnowski Conjecture:

(1) *An integer optimization problem for non-Hamiltonian periodic flows*
Á. Pelayo, S. Sabatini
arXiv:1307.6766

(2) *Log-concavity and symplectic flows*
Y. Lin, Á. Pelayo
arXiv:1207.1335

(3) *Circle valued momentum maps for symplectic periodic flows*
Á. Pelayo, T. Ratiu
L'Enseign. Math. 58 (2012) 205-219

Homotopy Type Theory:

(1) *Voevodsky's Univalence Axiom in homotopy type theory*
S. Awodey, Á. Pelayo, M. Warren
Notices Amer. Math. Soc. 60 (2013) 1164-1167

(2) *A preliminary univalent formalization of the p-adic numbers*
Á. Pelayo, V. Voevodsky, M. Warren
arXiv:1302.1207

(3) *Homotopy type theory and Voevodsky's univalent foundations*
Á. Pelayo, M. Warren
arXiv:1210.5658

Symplectic embeddings and Hamiltonian Dynamics:

(1) *Sharp symplectic embeddings of cylinders*
Á. Pelayo, S.Vũ Ngọc
arXiv:1304.5250

(2) *The Hofer question on intermediate symplectic capacities*
Á. Pelayo, S.Vũ Ngọc
arXiv:1210.1537

Moduli Spaces:

(1) *Moduli spaces of toric manifolds*
Á. Pelayo, A.R. Pires, T. Ratiu, S. Sabatini
Geometriae Dedicata, to appear
arXiv:1207.0092

(2) *Symplectic geometry on moduli spaces of J-holomorphic curves*
J .Coffey, L. Kessler, Á. Pelayo
Ann. Glob. Anal. Geom. 41 (2012) 265-280

Books

(1) Geometric Quantization
A Lie Theory Approach
B. Kostant, Á. Pelayo
Springer Universitext, book to appear, 2013

(2) Homotopy Type Theory:
Univalent Foundations of Mathematics
The Univalent Foundations Program
Institute for Advanced Study, 2013
arXiv:1308.0729