LOG-CONCAVITY AND SYMPLECTIC FLOWS

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Abstract. We prove the logarithmic concavity of the Duistermaat--Heckman measure of an Hamiltonian $(n-2)$-dimensional torus action for which there exists an effective commuting symplectic action of a 2--torus with symplectic orbits. Using this, we show that any symplectic $(n-2)$-torus action with non-empty fixed point set which satisfies this additional 2-torus condition must be Hamiltonian.

1. Introduction

One leading problem in group actions is to understand the relationship between: (i) a symplectic action being Hamiltonian; and (ii) the existence of action fixed points. Frankel [8] proved that on compact connected Kähler manifolds, a 2π-periodic vector field whose flow preserves the Kähler form (equivalently, a Kähler S^1-action) is Hamiltonian if and only if it has a fixed point. This implies that (i) and (ii) are equivalent for any torus T in the Kähler setting (see eg. [26, Proof of Theorem 3]). McDuff [21] proved that if the fixed point set of a symplectic S^1-action on a compact connected symplectic four manifold is non-empty, then the action is Hamiltonian. However, she also constructed a non-Hamiltonian symplectic S^1-action with non–discrete fixed point set. We will prove the following:

Theorem 1.1. *Let T be an $(n-2)$-dimensional torus which acts effectively with non-empty fixed point set on a compact, connected, symplectic $2n$-dimensional manifold (M,ω). Suppose that there is an effective commuting symplectic action of a 2-torus on M whose orbits are symplectic. Then the action of T on $(M,\,\omega)$ is Hamiltonian.*

The existence of a commuting action with symplectic orbits is critical for this paper (such actions are described in Section 6). This assumption is nevertheless satisfied in a number of interesting situations, and in particular Theorem 1.1 can be used to derive McDuff's theorem (with a different proof) and its subsequent extension by Kim (see Section 5).

Consider the momentum map $\mu\colon M \to \mathfrak{t}^*$ of the Hamiltonian T-action in Theorem 1.1, where $\mathfrak{t}$ is the Lie algebra of T, and $\mathfrak{t}^*$ its dual Lie algebra. If $V \subset M$ is an open set, the *Liouville measure on* M of V is defined as $\int_V \frac{\omega^n}{n!}$. The *Duistermaat-Heckman measure* on $\mathfrak{t}^*$ is the push-forward of the Liouville measure on M by μ. The Duistermaat-Heckman measure is absolutely continuous with respect to the Lebesgue measure, and its density function[1] $\mathrm{DH}_T\colon \mathfrak{t}^* \to [0, \infty)$ is the *Duistermaat-Heckman function* (see Section 2). Recall that if V is a vector space, and if a Borel measurable function $f : V \to \mathbb{R}$ is positive almost everywhere, we say that f is *log-concave* if its logarithm is a concave function.

Theorem 1.2. *Let T be an $(n-2)$-dimensional torus which acts effectively and Hamiltonianly on a compact, connected, symplectic $2n$-dimensional manifold (M, ω). Suppose that there is an effective commuting symplectic action of a 2-torus S on M whose orbits are symplectic. Then the Duistermaat-Heckman function DH_T of the Hamiltonian T-action is log-concave, i.e. its logarithm is a concave function.*

We obtain, as a consequence of Theorems 1.1 and 1.2, a theorem of Kim (Theorem 5.1). If $n = 2$, this statement is due to McDuff [21] (see Corollary 5.2). The literature on this topic is extensive, we refer to [32, 9, 20, 5, 18, 14, 19, 27] and the references therein for related results.

Structure of the paper. In Section 2 we review the basic elements of symplectic manifolds and torus actions that we need in the remaining of the paper. We prove Theorem 1.2 in Section 3, and we prove Theorem 1.1 in Section 4. In Section 5 we explain how our theorems can be used to derive proofs of some well-known results.

2. Duistermaat-Heckman theory preliminares

Symplectic group actions. Let (M, ω) be a symplectic manifold, i.e. the pair consisting of a smooth manifold M and a symplectic form ω on M (a non-degenerate closed 2-form on M). Let T be a torus, i.e. a compact, connected, commutative Lie group with Lie algebra $\mathfrak{t}$ (T is isomorphic, as a Lie group, to a finite product of circles S^1). Suppose that T acts on a symplectic manifold (M, ω) symplectically (i.e., by diffeomorphisms which preserve the symplectic form). We denote by $(t, m) \mapsto t \cdot m$ the action $T \times M \to M$ of T on M. Any element $X \in \mathfrak{t}$ generates a vector field X_M on M, called the *infinitesimal generator*, given by $X_M(m) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(tX)\cdot m$, where $\exp\colon \mathfrak{t} \to T$ is the exponential map of Lie theory and $m \in M$. As usual, we write $\iota_{X_M}\omega := \omega(X_M, \cdot) \in \Omega^1(M)$ for the contraction 1-form. The T-action on (M, ω) is said to be *Hamiltonian* if there exists a smooth invariant map $\mu\colon M \to \mathfrak{t}^*$, called the *momentum map*, such that for all $X \in \mathfrak{t}$ we have that

$$\iota_{X_M}\omega = \mathrm{d}\langle \mu, X\rangle, \tag{1}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$ is the duality pairing.

Remark 2.1. Theorem 1.1 says that there exists a smooth map invariant map $\mu\colon M \to \mathfrak{t}^*$ satisfying (1), where $\mathfrak{t}^*$ is the dual Lie algebra of T^{n-2}.

[1] which is well defined once the normalization of the Lebesgue measure is declared.

We say that the T-action on M *has fixed points* if

$$M^T := \{m \in M \mid t \cdot m = m, \text{ for all } t \in T\} \neq \varnothing. \tag{2}$$

The T-action is *effective* if the intersection of all stabilizer subgroups $T_m := \{t \in T \,|\, t \cdot m = m\},\ m \in M$, is the trivial group. The T-action is *free* if T_m is the trivial group for all points $m \in N$. The T-action is *semifree* if it is free on $M \setminus M^T$, where M^T is given by (2). Note that if a T-action is semifree, then it either leaves every point in M fixed, or it is effective. Finally, the action of T is called *quasi-free* if the stabilizer subgroup of every point is connected.

Atiyah-Guillemin-Sternberg convexity. We will need the following:

Theorem 2.2 (Atiyah [2], Guillemin and Sternberg [12])**.** *If a torus T with Lie algebra $\mathfrak{t}$ acts on a compact, connected $2n$-dimensional symplectic manifold (M, ω) in a Hamiltonian fashion, then the image $\mu(M)$ under the momentum map $\mu\colon M \to \mathfrak{t}^*$ of the action is a convex polytope $\Delta \subset \mathfrak{t}^*$.*

Reduced symplectic forms and Duistermaat-Heckman formula. Consider the Hamiltonian action of a torus T on a symplectic manifold (M, ω). Let $a \in \mathfrak{t}^*$ be a regular value of the momentum map $\mu : M \to \mathfrak{t}^*$ of the action. When the action of T on M is not quasi-free, the quotient $M_a := \mu^{-1}(a)/T$ taken at a regular value of the momentum map is not a smooth manifold in general. However, M_a admits a smooth orbifold structure in the sense of Satake [31]. Even though smooth orbifolds are not necessarily smooth manifolds, they carry differential structures such as differential forms, fiber bundles, etc. In fact, the usual definition of symplectic structures extends to the orbifold case. In particular, the restriction of the symplectic form ω to the fiber $\mu^{-1}(a)$ descends to a symplectic form ω_a on the quotient space M_a, cf. Weinstein [34]. We refer to [1, 22] and to [25, Appendix] background on orbifolds. We are ready to state an equivariant version of the celebrated Duistermaat-Heckman theorem.

Theorem 2.3 (Duistermaat-Heckman [6])**.** *Consider an effective Hamiltonian action of a k-dimensional torus T^k on a compact, connected, $2n$-dimensional symplectic manifold (M, ω) with momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that there is another symplectic action of 2-torus T^2 on M that commutes with the action of T^k. Then the following hold:*

(a) *at a regular value $a \in \mathfrak{t}^*$ of μ, the Duistermaat-Heckman function is given by*
$$\mathrm{DH}_{T^k}(a) = \int_{M_a} \frac{\omega_a^{n-k}}{(n-k)!},$$
where $M_a := \mu^{-1}(a)/T^k$ is the symplectic quotient, ω_a is the corresponding reduced symplectic form, and M_a has been given the orientation of ω_a^{n-k};

(b) *if $a, a_0 \in \mathfrak{t}$ lie in the same connected component of the set of regular values of the momentum map μ, then there is a T^2-equivariant diffemorphism $F\colon M_a \longrightarrow M_{a_0}$, where $M_{a_0} := \mu^{-1}(a_0)/T^k$ and $M_a := \mu^{-1}(a)/T^k$. Furthermore, using F to identify M_a with M_{a_0}, the reduced symplectic form on M_a may be identified with $\omega_a = a_0 + \langle c, a - a_0\rangle$, where $c \in \Omega^2(M, \mathfrak{t}^*)$ is a closed $\mathfrak{t}^*$–valued two--form representing the Chern class of the principal torus bundle $\mu^{-1}(a_0) \to M_{a_0}$.*

Primitive decomposition of differential forms. Here we assume that (X, ω) is a $2n$-dimensional symplectic orbifold. Let $\Omega^k(X)$ be the space of differential forms of degree k on X. Let $\Omega(X)$ be the space corresponding to the collection of all $\Omega^k(X)$, for varying k's.

We note that, in analogy with the case of symplectic manifolds, there are three natural operators on $\Omega(X)$ given as follows:

$$\begin{aligned} &\mathrm{L} : \Omega^k(X) \to \Omega^{k+2}(X), \;\; \alpha \mapsto \omega \wedge \alpha, \\ &\Lambda : \Omega^k(X) \to \Omega^{k-2}(X), \;\; \alpha \mapsto \iota_\pi \alpha, \\ &\mathrm{H} : \Omega^k(X) \to \Omega^{k}(X), \;\; \alpha \mapsto (n-k)\alpha, \end{aligned}$$

where $\pi = \omega^{-1}$ is the Poisson bi-vector induced by the symplectic form ω. These operators satisfy the following bracket relations.

$$[\Lambda, \mathrm{L}] = \mathrm{H}, \;\; [\mathrm{H}, \Lambda] = 2\Lambda, \;\; [\mathrm{H}, \mathrm{L}] = -2\mathrm{L}.$$

Therefore they define a representation of the Lie algebra sl(2) on $\Omega(X)$. A primitive differential form in $\Omega(X)$ is by definition a highest weight vector in the sl_2-module $\Omega(X)$. Equivalently, for any integer $0 \leqslant k \leqslant n$, we say that a differential k-form α on X is primitive if and only if $\omega^{n-k+1} \wedge \alpha = 0$. Although the sl_2-module $\Omega(X)$ is infinite dimensional, there are only finitely many eigenvalues of H. There is a detailed study of sl_2-modules of this type in [36]. The following result is an immediate consequence of [36, Corollary 2.5].

Lemma 2.4. *A differential form $\alpha \in \Omega^k(X)$ admits a unique primitive decomposition*

$$\alpha = \sum_{r \,\geqslant\, \max(k-n,0)} \frac{\mathrm{L}^r}{r!}\, \beta_{k-2r}, \tag{3}$$

where β_{k-2r} is a primitive form of degree $k - 2r$.

3. Proof of Theorem 1.2

Toric fibers and reduction ingredients.

Lemma 3.1. *Let a torus T with Lie algebra $\mathfrak{t}$ act on a compact, connected, symplectic manifold (M, ω) in a Hamiltonian fashion with momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that the action of another torus S on M is symplectic* *and commutes with the action of T. Then for any $a \in \mathfrak{t}^*$, the action of S preserves the level set $\mu^{-1}(a)$ of the momentum map.*

Proof. Let X and Y be two vectors in the Lie algebra of S and T respectively, and let X_M and Y_M be the vector fields on M induced by the infinitesimal action of X and Y respectively. Let $\mathcal{L}_{X_M}\mu^Y$ denote the Lie derivative of $\mu^Y := \langle \mu, Y\rangle \colon M \to \mathbb{R}$ with respect to X_M. Let $[\cdot, \cdot]$ denote the Lie bracket on vector fields. It suffices to show that $\mathcal{L}_{X_M}\mu^Y = 0$.

Since the action of T and S commute, we have that $[X_M, Y_M] = 0$. By the Cartan identities, we have

$$\begin{aligned}
0 = \iota_{[X_M,Y_M]}\omega &= \mathcal{L}_{X_M}\iota_{Y_M}\omega - \iota_{Y_M}\mathcal{L}_{X_M}\omega \\
&= \mathcal{L}_{X_M}\iota_{Y_M}\omega \quad \text{(because the action of } S \text{ is symplectic.)} \\
&= \mathcal{L}_{X_M}(\mathrm{d}\mu^Y) = \mathrm{d}\mathcal{L}_{X_M}\mu^Y.
\end{aligned}$$

This proves that $\mathcal{L}_{X_M}\mu^Y$ must be a constant. On the other hand, we have

$$\mathcal{L}_{X_M}\mu^Y = \iota_{X_M}\mathrm{d}\mu^Y = \iota_{X_M}\iota_{Y_M}\omega = \omega(Y_M, X_M).$$

Since the action of T is Hamiltonian and the underlying manifold M is compact, there must exist a point $m \in M$ such that Y_M vanishes at m. Thus $(\mathcal{L}_{X_M}\mu^Y)(m) = 0$, and therefore $\mathcal{L}_{X_M}\mu^Y$ is identically zero on M. □

Remark 3.2. It follows from the above argument that if the action of T is symplectic with a generalized momentum map, cf. Definition 4.2, and if the fixed point set is non-empty, the assertion of Lemma 3.1 still holds.

Lemma 3.3. *Let an $(n-2)$-dimensional torus T^{n-2} with Lie algebra $\mathfrak{t}$ act effectively on a compact connected symplectic manifold (M, ω) in a Hamiltonian fashion with momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that there is an effective commuting T^2-action on (M, ω) which has a symplectic orbit. Then for any regular value $a \in \mathfrak{t}^*$, the symplectic quotient $M_a := \mu^{-1}(a)/T^{n-2}$ inherits an effective symplectic T^2-action with symplectic orbits.*

Proof. Let a be a regular value of the momentum map $\mu\colon M \to \mathfrak{t}^*$. Then the symplectic quotient M_a is a four dimensional symplectic orbifold. By assumption, the T^2-action on M has a symplectic orbit. Hence all the orbits of the T^2-action on M are symplectic [25, Corollary 2.2.4]. By Lemma 3.1, the T^2-action preserves the level set $\mu^{-1}(a)$, and thus it descends to an action on the reduced space M_a. It follows that the orbits of the induced T^2-action on M_a are symplectic two dimensional orbifolds. Thus, by dimensional considerations, any isotropy subgroup of the T^2-action on M_a is zero-dimensional. Since these isotropy subgroups must be closed subgroups of T^2, they must be finite subgroups of T^2. Now let H be the intersection of the all isotropy subgroups of T^2. Then H is a finite subgroup of T^2 itself. Note that the quotient group T^2/H is a two-dimensional compact, connected commutative Lie group. Hence T^2/H is isomorphic to T^2. Therefore M_a admits an effective action of $T^2 \cong T^2/H$ with symplectic orbits. □

Symplectic T-model of (M,ω). Let (X,σ) be a connected symplectic 4-manifold equipped with an effective symplectic action of a 2-torus T^2 for which the T^2-orbits are symplectic. We give a concise overview of a model of (X,σ), cf. Section 6. Consider the quotient map $\pi : X \to X/T^2$. Choose a base point $x_0 \in X$, and let $p_0 := \pi(x_0)$. For any homotopy class $[\gamma] \in \widetilde{X/T^2}$, i.e., the homotopy class of a loop γ in X/T^2 with base p_0, denote by $\lambda_\gamma : [0,1] \to X$ the unique horizontal lift of γ ,with respect to the flat connection Ω of symplectic orthogonal complements to the tangent spaces to the T-orbits, such that $\gamma(0) = x_0$, cf. [25, Lemma 3.4.2]. Then the map

$$\Phi : \widetilde{X/T^2} \times T^2 \to X, \ \ ([\gamma], t) \mapsto t \cdot \lambda_\gamma(1)$$

is a smooth covering map and it induces a T^2-equivariant symplectomorphism $\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2 \to X$. We refer to see Section 6, and in particular Theorem 6.1 and the remarks below it, for a more thorough description.

Kähler ingredients.

Lemma 3.4. *Consider an effective symplectic action of a 2-torus T^2 on a connected [2] symplectic 4-manifold (X,σ) with a symplectic orbit. Then X admits a T^2-invariant Kähler structure. It consists of a T^2-invariant complex structure, and a Kähler form equal to σ.*

Proof. If X is compact, the lemma is a particular case of Duistermaat-Pelayo [7, Theorem 1.1]. The proof therein is given by constructing the complex structure on the model $\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2$ of M. In fact, the same proof therein applies even if X is not compact. We review it for completeness.

The orbifold universal covering $\widetilde{X/T^2}$ of the orbisurface X/T^2 may be identified with the Riemann sphere, the Euclidean plane, or the hyperbolic plane, on which the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/T^2)$ of X/T^2 acts by means of orientation preserving isometries, see Thurston [33, Section 5.5]. Provide $\widetilde{X/T^2}$ with the standard complex structure. Let σ^{T^2} be the unique T^2-invariant symplectic form on T^2 defined by $\sigma^{\mathfrak{t}}$ at the beginning of Section 6: the restriction of σ to any T-orbit. Equip T^2 with a T^2-invariant complex structure such that σ^T is equal to a Kähler form. In this way we obtain a T^2-invariant complex structure on $\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2$ with symplectic form σ equal to the Kähler form. □

Suppose that a is a regular value of $\mu : M \to \mathfrak{t}^*$. Let M_a^{reg} be the set of smooth points in M_a. Then it follows from Lemma 3.4 that M_a^{reg} is a smooth manifold that admits a T^2 invariant Kähler structure, which consists of a T^2-invariant complex structure and a T^2-invariant Kähler form ω_a.

[2] not necessarily compact

Duistermaat-Heckman theorem in a complex setting. The following lemma is a key step in our approach.

Lemma 3.5. *Consider an effective Hamiltonian action of an $(n-2)$-torus T^{n-2} with Lie algebra $\mathfrak{t}$ on a compact, connected, $2n$-dimensional symplectic manifold (M,ω) with momentum map $\mu\colon M\to\mathfrak{t}^*$. Suppose that there is an effective commuting symplectic T^2-action on M which has a symplectic orbit, that a and b are two regular values of μ which lie in the same connected component of the regular values of the momentum map, and that*

$$M_a:=\mu^{-1}(a)/T^{n-2} \text{ and } M_b:=\mu^{-1}(b)/T^{n-2}$$

are symplectic quotients with reduced symplectic structure ω_a and ω_b respectively. Let M_a^{reg} and M_b^{reg} be the complement of the sets of orbifold singularities in M_a and M_b, respectively. Then both M_a^{reg} and M_b^{reg} are T^2--invariant Kähler manifolds as in Lemma 3.4, whose Kähler forms are ω_a and ω_b respectively. Moreover, there is a diffemorphism $F:M_a\to M_b$ such that $F^*\omega_b$ is a $(1,1)$-form on the Kähler manifold M_b^{reg}.

Proof. By Lemma 3.3, both (M_a,ω_a) and (M_b,ω_b) are symplectic four orbifolds that admit an effective symplectic T^2 action with symplectic orbits. The smooth parts M_a^{reg} and M_b^{reg} are symplectic manifolds that admit an effective symplectic action of T^2 with symplectic orbits. It follows from Lemma 3.4 that $(M_a^{\mathrm{reg}},\omega_a)$ and $(M_b^{\mathrm{reg}},\omega_b)$ admit a T^2-invariant complex structure and the corresponding Kähler form is equal to ω_a,ω_b, respectively. We prove that $F^*\omega_a$ is a $(1,1)$-form on M_b^{reg} by writing down a local expression of F using holomorphic coordinates on M_a^{reg} and M_b^{reg} respectively.

Let Ω_a be the flat connection given by the ω_a-orthogonal complements to the tangent spaces to the T^2-orbits on M_a^{reg}. Similarly for Ω_b. Let $\Sigma_a:=M_a^{\mathrm{reg}}/T^2$, $\Sigma_b:=M_b^{\mathrm{reg}}/T^2$, and let $\widetilde{\Sigma_a}$ and $\widetilde{\Sigma_b}$ be the orbifold universal covers of Σ_a and Σ_b, respectively, at some base points. We have two smooth covering maps $\Phi_a:\widetilde{\Sigma_a}\times T^2\to M_a^{\mathrm{reg}}$ and $\Phi_b:\widetilde{\Sigma_b}\times T^2\to M_b^{\mathrm{reg}}$. Suppose $p_0=t_0\lambda_{\gamma_0}(1)\in M_a^{\mathrm{reg}}$ is an arbitrary point. Choose an open neighborhood U of $[\gamma_0]$ and V of t_0 such that the restriction of Φ_a to $U\times V$ is a diffeomorphism. By Theorem 2.3, there exists a T^2-equivariant diffeomorphism $F:M_a^{\mathrm{reg}}\to M_b^{\mathrm{reg}}$. We have that for any $[\gamma]\in U$ and $t\in V$, $F(t\lambda_\gamma(1))=t\cdot F(\lambda_\gamma(1))$.

On the other hand, F induces a diffeomorphism between two 2-dimensional orbifolds

$$\varphi_{ab}:\,\Sigma_a\to\Sigma_b. \tag{4}$$

These orbifolds can be shown to be good orbifolds [25, Lemma 3.4.1], and hence their orbifold universal covers are smooth manifolds. Then we have an induced diffeomorphism between smooth manifolds $\widetilde{\varphi}_{ab}:\widetilde{\Sigma_a}\to\widetilde{\Sigma_b}$. Note that $\widetilde{\varphi}_{ab}(\gamma)$ is a loop based at $\widetilde{\varphi}_{ab}(p_0)$. Denote by $\lambda_{\widetilde{\varphi}_{ab}(\gamma)}$ its horizontal lift with respect to Ω_b. Then we have $\lambda_{\widetilde{\varphi}_{ab}(\gamma)}(1)=\tau_\gamma\cdot F(\lambda_\gamma(1))$, for some $\tau_\gamma\in T$. Therefore, using covering maps Φ_a and Φ_b, the diffemorphism $F:M_a^{\mathrm{reg}}\to M_b^{\mathrm{reg}}$ has a local expression $([\gamma],t)\mapsto([\widetilde{\varphi}_{ab}(\gamma)],\,\tau_\gamma t)$. The proposition follows from the above expression for F, and how the complex structures and symplectic structures are constructed on M_a and M_b, cf. Lemma 3.4. □

Walls and Lerman-Guillemin-Sternberg jump formulas. Consider the effective Hamiltonian action of a k-torus T^k on a $2n$-dimensional compact symplectic manifold (M, ω) with momentum map $\mu : M \to \mathfrak{t}^*$. By Theorem 2.2, the image of the momentum map $\Delta := \mu(M)$ is a convex polytope. In fact, Δ is a union of convex subpolytopes with the property that the interiors of the subpolytopes are disjoint convex open sets and constitute the set of regular values of μ. These convex subpolytopes are called the *chambers of* Δ.

For any circle S^1 of T^k, and for any connected component $\mathcal{C}$ of the fixed point submanifold of S^1, the image of $\mathcal{C}$ under the momentum map μ is called an $(k-1)$-dimensional wall, or simply a *wall of* Δ. Moreover, $\mu(\mathcal{C})$ is called an *interior wall of* Δ if $\mu(\mathcal{C})$ is not a subset of the boundary of Δ. Now choose a T^k-invariant inner product on $\mathfrak{t}^*$ to identify $\mathfrak{t}^*$ with $\mathfrak{t}$. Suppose that a is a point on a codimension one interior wall W of Δ, and that v is a normal vector to W such that the line segment $\{a + tv\}$ is transverse to the wall W. For t in a small open interval near 0, write

$$g(t) := \mathrm{DH}_T(a + tv), \tag{5}$$

where DH_T is the Duistermaat-Heckman function of T.

Let S^1 be a circle sitting inside T^k generated by v. Let H be a $(k-1)$-dimensional torus H such that $T^k = S^1 \times H$. Suppose that X is a connected component of the fixed point submanifold of the S^1-action on M such that $\mu(X) = W$. Then X is invariant under the action of H. Moreover, the action of H on X is Hamiltonian. In fact, let $\mathfrak{h}$ be the Lie algebra of H, and let $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ be the canonical projection map. Then the composite $\mu_H := \pi \circ \mu\,|_X : X \to \mathfrak{h}^*$ is a momentum map for the action of H on X.

By assumption, a is a point in $W = \mu(X)$ such that $\pi(a)$ is a regular value of μ_H. There are two symplectic quotients associated to a: one may reduce M (viewed as an H-space) at $\pi(a)$, and one may reduce X (viewed as a T^k/S^1-space) at a. We denote these symplectic quotients by M_a and X_a respectively. It is immediate that M_a inherits a Hamiltonian S^1-action and that X_a is the fixed point submanifold of the S^1-action on M_a.

Guillemin-Lerman-Sternberg gave the following formula for computing the jump in the Duistermaat-Heckman function across the wall $\mu(X)$ of Δ.

Theorem 3.6 (Guillemin-Lerman-Sternberg [11])**.** *The jump of the function $g(t)$ in expression (5) when moving across the interior wall W is given by*

$$g_+(t) - g_-(t) \;=\; \sum \mathrm{volume}(X_a) \left(\prod_{i=1}^{k} \alpha_k^{-1} \right) \frac{t^{k-1}}{(k-1)!},$$

plus an error term of order $\mathcal{O}(t^k)$. Here the α_k's are the weights of the representation of S^1 on the normal bundle of X, and the sum is taken over the symplectic quotients of all the connected components X of $M^{S^1} \cap \mu^{-1}(\alpha)$ with respect to the T^k/S^1 action at a.

Building on Theorem 3.6, Graham established the following result, cf. [10, Section 3].

Proposition 3.7 (Graham [10])**.** *Suppose that $\mu : M \to \mathfrak{t}^*$ is the momentum map of an effective Hamiltonian action of torus T on a compact, connected symplectic manifold (M, ω). Let a be a point in a codimension one interior wall of $\mu(M)$, and let $v \in \mathfrak{t}^*$ be a vector such that the line segment $\{a + tv\}$ is transverse to the wall. For t in a small open interval near 0, write $g(t) = \mathrm{DH}_T(a + tv)$, where DH_T is the Duistermaat-Heckman function. Then we have that $g'_+(0) \leqslant g'_-(0)$.*

Hodge-Riemann bilinear relations and last step in proof of Theorem 1.2. Recall that if V is a vector space and and $A \subset V$ is a convex open subset, and if $f\colon A \to \mathbb{R}$ is a Borel measurable map that is positive on A almost everywhere, then we say that f is *log-concave* on A if and only if $\log f$ is a concave function on A. Moreover, if f is smooth on A, and if a is a fixed point in A, then a simple calculation shows that f is log-concave on A if and only if

$$f|''_{\{a+tv\}}(a + tv) \cdot f|_{\{a+tv\}}(a + tv) - (f|'_{\{a+tv\}})^2(a + tv) \leqslant 0, \text{ for all } \; v \in A$$

where $\{a + tv\}$ is a line segment through a, where t lies in some small open interval containing 0.

Proof of Theorem 1.2. By Proposition 3.7, to establish the log-concavity of DH_T on $\mu(M)$, it suffices to show that the restriction of $\log \mathrm{DH}_T$ to each connected component of the set of regular values of μ is concave. Let $\mathcal{C}$ be such a component, let $v \in \mathfrak{t}^*$, and let $\{a + tv\}$ be a line segment in $\mathcal{C}$ passing through a point $a \in \mathcal{C}$, where the parameter t lies in some small open interval containing 0. We need to show that $g(t) := \mathrm{DH}_T(a + tv)$ is log-concave, or equivalently, that

$$g''g - (g')^2 \leqslant 0. \tag{6}$$

Since the point a is arbitrary, it suffices to show that equation (6) holds at $t = 0$. By Theorem 2.3, at $a + tv \in \mathfrak{t}^*$, the Duistermaat-Heckman function is

$$\mathrm{DH}_T(a + tv) = \int_{M_a} \frac{1}{2} (\omega_a + tc)^2.$$

(We identify M_{a+tv} with M_a). Here $M_a = \mu^{-1}(a)/T$ is the symplectic quotient taken at a and c is a closed two-form depending only on $v \in \mathfrak{t}^*$. To prove (6), we must show that

$$\int_{M_a} c^2 \int_{M_a} \omega_a^2 \;\leqslant\; 2 \left(\int_{M_a} c\, \omega_a \right)^2. \tag{7}$$

Consider the primitive decomposition of the two-form c as $c = \gamma + s\omega_a$, where s is a real number, and γ is a primitive two-form. By primitivity, we have that $\gamma \wedge \omega_a = 0$. A simple calculation shows that (7) for γ implies (7) for c. But then (7) becomes

$$\int_{M_a} \gamma^2 \leqslant 0. \tag{8}$$

Note that in a symplectic orbifold the subset of orbifold singularities is of codimension greater than or equal to 2, cf. [3, Prop. III.2.20]. In particular, it is of measure zero with respect to the Liouville measure on the symplectic orbifold. Let M_a^{reg} be the complement of the set of orbifold singularities in M_a. It follows that $\int_{M_a} \gamma^2 = \int_{M_a^{\mathrm{reg}}} \gamma^2$. Note that M_a^{reg} is a connected symplectic four-manifold which admits the effective symplectic action of T^2 with symplectic orbits. By Lemma 3.4, M_a^{reg} is a Kähler manifold. Moreover, ω_a is the Kähler form on M_a^{reg}. In particular, it must be a $(1,1)$-form.

Let $\langle \cdot, \cdot \rangle$ be the metric on the space of $(1,1)$-forms induced by the Kähler metric, and let $*$ be the Hodge star operator induced by the Kähler metric. By Lemma 3.5, c must be a $(1,1)$-form. As a result, γ must be a real primitive $(1,1)$-form. Applying Weil's identity, cf. [35, Thm. 3.16], we get that $\gamma = - * \gamma$. Consequently, we have that

$$\int_{M_a^{\mathrm{reg}}} \gamma^2 = - \int_{M_a^{\mathrm{reg}}} \gamma \wedge * \gamma = - \int_{M_a^{\mathrm{reg}}} \langle \gamma, \gamma \rangle \; \leqslant \; 0.$$

This completes the proof of Theorem 1.2. □

Remark 3.8. The log-concavity of the Duistermaat-Heckman function was proved by Okounkov [23, 24] for projective algebraic varieties, and by Graham [10] for arbitrary compact Kähler manifolds. In view of these positive results, it was conjectured independently by Ginzburg and Knudsen in 1990's that the Duistermaat-Heckman function of any Hamiltonian torus action on a compact symplectic manifold is log-concave. However, Karshon [15] disproved the conjecture in general.

Log-concavity conjecture for complexity one Hamiltonian torus actions. We note that our proof of Theorem 1.2 and Theorem 1.1 depend on the classification of symplectic four manifolds with a symplectic T^2 action [25]. However, in the following special case, we have an elementary proof which makes no uses of the classification results therein.

Theorem 3.9. *Consider an effective Hamiltonian action of an $(n-2)$-dimensional torus T^{n-2} on a compact, connected, $(2n-2)$-dimensional symplectic manifold (M, ω). Let T^2 be a 2-dimensional torus equipped with the standard symplectic structure. We consider the following action of T^{n-2} on the $2n$-dimensional product symplectic manifold $M \times T^2$,*

$$g \cdot (m, t) = (g \cdot m, t), \; \text{ for all } \; g \in T^{n-2}, \, m \in M, \, t \in T^2. \tag{9}$$

Then the Duistermaat-Heckman function $\mathrm{DH}_{T^{n-2}}$ *of the* T^{n-2} *action on* $M \times T^2$ *is log-concave.*

Proof. Note that there is an obvious commuting T^2 symplectic action on M

$$h \cdot (m, t) = (m, h \cdot t), \text{ for all } h \in T^2, m \in M, t \in T^2, \tag{10}$$

which is effective and which has symplectic orbits. Let μ be the momentum map of the T^{n-2} action on $M \times T^2$, and $\mathrm{DH}_{T^{n-2}}$ the Duistermaat-Heckman function.

To prove that $\mathrm{DH}_{T^{n-2}}$ is log-concave, in view of Proposition 3.7, it suffices to show that the restriction of $\ln \mathrm{DH}_{T^{n-2}}$ to each connected component of the set of regular values of μ is concave. Using the same notation as in the proof of Theorem 1.2, we only need to establish the validity of inequality (6). However, a calculation shows that in the current case the function g must be a degree one polynomial, and so the inequality (6) holds automatically. □

The following corollary is an immediate consequence of Theorem 3.9.

Corollary 3.10. *Suppose that there is an effective Hamiltonian action of an* $(n-1)$*-dimensional torus* T^{n-1} *on a compact, connected,* $2n$*-dimensional symplectic manifold* (M, ω)*. Then the Duistermaat-Heckman function* DH_T *of the* T*-action is log-concave.*

4. Proof of Theorem 1.1

Logarithmic concavity of torus valued functions. We first extend the notion of log-concavity to functions defined on a k-dimensional torus $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$. Consider the covering map

$$\exp : \mathbb{R}^k \to T^k, \quad (t_1, \cdots, t_k) \mapsto (\mathrm{e}^{\mathrm{i}2\pi t_1}, \cdots, \mathrm{e}^{\mathrm{i}2\pi t_k}).$$

Definition 4.1. We say that a map $f : T^k \to (0, \infty)$ is *log-concave at a point* $p \in T^k$ if there is a point $x \in \mathbb{R}^k$ with $\exp(x) = p$ and an open set $V \subset \mathbb{R}^k$ such that $\exp|_V \colon V \to \exp(V)$ is a diffeomorphism and such that the logarithm of $\tilde{f} := f \circ \exp$ is a concave function on V. We say that f is *log-concave on* T^k if it is log-concave at every point of T^k.

The log-concavity of the function f does not depend on the choice of V. Indeed, suppose that there are two points $x_1, x_2 \in \mathbb{R}^k$ such that $\exp x_1 = \exp x_2 = p$, and two open sets $x_1 \in V_1 \subset \mathbb{R}^k$ and $x_2 \in V_2 \subset \mathbb{R}^k$ such that both $\exp|_{V_1} \colon V_1 \to \exp(V_1)$ and $\exp|_{V_2} \colon V_2 \to \exp(V_2)$ are diffeomorphisms. Set $\tilde{f}_i = f \circ \exp|_{V_i}$, $i = 1, 2$. Then there exists $(n_1, \cdots n_k) \in \mathbb{Z}^k$ such that

$$\tilde{f}_1(t_1, \cdots t_k) = \tilde{f}_2(t_1 + n_1, \cdots, t_k + n_k), \text{ for all } \in (t_1, \cdots, t_k) \in V_1.$$

Now let $T^k \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z}$ be a k dimensional torus, and λ the standard length form on $S^1 \cong \mathbb{R}/\mathbb{Z}$. Throughout this section we denote by Θ the $\mathfrak{t}^*$-valued form

$$\lambda \oplus \lambda \oplus \cdots \oplus \lambda \tag{11}$$

on T^k. Clearly, Θ is invariant under the T^k-action on T^k by multiplication.

Consider the action of a k-dimensional torus T^k on a $2n$-dimensional symplectic manifold (M, ω). Let $\mathfrak{t}$ be the Lie algebra of T^k, let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and let Θ be the canonical T^k-invariant $\mathfrak{t}^*$-valued one form given as in (11).

Definition 4.2. We say that $\Phi : M \to T^k$ is a *generalized momentum map* if $\iota_{X_M}\omega = \langle X, \Phi^*\Theta\rangle$, where $X \in \mathfrak{t}$, and X_M is the vector field on M generated by the infinitesimal action of X on M.

Duistermaat-Heckman densities of Torus valued momentum maps. The following fact is a straightforward generalization of a well-known result due to D. McDuff concerning the existence of circle valued momentum maps.

Theorem 4.3 (McDuff [21])**.** *Let a k-dimensional torus $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$ act symplectically on the compact symplectic manifold (M, ω). Assume that ω represents an integral cohomology class in $\mathrm{H}^2(M, \mathbb{Z})$. Then:*

1. *either the action admits a $\mathbb{R}^k$-valued momentum map or, if not,*
2. *there exists a T^k-invariant symplectic form ω' on M that admits a T^k-valued momentum map $\Phi\colon M \to T^k$.*

When ω is integral and $k = 1$, the 1-form $\iota_X\omega$ is also integral and the map in Theorem 4.3 is defined as follows. Pick a point $m_0 \in M$, let γ_m be an arbitrary smooth path connecting m_0 to m in M, and define the map $\Phi : M \to S^1$ by

$$\Phi(m) := \left[\int_{\gamma_m} \iota_X\omega\right] \in \mathbb{R}/\mathbb{Z}. \tag{12}$$

One can check that Φ is well-defined.

Remark 4.4. When $k = 1$, a detailed proof of Theorem 4.3 may be found in Pelayo-Ratiu [26]. The argument given in [26] extends to actions of higher dimensional tori, as pointed out therein. As noted in [21], the usual symplectic quotient construction carries through for generalized momentum maps.

Now we will need the following result, which is Theorem 3.5 in [30].

Theorem 4.5 (Rochon [30])**.** *Let a compact connected Lie group G act on a symplectic manifold (M, ω) symplectically. Then the G-action on (M, ω) is Hamiltonian if and only if there exists a symplectic form σ on M such that the G-action on (M, σ) is Hamiltonian.*

It follows from Theorem 4.5 that in order to show that a symplectic S^1-action on a symplectic manifold (M, ω) is Hamiltonian, it suffices to show it under the assumption that $[\omega]$ is integral, see [4, Remark 2.1].

Definition 4.6. Suppose that there is an effective symplectic action of a k-dimensional torus T^k on a $2n$-dimensional compact symplectic manifold

(M, ω) with a generalized momentum map $\Phi : M \to T^k$. The *Duistermaat-Heckman measure* is the push-froward of the Liouville measure on M by Φ. Its Density function is the *Duistermaat-Heckman function.*

Analogous to the case of Hamiltonian actions, we have the following result.

Proposition 4.7. *Suppose that there is an effective symplectic action of a k-dimensional torus T^k on a $2n$-dimensional compact symplectic manifold (M, ω) with a generalized momentum map $\Phi : M \to T^k$. Let $(T^k)_{\mathrm{reg}} \subset T^k$ be the set of regular values of Φ. Then we have that*

$$\mathrm{DH}_{T^k}(a) = \int_{M_a} \omega_a^{n-k}, \text{ for all } a \in (T^k)_{\mathrm{reg}},$$

where $M_a := \mu^{-1}(a)/T^k$ is the symplectic quotient taken at a, and ω_a is the reduced symplectic form.

Symplectic cuttings. We also need the construction of symplectic cutting which was first introduced by Lerman [17]. Next we present a review of this construction. Let $\mu : M \to \mathfrak{t}^*$ be a momentum map for an effective action of a k-dimensional torus T^k on a symplectic manifold (M, ω) and let $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{t}$ denote the integral lattice. Choose k vectors $v_j \in \mathfrak{t}_{\mathbb{Z}}$, $1 \leqslant j \leqslant k$. The form $\omega - \mathrm{i} \sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ is a symplectic form on the manifold $M \times \mathbb{C}^k$. The map $\nu : M \times \mathbb{C}^k \to \mathbb{R}^k$ with j-th component $\nu_j(m, z) = \langle \nu(m), v_j \rangle - |z_j|^2$ is a momentum map for the T^k-action on $M \times \mathbb{C}^k$. For any $b = (b_1, b_2, \cdots, b_k) \in \mathbb{R}^k$, consider a convex rational polyhedral set

$$P = \{x \in \mathfrak{t}^* \mid \langle x, v_i \rangle \geqslant b_i, \ \forall\, 1 \leqslant i \leqslant k\}. \tag{13}$$

The *symplectic cut of M with respect to P* is the reduction of $M \times \mathbb{C}^k$ at b. We denote it by M_P .

Similarly, for any two regular values $b = (b_1, \cdots, b_k), c = (c_1, \cdots, c_k) \in \mathbb{R}^k$ with $b_i \leqslant c_i$ for all $1 \leqslant i \leqslant k$, we define a convex rational polyhedral set

$$Q = \{x \in \mathfrak{t}^* \mid b_i \leqslant \langle x, v_i \rangle \leqslant c_i, \ \forall\, 1 \leqslant i \leqslant k\},$$

and define the *symplectic cut of M with respect to Q* to be the reduction of M_P (as given in the sentence below (13)) at c. We note that if M is in addition compact, then both M_P and M_Q are compact.

Concluding the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into three steps. Suppose that an $(n-2)$-dimensional torus T^{n-2} acts on a $2n$-dimensional compact connected symplectic manifold (M, ω) with fixed points, and that there is a commuting T^2 symplectic action on M which has a symplectic orbit.
Step 1. By Theorem 4.3 and Theorem 4.5, we may assume that there is a generalized momentum map $\Phi : M \to T^{n-2}$. To prove that the action of T^{n-2} is Hamiltonian, it suffices to prove that for any splitting of T^{n-2}

of the form $T^{n-2} = S^1 \times H$, where H is a $(n-3)$-dimensional torus, the action of S^1 is Hamiltonian. Let $\pi_1 : T^{n-2} \to S^1$ and $\pi_2 : T^{n-2} \to H$ be the projection maps. Then $\phi := \pi_1 \circ \Phi : M \to S^1$ and $\phi_H := \pi_2 \circ \Phi : M \to H$ are generalized momentum maps with respect to the S^1-action and the H-action on M respectively. Without loss of generality, we may assume that the generalized momentum map ϕ has no local extreme value, cf. [21, Lemma 2]. Let $h \in H$ be a regular value of ϕ_H. Denote by $\mathrm{DH}_{T^{n-2}} : T^{n-2} \to \mathbb{R}$ the Duistermaat-Heckman function of the T^{n-2} action on M. Define

$$f : S^1 \to \mathbb{R}, \quad f(x) := \mathrm{DH}_{T^{n-2}}(x, h).$$

To prove that the action of S^1 is Hamiltonian, it suffices to show that f is strictly log-concave. This is because that if f is strictly log-concave, then the generalized momentum map $\phi : M \to S^1$ cannot be surjective, and hence the action of S^1 on M is Hamiltonian.

We will divide the rest of the proof into two steps, and we will use the following notation. Consider the exponential map $\exp : \mathbb{R} \to S^1$, $t \mapsto \mathrm{e}^{\mathrm{i}2\pi t}$. By abuse of notation, for any open connected interval $I = (b, c) \subset \mathbb{R}$ such that $\exp(I) \nsubseteq S^1$, we will identify I with its image in S^1 under the exponential map.

Step 2. We first show that if $I_1 = (b_1, c_1) \nsubseteq S^1$ is an open connected set consisting of regular values of ϕ, then f is log-concave on I_1. Indeed, suppose that for any $2 \leqslant i \leqslant n-2$, $I_i = (b_i, c_i) \nsubseteq S^1$ is an open connected set such that $\mathcal{O} = I_2 \times \cdots \times I_{n-2} \ni h$ consists of regular values of ϕ_H. Then $I_1 \times \mathcal{O} \nsubseteq T^{n-2}$ is a connected open set that consists of regular values of $\Phi : M \to T^{n-2}$. Moreover, the action of T^{n-2} on $W = \Phi^{-1}(I \times \mathcal{O})$ is Hamiltonian. Set $Q = \{(x_1, \cdots, x_{n-2}) \in \mathbb{R}^{n-2} \mid b_i \leqslant x_i \leqslant c_i, \ \forall\, 1 \leqslant i \leqslant n-2\}$. The symplectic cutting W_Q of W with respect to Q is a compact Hamiltonian manifold from W, and, on the interior of Q, the Duistermaat-Heckman function of W_Q agrees with that of W. Consequently, the log-concavity of the function f follows from Theorem 1.2.

Step 3. We show that if $I_1 = (b_1, c_1) \nsubseteq S^1$ is a connected open set which contains a unique critical value a of ϕ, then f is strictly log-concave on I_1. We will use the same notation W and W_Q as in the Step 2. By definition, to show that f is strictly log-concave on I_1, it suffices to show that $\tilde{f} = f \circ \exp : \mathbb{R} \to \mathbb{R}$ is strictly log-concave on the pre-image of I_1 in $\mathbb{R}$ under the exponential map exp. Again, by abuse of notation, we would not distinguish I_1 and its pre-image in $\mathbb{R}$, and we will use the same notation a to denote the pre-image of $a \in I_1 \nsubseteq S^1$ in $\exp^{-1}(I_1)$. To see the existence of such interval I_1, we need the assumption that the fixed point set of T^{n-2} is non-empty. Since the fixed point submanifold of the action of T^{n-2} on M is a subset of M^{S^1}, it follows that M^{S^1} is non-empty. Let X be a connected component of M^{S^1}. Choose $a \in S^1$ to be the image of X under the generalized momentum map ϕ. Then a sufficiently small connected open subset that contains a will contain no other critical values of ϕ.

Note that X is a symplectic submanifold of M which is invariant under the action of H, and that there are two symplectic manifolds associated to the point $(a,h) \in T^k = S^1 \times H$: the symplectic quotient of M (viewed as an H-space) at h, and the symplectic quotient of X (viewed as T^k/S^1-space) at (a,h). We will denote these two quotient spaces by M_h and X_h respectively. An immediate calculation shows that the dimension of M_h is six. Since the action of S^1 commutes with that of H, by Lemma 3.1 and Remark 3.2, the action of S^1 preserves the level set of the generalized momentum map ϕ_H. Consequently, there is an induced action of S^1 on M_h. Since $\phi : M \to S^1$ has a constant value on any H orbit in M, it descends to a generalized momentum map $\tilde{\phi} : M_h \to S^1$. It is easy to see that X_h consists of the critical points of the generalized momentum map $\tilde{\phi} : M_h \to S^1$. Moreover, since ϕ has no local extreme values in M, it follows that $\tilde{\phi}$ has no local extreme value in M_h. Thus for dimensional reasons, X_h can only be of codimension four or six. Now observe that the commuting symplectic action of T^2 on M descends to a symplectic action on M_h which commutes with the action of S^1 on M^{S^1}. It follows that X_h is invariant under the induced action of T^2 on M_h and so must contain a T^2 orbit. As a result, X_h must have codimension four.

Applying Theorem 3.6 to the compact Hamiltonian manifold W_Q, we get that

$$\tilde{f}_+(a+t) - \tilde{f}_-(a-t) \;=\; \sum_{X_h} \frac{\mathrm{vol}\,(X_h)}{(d-1)!\prod_j \alpha_j}(t-a)^{d-1} + \mathcal{O}((t-a)^d).$$

In the above equation, X_h runs over the collection of all connected components of M^{S^1} that sits inside $\Phi^{-1}((a,h))$, the α_j's are the weights of the representation of S^1 on the normal bundle of X_h, and d is half of the real dimension of X_h. By our previous work, $d = 2$ for any X_h; moreover, the two non-zero weights of X_h must have opposite signs. So the jump in the derivative is strictly negative, i.e., $\tilde{f}'_+(a) - \tilde{f}'_-(a) < 0$. □

5. Examples and consequences of main results

Applications of Theorems 1.1 and 1.2. Our proof of Theorem 1.1 depends on Lemma 3.4. We note that the proof of Lemma 3.4 builds on the classification result of symplectic T^2 action on symplectic four manifolds, cf. [25] and [7]. However, we have a simpler proof in the following special case, which was the goal of the article [16] (see Section 1 therein).

Theorem 5.1 (Kim [16])**.** *Let $(M,\,\omega)$ be a compact, connected, symplectic $2n$-dimensional manifold. Then every effective symplectic action of an $(n-1)$-dimensional torus T^{n-1} on $(M,\,\omega)$ with non-empty fixed point set is Hamiltonian.*

Proof. We prove the equivalent statement that if $(N,\,\omega)$ is $(2n{-}2)$-dimensional compact, connected symplectic manifold, then every symplectic action of an

$(n-2)$-dimensional torus T^{n-2} on (N, ω) with non-empty fixed point set is Hamiltonian. Indeed, suppose that there is a symplectic action of an $(n-2)$-dimensional torus on a $(2n-2)$-dimensional compact connected symplectic manifold N. By Theorem 4.3 and Theorem 4.5, without loss of generality we assume that there is a generalized momentum map $\phi_N : N \to T^{n-2}$. Consider the action of T^{n-2} on the product symplectic manifold $Z := N \times T^2$ as given in equation (9), where T^2 is equipped with the standard symplectic structure. Define $\Phi : N \times T^2 \to T^{n-2}$, $(x, t) \mapsto \phi_N(x)$ for all $x \in N$, $t \in T^2$. Then Φ is a generalized momentum map for the symplectic action of T^{n-2} on Z. Moreover, there is an effective commuting symplectic action of T^2 on M with symplectic orbits given as in equation (10). To show that the action of T^{n-2} on N is Hamiltonian, it suffices to show that the action of T^{n-2} on Z is Hamiltonian. Using the same notation as in the proof of Theorem 1.1, we explain that we can modify the argument there to prove that the action of T^{n-2} on Z is Hamiltonian without using Lemma 3.4. We first note that Step 3 in the proof of Theorem 1.1 does not use Lemma 3.4, and that the same argument there applies to the present situation, which is a special case of what is considered in Theorem 1.1. Lemma 3.4 is used in Step 2 in the proof of Theorem 1.1 to prove that f is log-concave on I_1. However, in the present situation, we can use Theorem 3.9 instead to establish the log-concavity of the function f on I_1. Since the proof of Theorem 3.9 does not use Lemma 3.4, Theorem 5.1 can be established without using Lemma 3.4 as well. $\square$

The following result [21, Proposition 2] is a special case of Theorem 5.1.

Corollary 5.2 (McDuff [21])**.** *If (M, ω) is 4-dimensional compact, connected symplectic manifold, then every effective symplectic S^1-action on (M, ω) with non-empty fixed point set is Hamiltonian.*

Using [29] we obtain the following result on measures. A measure defined on the measurable subsets of $\mathbb{R}^n$ is *log-concave* if for every pair A, B of convex subsets of $\mathbb{R}^n$ and for every $0 < t < 1$ such that $tA + (1-t)B$ is measurable we have $P(tA + (1-t)B) \geqslant (P(A))^t (P(B))^{1-t}$ where the sign $+$ denotes Minkowski addition of sets.

Theorem 5.3 (Prékopa [29])**.** *If the density function of a measure P defined on the measurable subsets of $\mathbb{R}^n$ is almost everywhere positive and is a log-concave function, then the measure P itself is log-concave.*

These notions extend immediately to our setting where $\mathbb{R}^n$ is replaced by the dual Lie algebra $\mathfrak{t}^*$, equipped with the push-forward of the Liouville measure on M, and we have, as a consequence of Theorem 1.2, the following.

Theorem 5.4. *Let T be an $(n-2)$-torus which acts effectively on a compact connected symplectic $2n$-manifold (M, ω) in a Hamiltonian fashion. Suppose that there is a commuting symplectic action of a 2-torus S on M whose orbits are symplectic. Then the Duistermaat-Heckman measure is log-concave.*

Examples. Next we give some more examples to which our theorems apply.

Example 5.5. The Kodaira-Thurston manifold is the symplectic manifold $\mathrm{KT} := (\mathbb{R}^2 \times T^2)/\mathbb{Z}^2$ (see [28, Section 2.4]). This is the case in Theorem 1.1 which is "trivial" since $n-2=0$, and hence $T^{n-2} = \{e\}$ (in fact this manifold is non-Kahler, and as such it does not admit any Hamiltonian torus action, of any non-trivial dimension). Nevertheless the KT example serves to illustrate a straightforward case which admits the symplectic transversal symmetry, and it may be used to construct lots of examples satisfying the assumptions of the theorem, eg. $\mathrm{KT} \times S^2$.

Example 5.6. This example is a generalization of Example 5.5. Theorem 1.1 covers an infinite class of symplectic manifolds with Hamiltonian S^1-actions, for instance

$$M := (\mathbb{CP}^1)^2 \times \mathbb{T}^2/\mathbb{Z}_2, \tag{14}$$

where S^1 acts Hamiltonianly on the left factor $(\mathbb{CP}^1)^2$, and $\mathbb{T}^2$ acts symplectically with symplectic orbits on the right factor; the quotient is taken with respect to the natural diagonal action of $\mathbb{Z}_2$ (by the antipodal action on a circle of $\mathbb{T}^2$, and by rotation by 180 degrees about the vertical axes of the sphere $S^2 \simeq \mathbb{CP}^1$), and the action on $(\mathbb{CP}^1)^2 \times \mathbb{T}^2$ descends to an action on M. Any product symplectic form upstairs also descends to the quotient.

Example 5.7. The $2n$-dimensional symplectic manifolds with $\mathbb{T}^n$-actions in Theorem 1.1 are a subclass of those classified in [25], which, given that $\mathbb{T}^2$ is 2-dimensional, are of the form we describe next. Let Z be any good $(2n-2)$--dimensional orbifold, and let $\widetilde{\Sigma}$ be its orbifold universal cover. Equip Z with any orbifold symplectic form, and $\widetilde{\Sigma}$ with the induced symplectic form. Equip $\mathbb{T}^2$ with any area form. Equip $\widetilde{Z} \times \mathbb{T}^2$ with the product symplectic form, and the symplectic $\mathbb{T}^2$-action with symplectic orbits by translations on the right most factor. This symplectic $\mathbb{T}^2$-action descends to a symplectic $\mathbb{T}^2$-action with symplectic orbits on

$$M_Z := \widetilde{Z} \times_{\pi_1^{\mathrm{orb}}(Z)} \mathbb{T}^2 \tag{15}$$

considered with the induced product symplectic form. An a priori intractable question is: *give an explicit list of manifolds M_Z which admit a Hamiltonian S^1-action.* Examples of such M_Z, and hence fitting in the statement of Theorem 1.1, may be constructed in all dimensions, eg. the manifold in (14). Theorem 1.1 says that any commuting symplectic $\mathbb{T}^{n-2}$ action on (15) *which has fixed points*, must be Hamiltonian.

6. Appendix: actions with symplectic orbits

This section is a review of [25, Sections 2, 3], with a new remark at the end. Let (X, σ) be a connected symplectic manifold equipped with an effective symplectic action of a torus T for which there is at least one T-orbit which is a $\dim T$-dimensional symplectic submanifold of (X, σ) (this implies that all are). *We do not assume that X is necessarily compact.* Then there exists a unique non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ on the Lie algebra $\mathfrak{t}$ of T such that $\sigma_x(u_X(x), v_X(x)) = \sigma^{\mathfrak{t}}(u, v)$, for every $u, v \in \mathfrak{t}$, and every $x \in X$. One can check that the stabilizer subgroup T_x of the T-action at every point $x \in X$ is a finite group.

Orbit space X/T. As usual, X/T denotes the orbit space of the T-action. Let $\pi \colon X \to X/T$ the canonical projection. The space X/T is provided with the maximal topology for which π is continuous; this topology is Hausdorff. Because X is connected, X/T is connected. Let $k := \dim X - \dim T$. By the tube theorem of Koszul (see eg. [13, Theorem B24]), for each $x \in X$ there exists a T-invariant open neighborhood U_x of the T-orbit $T \cdot x$ and a T-equivariant diffeomorphism Φ_x from U_x onto the associated bundle $T \times_{T_x} D_x$, where D_x is an open disk centered at the origin in $\mathbb{R}^k \cong \mathbb{C}^{k/2}$ and T_x acts by linear transformations on D_x. The action of T on $T \times_{T_x} D_x$ is induced by the action of T by translations on the left factor of $T \times D_x$. Because Φ_x is a T-equivariant diffeomorphism, it induces a homeomorphism $\widehat{\Phi}_x$ on the quotient $\widehat{\Phi}_x \colon D_x/T_x \to \pi(U_x)$, and there is a commutative diagram

$$\begin{tikzcd}
T \times D_x \arrow[r, "\pi_x"] & T \times_{T_x} D_x \arrow[r, "\Phi_x"] \arrow[d, "p_x"] & U_x \arrow[d, "\pi|_{U_x}"] \\
D_x \arrow[u, "i_x"] \arrow[r, "\pi'_x"] & D_x/T_x \arrow[r, "\widehat{\Phi}_x"] & \pi(U_x)
\end{tikzcd}$$

where π_x, π'_x, p_x are the canonical projection maps, and $i_x(d) := (\mathrm{e}, d)$ is the inclusion map. Let $\phi_x := \widehat{\Phi}_x \circ \pi'_x$. The collection of charts $\widehat{\mathcal{A}} := \{(\pi(U_x), D_x, \phi_x, T_x)\}_{x \in X}$ is an orbifold atlas for X/T. We call $\mathcal{A}$ the class of atlases equivalent to the orbifold atlas $\widehat{\mathcal{A}}$. We denote the orbifold X/T endowed with the class $\mathcal{A}$ by X/T, and the class $\mathcal{A}$ is assumed.

Flat connection. The collection $\Omega = \{\Omega_x\}_{x \in X}$ of subspaces $\Omega_x \subset \mathrm{T}_x X$, where Ω_x is the σ_x-orthogonal complement to $\mathrm{T}_x(T \cdot x)$ in $\mathrm{T}_x X$, for every $x \in X$, is a smooth distribution on X. The projection mapping $\pi \colon X \to X/T$ is a smooth principal T-orbibundle for which Ω is a T-invariant flat connection. Let $\mathcal{I}_x$ be the maximal integral manifold of the distribution Ω. The inclusion $i_x \colon \mathcal{I}_x \to X$ is an injective immersion between smooth manifolds and the composite $\pi \circ i_x \colon \mathcal{I}_x \to X/T$ is an orbifold covering map. Moreover, there exists a unique 2-form ν on X/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in X$. The form ν is symplectic, and so the pair $(X/T, \nu)$ is a connected symplectic orbifold.

Model for X/T. We define the space that we call the *T-equivariant symplectic model $(X_{\mathrm{model}, p_0}, \sigma_{\mathrm{model}})$ of (X, σ) based at a regular point $p_0 \in X/T$* as follows.

i) The space $X_{\mathrm{model},\,p_0}$ is $X_{\mathrm{model},\,p_0} := \widetilde{X/T} \times_{\pi_1^{\mathrm{orb}}(X/T,\,p_0)} T$, where the space $\widetilde{X/T}$ denotes the orbifold universal cover of the orbifold X/T based at a regular point $p_0 \in X/T$, and the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/T,\,p_0)$ acts on the Cartesian product $\widetilde{X/T} \times T$ by the diagonal action $x\,(y,\,t) = (x \star y^{-1},\, \mu(x) \cdot t)$, where $\star \colon \pi_1^{\mathrm{orb}}(X/T,\,p_0) \times \widetilde{X/T} \to \widetilde{X/T}$ denotes the natural action of $\pi_1^{\mathrm{orb}}(X/T,\,p_0)$ on $\widetilde{X/T}$, and $\mu \colon \pi_1^{\mathrm{orb}}(X/T,\,p_0) \to T$ denotes the monodromy homomorphism of Ω.

ii) The symplectic form σ_{model} is induced on the quotient by the product symplectic form on the Cartesian product $\widetilde{X/T} \times T$. The symplectic form on $\widetilde{X/T}$ is defined as the pullback by the orbifold universal covering map $\widetilde{X/T} \to X/T$ of ν. The symplectic form on the torus T is the unique T-invariant symplectic form σ^T determined by $\sigma^{\mathfrak{t}}$.

iii) The action of T on the space $X_{\mathrm{model},\,p_0}$ is the action of T by translations which descends from the action of T by translations on the right factor of the product $\widetilde{X/T} \times T$.

In this definition we are implicitly using that $\widetilde{X/T}$ is a smooth manifold and X/T is a good orbifold, which is proven by analogy with [25].

Model of (X, σ) with T-action. In [25], the following theorem was shown (see Theorem 3.4.3).

Theorem 6.1 (Pelayo [25])**.** *Let $(X,\,\sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T, for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(X,\,\sigma)$. Then $(X,\,\sigma)$ is T-equivariantly symplectomorphic to its T-equivariant symplectic model based at any regular point $p_0 \in X/T$.*

The purpose of this appendix was to point out that Theorem 6.1 with the word "compact" removed from the statement holds. We refer to [25] for a proof in the case that X is compact, which works verbatim in the case that X is not compact.

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