Moduli spaces of toric manifolds

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Abstract

We construct a distance on the moduli space of symplectic toric manifolds of dimension four. Then we study some basic topological properties of this space, in particular, connectedness, compactness and completeness. The construction of the distance is related to the Duistermaat-Heckman measure and the Hausdorff metric. While the moduli space, its topology and metric, may be constructed in any dimension, the tools we use in the proofs are four-dimensional, and hence so is our main result.

1 Introduction

A *toric integrable system*

$$\mu := (\mu_1, \ldots, \mu_n) \colon M \to \mathbb{R}^n$$

is an integrable system on a connected symplectic $2n$-dimensional manifold (M, ω) in which all the flows generated by the μ_i, $i = 1, \ldots, n$, are periodic of a fixed period. That is, there is Hamiltonian action of an n-dimensional torus $\mathbb{T}$ on M with momentum map μ. We will assume that this action is effective and that M is compact. In this case, the quadruple $(M, \omega, \mathbb{T}, \mu)$ is often called a *symplectic toric manifold* of dimension $2n$, to emphasize the connection with toric varieties (in fact, all symplectic toric manifolds are toric varieties, eg. see Remark 9 and [8, 10]).

The goal of the paper is to construct natural topologies on moduli spaces of compact symplectic toric 4-manifolds under natural equivalence relations and study some of their basic topological properties.

Throughout most of this paper we will assume that $n = 2$, but several definitions hold in more generality.

1.1 Conventions

Let, throughout this paper, $\mathbb{T} := \mathbb{T}^n$ denote the n-dimensional standard torus $(\mathbb{T}^1)^n$, that is, the Cartesian product of n copies of the circle $(\mathbb{T}^1, +)$ equipped with the product operation. Denote by $\mathfrak{t}$ the Lie algebra $\mathrm{Lie}(\mathbb{T})$ of $\mathbb{T}$ and by $\mathfrak{t}^*$ the dual Lie algebra of $\mathfrak{t}$. Strictly speaking, the momentum map of the Hamiltonian action of $\mathbb{T}$ on a manifold N, is a map $N \to \mathfrak{t}^*$. However, the presentation is simpler (and the notation in the proofs), if from the beginning we consider this map as a map $N \to \mathbb{R}^n$. How to do this is a standard, but not canonical, procedure. Choose an epimorphism $E \colon \mathbb{R} \to \mathbb{T}^1$ which for instance we take to be $x \mapsto e^{2\sqrt{-1}x}$. This epimorphism induces an isomorphism between $\mathrm{Lie}(\mathbb{T}^1)$ and $\mathbb{R}$, $\frac{\partial}{\partial x} \mapsto 1/2$, giving rise to a new isomorphism $\mathfrak{t} \to \mathbb{R}^n$, $\frac{\partial}{\partial x_k} \mapsto 1/2\, e_k$, by canonically identifying $\mathfrak{t}$ with the product of n copies of $\mathrm{Lie}(\mathbb{T}^1)$. By choosing any bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}$, we obtain an isomorphism $\mathfrak{t}^* \to \mathfrak{t}$, and hence by composing with the isomorphism $\mathfrak{t} \to \mathbb{R}^n$, an isomorphism $\mathfrak{t}^* \to \mathbb{R}^n$. With this convention, we can define the space $\mathcal{M}_{\mathbb{T}}$. If we choose a different identification $\mathfrak{t}^* \to \mathbb{R}^n$, then the resulting moduli space is a different set that $\mathcal{M}_{\mathbb{T}}$. However, these spaces are in bijective correspondence by a map which preserves all the structures that this paper deals with (Section 1.4). An alternative (equivalent approach) to this convention suggested by the referee is to define $\mathcal{M}_{\mathbb{T}}$ to be a space of pairs, where the second element of the pair concerns the Lie algebra identification. We use this convention throughout the paper (see Section 2.2).

1.2 The moduli space $\mathcal{M}_{\mathbb{T}}$

Let $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ be symplectic toric manifolds, with effective symplectic actions $\rho\colon \mathbb{T} \to \mathrm{Sympl}(M, \omega)$ and $\rho'\colon \mathbb{T} \to \mathrm{Sympl}(M', \omega')$. Two symplectic toric manifolds are *isomorphic* if there exists an equivariant symplectomorphism $\varphi\colon M \to M'$ such that

$$\mu' \circ \varphi = \mu.$$

We denote by $\mathcal{M}_{\mathbb{T}}$ the *moduli space of* $2n$*-dimensional symplectic toric manifolds under this equivalence relation.*

The convexity theorem of Atiyah [1] and Guillemin-Sternberg [13] asserts that the image of the momentum map is a convex polytope. In addition, if the action is toric (the acting torus is precisely half the dimension of the manifold) the momentum image is a Delzant polytope (see Section 2). As explained in Section 2.2, Delzant's classification theorem [8] gives a bijection

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{T}} & \longrightarrow & \mathcal{D}_{\mathbb{T}} \\ [(M, \omega, \mathbb{T}, \mu)] & \mapsto & \mu(M). \end{array} \tag{1}$$

For simplicity we usually write $(M, \omega, \mathbb{T}, \mu)$ instead of $[(M, \omega, \mathbb{T}, \mu)]$.

1.3 The moduli space $\widetilde{\mathcal{M}_{\mathbb{T}}}$

Following [18], we say that two symplectic toric manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are *equivariantly isomorphic* if there exists an automorphism of the torus $h\colon \mathbb{T} \to \mathbb{T}$ and an h-equivariant symplectomorphism $\varphi\colon M \to M'$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} \times M & \xrightarrow{\rho*} & M \\ \Big\downarrow{\scriptstyle (h,\varphi)} & & \Big\downarrow{\scriptstyle \varphi} \\ \mathbb{T} \times M' & \xrightarrow{\rho'*} & M'. \end{array} \tag{2}$$

We denote by $\widetilde{\mathcal{M}_{\mathbb{T}}}$ the *moduli space of equivariantly isomorphic* $2n$*-dimensional symplectic toric manifolds* (Section 2.2 provides more details).

Two equivariantly isomorphic toric manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are isomorphic if and only if h in (2) is the identity and $\mu' = \mu \circ \varphi$.

1.4 Topologies and metrics

We consider the space of Delzant polytopes $\mathcal{D}_{\mathbb{T}}$ and make it a metric space by endowing it with the distance function given by the volume of the symmetric difference

$$(\Delta_1 \smallsetminus \Delta_2) \cup (\Delta_2 \smallsetminus \Delta_1)$$

of any two polytopes.

The map (1) allows us to define a metric $d_{\mathbb{T}}$ on $\mathcal{M}_{\mathbb{T}}$ as the pullback of the metric defined on $\mathcal{D}_{\mathbb{T}}$, thereby getting the metric space $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$. *This metric induces a topology ν on $\mathcal{M}_{\mathbb{T}}$* (and by definition it follows that $(\mathcal{M}_{\mathbb{T}}, \nu)$ is a metrizable topological space).

Let $\mathrm{AGL}(n,\mathbb{Z}) = \mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n$ be the group of affine transformations of $\mathbb{R}^n$ given by

$$x \mapsto Ax + c,$$

where $A \in \mathrm{GL}(n,\mathbb{Z})$ and $c \in \mathbb{R}^n$. We say that two Delzant polytopes Δ_1 and Δ_2 are $\mathrm{AGL}(n,\mathbb{Z})$--*equivalent* if there exists $\alpha \in \mathrm{AGL}(n,\mathbb{Z})$ such that $\alpha(\Delta_1) = \Delta_2$. Let $\widetilde{\mathcal{D}_{\mathbb{T}}}$ be the moduli space of Delzant polytopes modulo the equivalence relation given by $\mathrm{AGL}(n,\mathbb{Z})$; we endow this space with the quotient topology induced by the projection map

$$\pi\colon \mathcal{D}_{\mathbb{T}} \to \widetilde{\mathcal{D}_{\mathbb{T}}} \simeq \mathcal{D}_{\mathbb{T}} / \mathrm{AGL}(n,\mathbb{Z})\ .$$

As we will see in Section 2.2, there exists a bijection Ψ between $\widetilde{\mathcal{M}_{\mathbb{T}}}$ and $\widetilde{\mathcal{D}_{\mathbb{T}}}$ (in fact, it is induced by (1)); thus $\widetilde{\mathcal{M}_{\mathbb{T}}}$ is also a topological space, with topology $\widetilde{\nu}$ induced by Ψ. *We refer to this topological space as the pair $(\widetilde{\mathcal{M}_{\mathbb{T}}}, \widetilde{\nu})$.*

1.5 Main Theorem

Let $\mathfrak{B}(\mathbb{R}^n)$ be the σ-algebra of Borel sets of $\mathbb{R}^n$, and let $\lambda\colon \mathfrak{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be the Lebesgue measure on $\mathbb{R}^n$. Let $\mathfrak{B}'(\mathbb{R}^n) \subset \mathfrak{B}(\mathbb{R}^n)$ be the Borel sets with finite Lebesgue measure, and define

$$d(A,B) := \|\chi_A - \chi_B\|_{\mathrm{L}^1}, \tag{3}$$

where $\chi_C\colon \mathbb{R}^n \to \mathbb{R}$ denotes the characteristic function of $C \in \mathfrak{B}'(\mathbb{R}^n)$. This extends the distance function defined above on $\mathcal{D}_{\mathbb{T}}$, but it is *not* a metric on $\mathfrak{B}'(\mathbb{R}^n)$. Identifying the sets A, B in $\mathfrak{B}'(\mathbb{R}^n)$ for which $d(A,B) = 0$, we obtain a metric in the resulting quotient space of $\mathfrak{B}'(\mathbb{R}^n)$ (see Section 2.1 for details).

For every $A \in \mathfrak{B}'(\mathbb{R}^2)$, we denote by $[A]$ the equivalence class of A with respect to this identification. Let $\mathcal{C}$ be the space of convex compact subsets of $\mathbb{R}^2$ with positive Lebesgue measure, $\varnothing$ be the class of zero Lebesgue measure elements, and

$$\hat{\mathcal{C}} := \{[A] \mid A \in \mathcal{C}\} \cup \varnothing.$$

The space

$$\hat{\mathcal{C}} \subset \mathfrak{B}'(\mathbb{R}^2)/\sim$$

equipped with the metric d as in (3) is a metric space.

We prove the following theorem.

Theorem 1. *Let $\mathcal{M}_{\mathbb{T}}$ and $\widetilde{\mathcal{M}_{\mathbb{T}}}$ be the moduli spaces of toric four-dimensional manifolds, under isomorphisms and equivariant isomorphisms, respectively. Then:*

(a) $(\widetilde{\mathcal{M}_{\mathbb{T}}}, \widetilde{\nu})$ *is connected;*

(b) $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$ *is a non-locally compact, non-complete metric space. Its completion can be identified with the metric space $(\hat{\mathcal{C}}, d)$ in the following sense: we identify $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$ with $(\mathcal{D}_{\mathbb{T}}, d)$ via (1), and the completion of $(\mathcal{D}_{\mathbb{T}}, d)$ is $(\hat{\mathcal{C}}, d)$.*

Remark 2. Metric spaces are Tychonoff (that is, completely regular and Hausdorff), therefore $\mathcal{M}_{\mathbb{T}}$ is Tychonoff. The Stone-Čech compactification [29, 5] holds for Tychonoff spaces. The Stone-Čech compactification in general gives rise to a compactified space which is Hausforff and normal. Hence $\mathcal{M}_{\mathbb{T}}$ admits a Hausdorff compactification. ⊘

Remark 3. Theorem 1 answers the case $2n = 4$ of Problem 2.42 in [26]. We do not know if the analogue statement to Theorem 1 holds in dimensions greater than or equal to six. Note that the constructions of the moduli spaces $\mathcal{M}_{\mathbb{T}}$ and $\widetilde{\mathcal{M}_{\mathbb{T}}}$ do not depend on dimension. ⊘

Structure of the paper

In Section 2 we introduce the topological spaces we are going to work with, involving Delzant polytopes and symplectic toric manifolds, under certain equivalence relations. The ingredient that allows us to relate these two categories of spaces is the Delzant classification theorem (Theorem 7).

Section 3 starts with a detailed analysis of how to construct Delzant polygons (i.e. polytopes of dimension 2) following a simple recursive recipe presented in [18]. This recipe is a main technical tool for the present paper, and no such method is known for polytopes of dimension greater than or equal to three. The remainder of this section and Section 4 are devoted to proving the connectedness and metric properties of the space of Delzant polygons (or rather, a natural quotient of it); the main theorem of the paper is implied by the results proven in these sections.

Finally, in Section 5, several open problems are presented. The appendix contains a brief review of the polytope terms we use in the paper.

2 Delzant polytopes and toric manifolds

2.1 A metric on the space of Delzant polytopes $\mathcal{D}_{\mathbb{T}}$

In this paper we are interested only in convex full dimensional polytopes, which we will simply call polytopes. We refer to Section 6 for the basic definitions on polytopes.

Definition 4. (following [4]) A convex polytope Δ in $\mathbb{R}^n$ is a *Delzant polytope* if it is simple, rational and smooth:

(i) Δ is *simple* if there are exactly n edges meeting at each vertex $v \in V$;

(ii) Δ is *rational* if for every vertex $v \in V$, the edges meeting at v are of the form $v + tu_i$, $t \geq 0$, and $u_i \in \mathbb{Z}^n$;

(iii) A vertex $v \in V$ is *smooth* if the edges meeting at v are of the form $v + tu_i$, $t \geq 0$, where the vectors $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$ basis of $\mathbb{Z}^n$. Δ is *smooth* if every vertex $v \in V$ is smooth.

Let $\mathcal{D}_{\mathbb{T}}$ denote the *space of Delzant polytopes in* $\mathbb{R}^n$, where $n = \dim \mathbb{T}$. We construct a topology on $\mathcal{D}_{\mathbb{T}}$, coming from a metric.

Recall that the *symmetric difference* of two subsets A and B of $\mathbb{R}^n$ is

$$A\Delta B := (A \smallsetminus B) \cup (B \smallsetminus A).$$

Let $\mathfrak{B}(\mathbb{R}^n)$ be the σ-algebra of Borel sets of $\mathbb{R}^n$, and let

$$\lambda \colon \mathfrak{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

be the Lebesgue measure on $\mathbb{R}^n$.

Definition 5. Let $\mathfrak{B}'(\mathbb{R}^n) \subset \mathfrak{B}(\mathbb{R}^n)$ be the Borel sets with finite Lebesgue measure, and define

$$d \colon \mathfrak{B}'(\mathbb{R}^n) \times \mathfrak{B}'(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$$

to be

$$d(A, B) := \lambda(A\Delta B) = \int_{\mathbb{R}^n} |\chi_A - \chi_B| \, \mathrm{d}\lambda = \|\chi_A - \chi_B\|_{\mathrm{L}^1}, \tag{4}$$

where $\chi_C \colon \mathbb{R}^n \to \mathbb{R}$ denotes the characteristic function of $C \in \mathfrak{B}'(\mathbb{R}^n)$.

Observe that d is symmetric and satisfies the triangle inequality, since

$$A\Delta C \subset A\Delta B \cup B\Delta C.$$

However, in this space, $d(A, B) = 0$ does not necessarily imply that $A = B$. We introduce in $\mathfrak{B}'(\mathbb{R}^n)$ the equivalence relation $\sim$, where

$$A \sim B \quad \text{if and only if} \quad \lambda(A\Delta B) = 0\,,$$

and now clearly have that the induced $d \colon (\mathfrak{B}'(\mathbb{R}^n)/\sim) \times (\mathfrak{B}'(\mathbb{R}^n)/\sim) \to \mathbb{R}_{\geq 0}$ is a metric (associated to the L^1 norm).

Observe that if $A, B \in \mathcal{D}_{\mathbb{T}} \subset \mathfrak{B}'(\mathbb{R}^n)$ and $A \sim B$ then $A = B$. Therefore

$$(\mathcal{D}_{\mathbb{T}}/\sim) = \mathcal{D}_{\mathbb{T}}$$

and the restriction of d to $\mathcal{D}_{\mathbb{T}}$ is a metric. Hence $(\mathcal{D}_{\mathbb{T}}, d)$ is a metric space, endowed with the topology induced by d.

2.2 Symplectic toric manifolds

We review the ingredients from the theory of symplectic toric manifolds which we need for this paper, in particular the Delzant classification theorem. We follow the conventions in Section 1.1.

A *symplectic manifold* (M, ω) is a pair consisting of a smooth manifold M and a *symplectic form* ω, i.e. a non-degenerate closed 2-form on M. Suppose that the n-dimensional torus $\mathbb{T}$ acts on $(M,\, \omega)$ symplectically (i.e. by diffeomorphisms which preserve the symplectic form). The action

$$\mathbb{T} \times M \to M$$

of $\mathbb{T}$ on M is denoted by $(t, m) \mapsto t \cdot m$.

A vector X of the Lie algebra $\mathfrak{t}$ generates a smooth vector field X_M on M, called the *infinitesimal generator*, given by

$$X_M(m) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tX) \cdot m,$$

where $\exp\colon \mathfrak{t} \to \mathbb{T}$ is the exponential map of Lie theory and $m \in M$. We write $\iota_{X_M}\omega := \omega(X_M, \cdot) \in \Omega^1(M)$ for the contraction 1-form.

Let $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$ be the duality pairing. The $\mathbb{T}$-action on (M, ω) is said to be *Hamiltonian* if there exists a smooth invariant map $\mu\colon M \to \mathfrak{t}^*$, called the *momentum map*, such that for all $X \in \mathfrak{t}$ we have that

$$\iota_{X_M}\omega = \mathrm{d}\langle \mu, X \rangle. \tag{5}$$

As defined in Section 1, a *symplectic toric manifold* $(M, \omega, \mathbb{T}, \mu)$ is a symplectic compact connected manifold (M, ω) of dimension $2n$ endowed with an effective Hamiltonian action of an n-dimensional torus $\mathbb{T}$ and with momentum map $\mu : M \to \mathfrak{t}^*$. With the conventions of Section 1.1, the map $\mu : M \to \mathfrak{t}^*$ gives rise (in a non-canonical way) to a map $M \to \mathfrak{t}^* \to \mathbb{R}^n$, which for simplicity we also denote by $\mu\colon M \to \mathbb{R}^n$.

Definition 6. Let $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ be symplectic toric manifolds, with effective symplectic actions $\rho\colon \mathbb{T} \to \mathrm{Sympl}(M, \omega)$ and $\rho'\colon \mathbb{T} \to \mathrm{Sympl}(M', \omega')$. We say that $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are *isomorphic*[1] if there exists an equivariant symplectomorphism $\varphi\colon M \to M'$ such that

$$\mu' \circ \varphi = \mu.$$

We denote by $\mathcal{M}_{\mathbb{T}}$ the moduli space of $2n$-dimensional isomorphic toric manifolds.

The following is an influential theorem by T. Delzant ([8]).

Theorem 7 (Delzant's Theorem)**.** *There is a one-to-one correspondence between isomorphism classes of symplectic toric manifolds and Delzant polytopes, given by:*

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{T}} & \longrightarrow & \mathcal{D}_{\mathbb{T}} \\ [(M, \omega, \mathbb{T}, \mu)] & \mapsto & \mu(M). \end{array} \tag{6}$$

As a consequence of this bijection, we can endow $\mathcal{M}_{\mathbb{T}}$ with the pullback metric. Namely, let $\mathbf{M}_1 = (M_1, \omega_1, \mathbb{T}, \mu_1)$ and $\mathbf{M}_2 = (M_2, \omega_2, \mathbb{T}, \mu_2)$ be two symplectic toric manifolds. Then

$$d_{\mathbb{T}}(\mathbf{M}_1, \mathbf{M}_2) := \lambda(\mu_1(M_1) \,\Delta\, \mu_2(M_2)), \tag{7}$$

where λ denotes the Lebesgue measure.

Remark 8. Observe that the metric $d_{\mathbb{T}}$ defined in (7) is related to the *Duistermaat-Heckman measure* ([9]). Indeed, for a symplectic toric manifold M with momentum map μ, the induced Duistermaat-Heckman measure of a Borel set $U \subset \mathbb{R}^n \simeq \mathfrak{t}^*$ is simply given by

$$m_{\mathrm{DH}}(U) = \lambda(U \cap \mu(M)).$$

[1] In the literature, these manifolds are usually called *equivariantly symplectomorphic*. However, the same name is sometimes also used for the notion in Definition 11, and so we use different names to distinguish the two.

$\oslash$

Remark 9. T. Delzant [8, Section 5] observed that a Delzant polytope gives rise to a fan ("éventail" in French), and that the symplectic toric manifold with associated Delzant polytope Δ is $\mathbb{T}$-equivariantly diffeomorphic to the *toric variety* defined by the fan.

The toric variety is an n-dimensional complex analytic manifold, and the action of the real torus $\mathbb{T}$ on it has an extension to a complex analytic action on of the complexification $\mathbb{T}_{\mathbb{C}}$ of $\mathbb{T}$. $\oslash$

Remark 10. In dimension 4, there is another class of integrable systems which is classified: those called semitoric [24, 25]. $\oslash$

Now we introduce a weaker notion of equivalence between toric manifolds, following [18].

Definition 11. Two symplectic toric manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are *equivariantly isomorphic* if there exists an automorphism of the torus $h\colon \mathbb{T} \to \mathbb{T}$ and an h-equivariant symplectomorphism $\varphi\colon M \to M'$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{T} \times M \arrow[r, "\rho*"] \arrow[d, "(h,\varphi)"'] & M \arrow[d, "\varphi"'] \\ \mathbb{T} \times M' \arrow[r, "\rho'^*"] & M'. \end{tikzcd} \tag{8}$$

We denote by $\widetilde{\mathcal{M}_{\mathbb{T}}}$ the moduli space of equivariantly isomorphic $2n$-dimensional toric manifolds.

It is easy to see that two equivariantly isomorphic toric manifolds are isomorphic if h is the identity.

Recall that the automorphism group of the torus $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ is given by $\mathrm{GL}(n, \mathbb{Z})$; thus the automorphism h is represented by a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$. Let $\mathrm{AGL}(n, \mathbb{Z})$ be the group of affine transformations of $\mathbb{R}^n$ given by

$$x \mapsto Ax + c,$$

where $A \in \mathrm{GL}(n, \mathbb{Z})$ and $c \in \mathbb{R}^n$. We have the following:

Proposition 12. ([18, Proposition 2.3 (2)]) *Two symplectic toric manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are equivariantly isomorphic if and only if their momentum map images are* $\mathrm{AGL}(n, \mathbb{Z})$-*congruent.*

Thus, if we define $\widetilde{\mathcal{D}_{\mathbb{T}}}$ to be the moduli space of $\mathrm{AGL}(n, \mathbb{Z})$-equivalent Delzant polytopes,

$$\widetilde{\mathcal{D}_{\mathbb{T}}} := \mathcal{D}_{\mathbb{T}} / \mathrm{AGL}(n, \mathbb{Z}),$$

Proposition 12 implies that the isomorphism in (6) descends to an isomorphism

$$\widetilde{\mathcal{M}_{\mathbb{T}}} \longrightarrow \widetilde{\mathcal{D}_{\mathbb{T}}}. \tag{9}$$

Let

$$\pi\colon \mathcal{D}_{\mathbb{T}} \to \widetilde{\mathcal{D}_{\mathbb{T}}}$$

be the projection map; we endow $\widetilde{\mathcal{D}_{\mathbb{T}}}$ with the quotient topology $\widetilde{\delta}$.

We endow $\widetilde{\mathcal{M}_{\mathbb{T}}}$ with the topology $\widetilde{\nu}$ induced by the isomorphism (9).

3 Connectedness of the space $(\widetilde{\mathcal{D}_{\mathbb{T}}}, \widetilde{\delta})$

3.1 Classification of Delzant polygons in $\mathbb{R}^2$

We introduce the definitions of rational length and corner chopping of size ε, which will be used in the classification of the Delzant polytopes in $\mathbb{R}^2$:

Definition 13.(following [18, 2.4 and 2.11])

(i) The *rational length* of an interval I of rational slope in $\mathbb{R}^n$ is the unique number $l = \text{length}(I)$ such that the interval is $\text{AGL}(n,\mathbb{Z})$-congruent to an interval of length l on a coordinate axis.

(ii) Let Δ be a Delzant polytope in $\mathbb{R}^n$ and v a vertex of Δ. Let

$$\{v + tu_i \mid 0 \leq t \leq \ell_i\}$$

be the set of edges emanating from v, where the $u_1, \ldots, u_n$ generate the lattice $\mathbb{Z}^n$ and $\ell_i = \text{length}(u_i)$. For $\varepsilon > 0$ smaller than the ℓ_i's, the *corner chopping of size* ε of Δ at v is the polytope Δ' obtained from Δ by intersecting with the half space

$$\{v + t_1u_1 + \cdots + t_nu_n \mid t_1 + \cdots + t_n \geq \varepsilon\}.$$

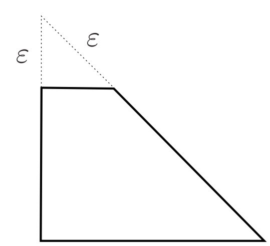

Figure 1: A corner chopping of size ε.

In $\mathbb{R}^2$, all Delzant polygons can be obtained by a recursive recipe, which can be found in [18, Lemma 2.15]:

Lemma 14. *The following hold.*

1. *Any Delzant polygon $\Delta \subset \mathbb{R}^2$ with three edges is $\text{AGL}(2,\mathbb{Z})$-congruent to the Delzant triangle Δ_λ for a unique $\lambda > 0$ (Example 15).*

2. *For any Delzant polygon $\Delta \subset \mathbb{R}^2$ with $4 + s$ edges, where s is a non-negative integer, there exist positive numbers $a \geq b > 0$, an integer $0 \leq k \leq 2a/b$, and positive numbers $\varepsilon_1, \ldots, \varepsilon_s$, such that Δ is $\text{AGL}(2,\mathbb{Z})$-congruent to a Delzant polygon that is obtained from the Hirzebruch trapezoid $H_{a,b,k}$ (see Example 15) by a sequence of corner choppings of sizes $\varepsilon_1, \ldots, \varepsilon_s$.*

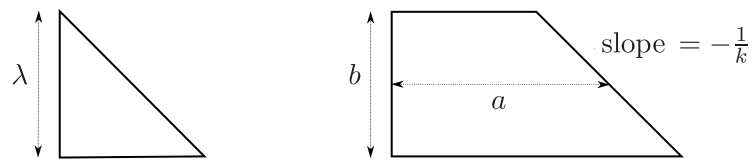

Figure 2: The Delzant triangle Δ_λ and the Hirzebruch trapezoid $H_{a,b,k}$.

Example 15. Figure 2 shows the Delzant triangle Δ_λ and the Hirzebruch trapezoid $H_{a,b,k}$. The *Delzant triangle*,

$$\Delta_\lambda := \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, \ \ x_1 + x_2 \leq \lambda\},$$

is the momentum map image of the standard $\mathbb{T}^2$ action on $\mathbb{CP}^2$ endowed with the Fubini-Study symplectic form. This form is normalized so that the symplectic area of each of the $\mathbb{CP}^1$'s corresponding to the edges is $2\pi\lambda$.

The *Hirzebruch trapezoid*,

$$H_{a,b,k} := \{(x_1, x_2) \mid -\frac{b}{2} \leq x_2 \leq \frac{b}{2}, \ \ 0 \leq x_1 \leq a - kx_2\},$$

is the momentum map image of the standard toric action on a Hirzebruch surface. Here b is the height of the trapezoid, a is its average width, and k is a non-negative integer such that the right edge has slope $-1/k$ or is vertical if $k = 0$. Moreover a and b have to satisfy $a \geq b$ and $a - k\frac{b}{2} > 0$. $\oslash$

3.2 Proof of Theorem 1 (a)

Recall that $(\mathcal{D}_\mathbb{T}, d)$ is the space of Delzant polytopes in $\mathbb{R}^2$ together with the distance function given by the area of the symmetric difference and that $\widetilde{\mathcal{D}_\mathbb{T}}$ is the quotient space

$$\mathcal{D}_\mathbb{T}/\mathrm{AGL}(2, \mathbb{Z})$$

with the quotient topology $\widetilde{\delta}$, induced by the quotient map $\pi\colon \mathcal{D}_\mathbb{T} \to \widetilde{\mathcal{D}_\mathbb{T}}$.

In order to prove Theorem 1 (a), given the isomorphism (9), it is enough to prove the following.

Theorem 16. *The space* $(\widetilde{\mathcal{D}_\mathbb{T}}, \widetilde{\delta})$ *is connected.*

Proof. Let $\mathcal{S} \subset \mathcal{D}_\mathbb{T}$ be the subset that contains all Delzant triangles Δ_λ for $\lambda \in \mathbb{R}^+$, all Hirzebruch trapezoids $H_{a,b,k}$ with $a, b \in \mathbb{R}$ such that $a \geq b > 0$ and $k \leq 2a/b$ a non-negative integer, and also all Delzant polygons obtained from Hirzebruch trapezoids by a sequence of corner choppings (cf. Lemma 14). We will prove that $\mathcal{S}$ is path connected in $\mathcal{D}_\mathbb{T}$.

First of all, observe that the intuitive paths from a Delzant polygon P to a translation of P or a scaling of P by a positive factor (or a composition of the two) are clearly continuous with respect to the topology induced by d. Furthermore if P is in $\mathcal{S}$ then so is the entire path. The same holds when moving an edge parallel to itself without changing the total number of edges (see Figure 3).

In particular, all Delzant triangles Δ_λ are connected by a path in $\mathcal{S}$ and so are the Hirzebruch trapezoids $H_{a,b,k}$, for each fixed k.

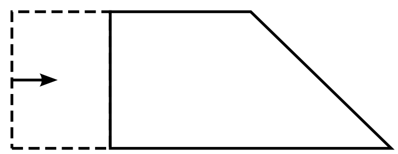

Figure 3: Moving an edge parallel to itself.

Secondly, if P_ε is obtained from P by a corner chopping of size ε at a vertex $v \in P$, then P and P_ε are path connected in $\mathcal{D}_{\mathbb{T}}$; the path is simply given by $\gamma\colon [0,1] \to \mathcal{D}_{\mathbb{T}}$ where

$$\gamma(t) := P_{t\varepsilon},$$

which is still in $\mathcal{D}_{\mathbb{T}}$. Thus any Delzant polygon obtained from $H_{a,b,k}$ by a sequence of corner choppings is connected to $H_{a,b,k}$ via a path in $\mathcal{S}$.

Let us now see that for $k \geq 0$ there is a continuous path between the Hirzebruch trapezoid $H_{a,b,k}$ and the Hirzebruch trapezoid $H_{a,b,k+1}$, where now $0 \leq k, k+1 \leq 2a/b$. The first half of the path connects $H_{a,b,k}$ to the polygon $H'_{a,b,k}$ by corner chopping at the top right vertex, and the inverse of the second half connects $H_{a,b,k+1}$ to $H'_{a,b,k}$ by corner chopping at the bottom right vertex (cf. Figure 4).

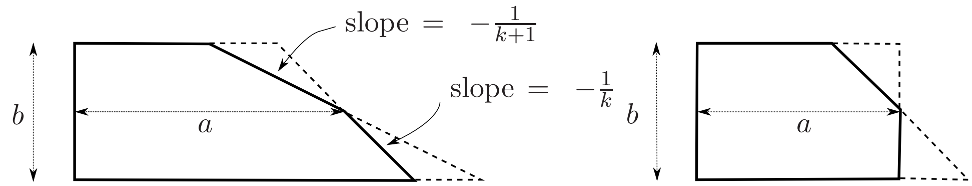

Figure 4: $H'_{a,b,k}$ for $k \geq 1$ and $H'_{a,b,0}$.

Note that $H'_{a,b,k}$ is still a Delzant polygon; it suffices to check smoothness at the new vertex, and indeed

$$\det \begin{bmatrix} -(k+1) & k \\ 1 & -1 \end{bmatrix} = 1.$$

Combining with previous observations, we conclude that all Hirzebruch trapezoids with $k \geq 0$ lie in the same path connected component of $\mathcal{S}$.

In order to conclude that $\mathcal{S}$ is path connected, it remains to check that there exists a continuous path between, for example, $H_{\lambda,\lambda,0}$ and Δ_λ. Let

$$\gamma\colon [0,1] \to \mathcal{S}$$

be the path such that:

(i) $\gamma(t)$ is the corner chopping of size λt at the top right vertex of the square $H_{\lambda,\lambda,0}$ for $0 < t < 1$,

(ii) $\gamma(0) := H_{\lambda,\lambda,0}$, and

(iii) $\gamma(1) := \Delta_\lambda$.

The path γ is continuous with respect to the topology on $\mathcal{D}_{\mathbb{T}}$.

By Lemma 14, we know that every element of $\widetilde{\mathcal{D}_{\mathbb{T}}}$ has a representative in $\mathcal{S}$, therefore $\mathcal{S}$ being path connected implies that $\widetilde{\mathcal{D}_{\mathbb{T}}}$ is path connected, and hence connected. $\square$

4 Topology of the space $(\mathcal{D}_{\mathbb{T}}, d)$

In this section we prove Theorem 1 (b). By isomorphism (9), it suffices to study the topological properties of $(\mathcal{D}_{\mathbb{T}}, d)$.

Let $(\mathfrak{B}'(\mathbb{R}^2)/\sim, d)$ be the metric space introduced in Section 2.1.

Proposition 17. *The space $(\mathcal{D}_{\mathbb{T}}, d)$ is not complete.*

Proof. We prove that $(\mathcal{D}_{\mathbb{T}}, d)$ is not complete, by giving an example of a Cauchy sequence in $(\mathcal{D}_{\mathbb{T}}, d)$ which converges in $(\mathfrak{B}'(\mathbb{R}^2)/\sim, d)$ whose limit is not a smooth polytope, hence not in $\mathcal{D}_{\mathbb{T}}$. For $k \neq 1$ consider the Hirzebruch surface $H_{a,b,k}$, and note that we can rewrite a as

$$a = c + \frac{bk}{2},$$

where c is the length of the top facet. Then, the sequence

$$H_{\frac{c}{n}+\frac{bk}{2},b,k}, \quad n = 1, 2, 3, \ldots$$

is Cauchy, but its limit is a right angle triangle that is not Delzant (see Figure 5).

$\square$

Remark 18. Note that $(\mathcal{D}_{\mathbb{T}}, d)$ is also non-compact, since compact metric spaces are automatically complete. $\oslash$

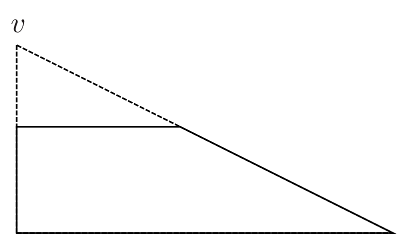

Figure 5: The vertex v is not smooth.

Let $\mathcal{C} \subset \mathfrak{B}'(\mathbb{R}^2)$ be the space of convex compact subsets of $\mathbb{R}^2$ with positive Lebesgue measure. Note that if $A, B \in \mathcal{C}$ and $d(A, B) = 0$ then $A = B$, and so d is a metric on $\mathcal{C}$. The same observations hold for $\mathcal{P}_2$, the space of convex 2-dimensional polygons in $\mathbb{R}^2$, and $\mathcal{P}_{\mathbb{Q}}$, the space of rational convex 2-dimensional polygons in $\mathbb{R}^2$. We have the following inclusions of metric spaces:

$$(\mathcal{D}_{\mathbb{T}}, d) \subset (\mathcal{P}_{\mathbb{Q}}, d) \subset (\mathcal{P}_2, d) \subset (\mathcal{C}, d).$$

As we will see in the proof of Theorem 24, the inclusions above are dense.

For every $A \in \mathfrak{B}'(\mathbb{R}^2)$, we denote by $[A]$ the equivalence class of A with respect to the equivalence relation $\sim$ defined in Section 2.1. We define $\varnothing$ to be the equivalence class of elements of $\mathfrak{B}'(\mathbb{R}^2)$ with zero Lebesgue measure, and

$$\hat{\mathfrak{C}} = \{[A] \mid A \in \mathfrak{C}\} \cup \varnothing.$$

Observe that $\varnothing$ can also be represented by a convex set, namely a single point in $\mathbb{R}^2$. Thus $\hat{\mathfrak{C}} \subset \mathfrak{B}'(\mathbb{R}^2)/\sim$ and $(\hat{\mathfrak{C}}, d)$ is a metric space. We will prove in Theorem 20 that it is complete. For that we need the following Lemma.

Lemma 19. *Let $A \in \mathfrak{B}(\mathbb{R}^2)$ be convex and non-bounded. If $\lambda(A) > 0$, then $\lambda(A) = \infty$.*

Proof. Consider two points $\mathbf{p}, \mathbf{q} \in A$. By convexity of A, the segment $\ell_{\mathbf{pq}}$ connecting $\mathbf{p}$ to $\mathbf{q}$ is contained in A. Since $\lambda(A) > 0$, there exists a point $\mathbf{r} \in A$ non collinear to $\mathbf{p}$ and $\mathbf{q}$. Hence the whole triangle $\mathbf{pqr}$ is contained in A.

Let C be the circle inscribed in the triangle $\mathbf{pqr}$, and $\{a_n\}_{n\in\mathbb{N}}$ a sequence of points in A such that $\|a_n\|_{\mathbb{R}^2} \to \infty$. For each a_n, there exists a diameter D_n of C such that the triangle $T_n \subset A$ with basis D_n and third vertex a_n is isosceles, which guarantees that

$$\lambda(T_n) \to \infty,$$

and so $\lambda(A) = \infty$. $\square$

Theorem 20. *$(\hat{\mathfrak{C}}, d)$ is a complete metric space.*

Proof. Let $\{[A_n]\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\hat{\mathfrak{C}}$, with A_n convex. By definition of d (see (4)), the sequence $\{\chi_{A_n}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{L}^1(\mathbb{R}^2)$. By completeness of $\mathrm{L}^1(\mathbb{R}^2)$, there exists a function $f \in \mathrm{L}^1(\mathbb{R}^2)$ such that

$$\|\chi_{A_n} - f\|_{\mathrm{L}^1} \to 0.$$

Thus, there exists a subsequence $\chi_{A_{n_k}}$ and a zero measure set $E \subset \mathbb{R}^2$ such that

$$\chi_{A_{n_k}}(x) \to f(x)$$

for all $x \in \mathbb{R}^2 \setminus E$. Let

$$A = \{x \in \mathbb{R}^2 \setminus E \mid f(x) = 1\};$$

from the definitions it follows immediately that $\chi_{A_{n_k}}(x) \to \chi_A(x)$ for all $x \in \mathbb{R}^2 \setminus E$ and

$$\|\chi_{A_n} - \chi_A\|_{\mathrm{L}^1} \to 0.$$

It's easy to see that $\lambda(A) < \infty$, thus $A \in \mathfrak{B}'(\mathbb{R}^2)$. If $\lambda(A) = 0$ this means that $[A] = \varnothing$, which belongs to $\hat{\mathfrak{C}}$. Let us now assume that $\lambda(A) > 0$; we have to prove that A is almost everywhere equal to a convex compact subset of $\mathbb{R}^2$. Let A' be the convex hull of A. Then, for any $\mathbf{p} \in A'$ there exists $\mathbf{q}, \mathbf{r} \in A$ such that

$$\mathbf{p} = t\mathbf{q} + (1-t)\mathbf{r}$$

for some $t \in [0, 1]$. Since $\mathbf{q}, \mathbf{r} \in A$, there exists $N \in \mathbb{N}$ such that for all $l > N$

$$\chi_{A_{n_l}}(\mathbf{q}) = \chi_{A_{n_l}}(\mathbf{r}) = 1\ ,$$

that is $\mathbf{q}, \mathbf{r} \in A_{n_l}$ for all $l > N$. Since A_n is convex for all n, this means that $\mathbf{p} \in A_{n_l}$ for all $l > N$, which implies that

$$\chi_{A_{n_k}}(x) \to \chi_{A'}(x)$$

almost everywhere in $A' \cup (\mathbb{R}^2 \setminus A) = \mathbb{R}^2$. Hence

$$\lambda(A' \setminus A) = 0.$$

Now it is sufficient to observe that, since A' is convex, its boundary $\partial A'$ has Lebesgue measure zero (see [20]), and since $\overline{A'}$ is also convex with $\lambda(\overline{A'}) = \lambda(A) < \infty$, by Lemma 19 it is bounded, hence compact.

$\square$

Remark 21. Theorem 20 also proves that the completion of $(\mathcal{C}, d)$ is $(\hat{\mathcal{C}}, d)$. $\oslash$

Before investigating what the completion of $(\mathcal{D}_{\mathbb{T}}, d)$ is, we first prove an auxiliary result, the content of which is related to resolving singularities on toric varieties (see Remark 9 and [7]). Recall that a vector $u \in \mathbb{Z}^2$ is called *primitive* if, whenever $u = k\, v$ for some $k \in \mathbb{Z}$ and $v \in \mathbb{Z}^2$, then $k = \pm 1$.

Lemma 22. *Let Δ be a simple rational polytope in $\mathbb{R}^2$ that fails to be smooth only at one vertex $\mathbf{p}$: the primitive vectors $u, v \in \mathbb{Z}^2$ which direct the edges at $\mathbf{p}$ do not form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Then there is a simple rational polytope $\widetilde{\Delta}$ with at most*

$$|\det[u\, v]\,| - 1$$

more edges than Δ that is a smooth polytope and is equal to Δ except in a neighborhood of $\mathbf{p}$.

Proof. Let $u = (a, b)$ and $v = (c, d)$. We claim that there is always a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ and $(\alpha_0, \alpha_1) \in \mathbb{Z}^2$ a primitive vector such that

$$A \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0 \\ 0 & \alpha_1 \end{bmatrix}.$$

If we set $\alpha_1 = |\det[u\, v]\,|$, the claim is equivalent to being able to solve the following for α_0:

$$\begin{cases} a\,\alpha_0 \equiv c \mod \alpha_1 \\ b\,\alpha_0 \equiv d \mod \alpha_1 \end{cases}.$$

Note also that $\alpha_0 \not\equiv 0 \mod \alpha_1$, or it would contradict (α_0, α_1) being primitive.

The primitive vectors directing the edges of the polygon $A(\Delta)$ at the non-smooth vertex $A(\mathbf{p})$ are $(1, 0)$ and (α_0, α_1). We can additionally do a shear transformation via a matrix of the form

$$S_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and obtain a $\mathrm{GL}(2, \mathbb{Z})$-equivalent polygon $S_1 A(\Delta)$ for which the non-smooth vertex $S_1 A(\mathbf{p})$ has edge directing vectors $(1, 0)$ and (α_2, α_1) where $0 < \alpha_2 < \alpha_1$.

We now create a new vertical edge on the polygon $S_1 A(\Delta)$ as close to the vertex $S_1 A(\mathbf{p})$ as desired, and thereby eliminating that vertex. Call this new polygon Δ_1. Of the two vertices at the endpoints of this new edge, one is clearly smooth: the one with edge directing vectors $(1,0)$ and $(0,1)$. The other vertex has edge directing vectors $(0,-1)$ and (α_2,α_1) and is smooth if and only if

$$\det\begin{bmatrix} 0 & \alpha_2 \\ -1 & \alpha_1 \end{bmatrix} = \alpha_2 = 1.$$

If it is, the desired polytope $\widetilde{\Delta}$ is

$$A^{-1}S_1^{-1}(\Delta_1).$$

Otherwise, the process continues: let B be the matrix

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then the polygon $B(\Delta_1)$ is smooth except at one vertex, $\mathbf{p}_1$, which has edge directing vectors $B((0,-1)) = (1,0)$ and

$$B((\alpha_2,\alpha_1)) = (-\alpha_1,\alpha_2).$$

As before, we can apply a shear transformation to obtain $S_2B(\Delta_1)$ such that the edge directing vectors at the non-smooth vertex $S_2B(\mathbf{p}_1)$ are of the form $(1,0)$ and (α_3,α_2) with $0<\alpha_3<\alpha_2$. Proceeding exactly as before, we create a new vertical edge on the polygon $S_2B(\Delta_1)$ as close to the vertex $S_2B(\mathbf{p}_1)$ as desired, and thereby eliminating it. Call this new polygon Δ_2. Of the two vertices at the endpoints of the new edge, the one with edge directing vectors $(0,1)$ and $(1,0)$ is clearly smooth, the other one has edge directing vectors $(0,-1)$ and (α_3,α_2) and is smooth if and only if $\alpha_3 = 1$. If that is so, the desired polytope $\widetilde{\Delta}$ is

$$A^{-1}S_1^{-1}B^{-1}S_2^{-1}(\Delta_2),$$

otherwise we repeat the process.

Because $\alpha_1,\alpha_2,\alpha_3,\ldots$ is a strictly decreasing sequence of non-negative integers, it will reach 1 in at most α_1-1 steps, thus terminating the process and producing $\widetilde{\Delta}$ with at most α_1-1 more edges than Δ. $\square$

Remark 23. Note that the process described in the proof of Lemma 22 does not rely on Δ being smooth at all vertices other than $\mathbf{p}$. In fact, the result holds for any simple rational non-smooth polytope Δ, with any number of non-smooth vertices, except that the number of extra edges will be

$$\sum_{\mathbf{p}_i} \left(|\det[u_i\, v_i]| - 1\right),$$

where the $\mathbf{p}_i$s are the non-smooth vertices of Δ. The new polytope $\widetilde{\Delta}$ is equal to Δ except in neighborhoods of the vertices $\mathbf{p}_i$ that can be made as small as desired. $\oslash$

Now we are ready to prove the main theorem of this section.

Theorem 24. *The completion of* $(\mathcal{D}_{\mathbb{T}},d)$ *is* $(\hat{\mathcal{C}},d)$.

Proof. Recall that
$$(\mathcal{D}_{\mathbb{T}}, d) \subset (\mathcal{P}_{\mathbb{Q}}, d) \subset (\mathcal{P}_2, d) \subset (\mathcal{C}, d).$$
We begin by proving that the completion of $(\mathcal{D}_{\mathbb{T}}, d)$ contains $(\mathcal{P}_{\mathbb{Q}}, d)$, that the completion of $(\mathcal{P}_{\mathbb{Q}}, d)$ contains $(\mathcal{P}_2, d)$, and that the completion of $(\mathcal{P}_2, d)$ contains $(\mathcal{C}, d)$. Then by Remark 21 the conclusion follows.

Because d is a metric on $\mathcal{P}_{\mathbb{Q}}$ that coincides with the given metric on $\mathcal{D}_{\mathbb{T}}$, in order to prove that the completion of $\mathcal{D}_{\mathbb{T}}$ contains $\mathcal{P}_{\mathbb{Q}}$ it suffices to show that for each $\Delta \in \mathcal{P}_{\mathbb{Q}}$ there exists a polygon in $\mathcal{D}_{\mathbb{T}}$ as close to Δ as desired, relative to the metric d. Lemma 22 and in particular Remark 23 guarantee that this is so.

Because d is a metric on $\mathcal{P}_2$ that coincides with the given metric on $\mathcal{P}_{\mathbb{Q}}$, in order to prove that the completion of $(\mathcal{P}_{\mathbb{Q}}, d)$ contains $(\mathcal{P}_2, d)$ it suffices to show that for each $\Delta \in \mathcal{P}_2$ there exists a polygon $\Delta_{\mathbb{Q}} \in \mathcal{P}_{\mathbb{Q}}$ as close to Δ as desired, relative to the metric d. This rational polygon $\Delta_{\mathbb{Q}}$ can be obtained by approximating the irrational slopes of the edges of Δ by rational numbers and chosing for directing vectors of the edges the corresponding lattice vectors, and also by changing the vertex points accordingly. This way, we can make the symmetric difference between the original polygon Δ and the rational polygon $\Delta_{\mathbb{Q}}$ be contained in a ε-ball of the edges of Δ, the area of which can be made as small as needed by making ε small enough.

Because d is a metric on $\mathcal{C}$ that coincides with the given metric on $\mathcal{P}_2$, in order to prove that the completion of $(\mathcal{P}_2, d)$ contains $(\mathcal{C}, d)$ it suffices to show that for each $C \in \mathcal{C}$ there exists a polygon $\Delta_2 \in \mathcal{P}_2$ as close to C as desired, relative to the metric d. In order to do this, observe that given a compact convex set $C \in \mathcal{C}$ and $\varepsilon > 0$, there exists a collection of disjoint rectangles $\{[a_i, b_i) \times [c_i, d_i)\}_{i=1}^N$ contained in C such that
$$\|\chi_C - \sum_{i=1}^N \chi_{[a_i,b_i)\times[c_i,d_i)}\|_{\mathrm{L}^1} < \varepsilon.$$
Let Δ_2 be the convex hull of $\bigcup_{i=1}^N [a_i, b_i] \times [c_i, d_i]$. Since C is convex, we have
$$\bigcup_{i=1}^n [a_i, b_i) \times [c_i, d_i) \subset \Delta_2 \subset C,$$
and hence
$$\|\chi_C - \chi_{\Delta_2}\|_{\mathrm{L}^1} < \varepsilon,$$
which proves the claim. $\square$

Proposition 25. *The space $(\mathcal{D}_{\mathbb{T}}, d)$ is not locally compact.*

Proof. To prove that $(\mathcal{D}_{\mathbb{T}}, d)$ is not locally compact, we show that the closure of any open ball $B_\varepsilon(H_{1,1,0}) \subset \mathcal{D}_{\mathbb{T}}$ is not compact in $(\mathcal{D}_{\mathbb{T}}, d)$. For each fixed ε, let $\frac{\varepsilon}{2} < \delta < \varepsilon$ be an irrational number, and let Q_δ be the polygon in Figure 6. Note that $Q_\delta \in \mathcal{P}_2 \backslash \mathcal{D}_{\mathbb{T}}$. By a triangular inequality argument it's easy to see that
$$B_{\frac{\varepsilon}{2}}(Q_\delta) \subset B_\varepsilon(H_{1,1,0}).$$
Because $(\mathcal{D}_{\mathbb{T}}, d)$ is dense in $(\mathcal{P}_2, d)$ (see proof of Theorem 24), there exists a sequence of Delzant polygons
$$\{A_n\}_{n\in\mathbb{N}} \subset B_{\frac{\varepsilon}{2}}(Q_\delta)$$

that converges to Q_δ in $(\mathcal{P}_2, d)$. Thus any subsequence of $\{A_n\}$ also converges to Q_δ in $(\mathcal{P}_2, d)$, and hence does not converge in $(\mathcal{D}_\mathbb{T}, d)$. This proves that the closure of $B_\varepsilon(H_{1,1,0})$ in $\mathcal{D}_\mathbb{T}$ is not compact.

□

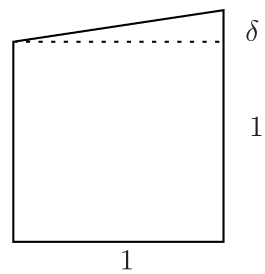

Figure 6: The polygon Q_δ.

Remark 26. There is another metric commonly used on $\mathcal{C}$, namely the *Hausdorff metric* d_H (see [3]). Given two elements $A, B \in \mathcal{C}$, we define

$$d_\mathrm{H}(A,B) := \max\{\sup_{y\in B}\inf_{x\in A} \|x-y\|,\ \sup_{x\in A}\inf_{y\in B}\|y-x\|\}\,.$$

As proved in [28], the metrics d and d_H are equivalent on $\mathcal{C}$, and consequently all the topological properties proved for $(\mathcal{C}, d)$ and $(\mathcal{D}_\mathbb{T}, d)$ also hold for $(\mathcal{C}, d_H)$ and $(\mathcal{D}_\mathbb{T}, d_H)$. We choose to work with d because on $\mathcal{D}_\mathbb{T}$ this is related to the Duistermaat-Heckmann measure (see Remark 8). ⊘

Remark 27. Other moduli spaces of polygons have been studied for example in [15, 16] by Hausmann and Knutson and in [17] by Kapovich and Millson. The former focuses on the space of polygons in $\mathbb{R}^k$ with a fixed number of edges up to translations and positive homotheties, whereas the latter studies the space of polygons in $\mathbb{R}^2$ with fixed side lengths up to orientation preserving isometries. However these different contexts completely change the flavor of the topological problem. ⊘

5 Further problems

Theorem 1 leads to further questions (not directly related among themselves).

Problem 28 (Completeness at manifold level)**.** This question attempts to make more explicit the relation between the completion at the level of polytopes with the completion at the level of manifolds given in Theorem 1. View $\mathcal{M}_\mathbb{T}$ as a subset of the set of (all) integrable systems on symplectic 4-manifolds

$$\mathcal{I} := \{(M, \omega, F) \mid F := (f_1, f_2)\colon M \to \mathbb{R}^2\}.$$

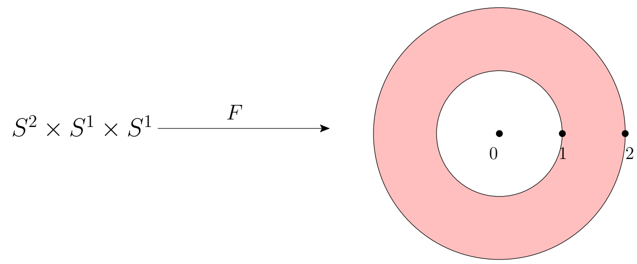

Figure 7: Image of a non-toric integrable system on $S^2 \times \mathbb{T}^2$. The image is not a polytope, it is not even convex (this example is studied in detail in [27, Example 4.10].

The map $v\colon \mathcal{I} \to \mathfrak{B}'(\mathbb{R}^2)$ given by

$$v(M, \omega, F) := F(M)$$

extends (1) (see Figure 7 for an example of this image). What can one say about the intersection

$$V := \hat{\mathcal{C}} \cap v(\mathcal{I})\,?$$

A different but related approach to the same problem is to enlarge the category of objects by relaxing the smoothness condition. For example, from the work of Lerman and Tolman in [21], we know that any rational convex polytope is the momentum map image of a symplectic toric orbifold. However we do not know of a similar identification for generic convex compact subsets of $\mathbb{R}^2$.

Problem 29 (Higher dimensional moduli spaces)**.** This paper addreses the case $2n = 4$ of Problem 2.42 in [26]. We do not have results on the higher dimensional case $2n \geq 6$.

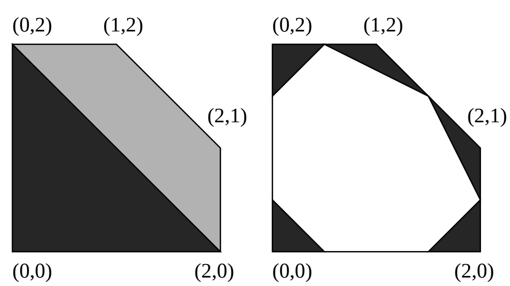

Figure 8: Ball packings of convex polytopes. These packings are not maximal density packings.

Problem 30 (Continuity of packing density function)**.** Consider the maximal density function

$$\Omega\colon \mathcal{M}_{\mathbb{T}}/\simeq \to (0,1]$$

which assigns to a symplectic toric manifold its maximal density by equivariantly embedded symplectic balls of varying radii (see [23, Definition 2.4] and Figure 8). The function is most interesting when considered on $\mathcal{M}_{\mathbb{T}}/\simeq$, where $\simeq$ corresponds to rescaling the symplectic form (since rescaling the symplectic form rescales the polytope and does not change the density).

This paper gives a quotient topology on $\mathcal{M}_{\mathbb{T}}/\simeq$. With respect to this topology, is Ω continuous? Ω is an interesting map even if one disregards topology, for instance the fiber over 1 consists of 2 points $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ (proven in [22, Theorem 1.7]) with scaled symplectic forms, but for any other $x \in (0,1)$ the fiber is uncountable [23, Theorems 1.2, 1.3].

Problem 31 (Toric actions and symplectic forms)**.** It would be interesting to define a torus action on the moduli spaces $\mathcal{M}_{\mathbb{T}}$ or $\widetilde{\mathcal{M}_{\mathbb{T}}}$. Similarly for a symplectic form (eg [6]).

If one has both a torus action and a symplectic form, then one can formulate a notion of Hamiltonian action and momentum map, and study connectivity and convexity properties of the image (see for instance [14]).

Problem 32 (Topological invariants)**.** Compute the topological invariants (fundamental group, higher homotopy groups, cohomology groups, etc.) of the connected space $\widetilde{\mathcal{M}_{\mathbb{T}}}$.

Some preliminary questions in this direction are:

(a) find non-trivial loop classes in $\pi_1(\widetilde{\mathcal{M}_{\mathbb{T}}})$;

(b) find non-trivial cohomology classes in $\mathrm{H}^1(\widetilde{\mathcal{M}_{\mathbb{T}}}, \mathbb{Z})$.

In view of the constructions of this paper, one should be able to compute these classes with the aid of the concrete description of the polytope space.

This problem is a particular case of [26, Problem 2.46].

6 Appendix: Polytopes

Let V be a finite dimensional real vector space. A *convex polytope* S in V is the closed convex hull of a finite set $\{v_1, \ldots, v_n\}$, i.e., the smallest convex set containing S or equivalently,

$$\mathrm{Conv}\{v_1, \ldots, v_n\} := \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in [0,1], \sum_{i=1}^{n} a_i = 1 \right\}.$$

The *dimension* of $\mathrm{Conv}\{v_1, \ldots, v_n\}$ is the dimension of the vector space

$$\mathrm{span}_{\mathbb{R}}\{v_1, \ldots, v_n\}.$$

A polytope is *full dimensional* if its dimension equals the dimension of V.

Note that the definition implies that a convex polytope is a compact subset of V. An *extreme point* of a convex subset $C \subseteq V$ is a point of C which does not lie in any open line segment joining two points of C. Thus, a convex polytope is the closed convex hull of its extreme points (by the Krein-Milman Theorem [19]) called *vertices*. In particular, the set of vertices is contained in $\{v_1, \ldots, v_n\}$. Clearly, there are infinitely many descriptions of the same polytope as a closed convex hull of a finite set of points. However, the description of a polytope as the convex hull of its vertices is minimal and unique.

There is another description of convex polytopes in terms of intersections of half-spaces. Let V^* be the dual of V and denote by

$$\langle \, , \rangle : V^* \times V \to \mathbb{R}$$

the natural non-degenerate duality pairing. The *positive (negative) half-space* defined by $\alpha \in V^*$ and $a \in \mathbb{R}$ is defined by

$$V^{\pm}_{\alpha,a} := \left\{ v \in V \;\middle|\; \langle \alpha, v \rangle \gtreqless a \right\} .$$

Traditionally, in the theory of convex polytopes, the half spaces are chosen to be of the form $V^{-}_{\alpha,a}$. With these definitions, a convex polytope is given as a finite intersection of half-spaces. As for the convex hull representation, there are infinitely many representations of the same convex polytope as a finite intersection of half-spaces, but there exists a distinguished one that is minimal, as we will see in the next paragraph.

A *face* of a convex polytope is an intersection with a half-space satisfying the following condition: the boundary of the half-space does not contain any interior point of the polytope. Thus the faces of a convex polytope are themselves polytopes (and hence compact sets) of dimension at most $m-1$, where m is the dimension of the polytope. The $(m-1)$-dimensional faces are called *facets*, the 1-dimensional faces are the *edges*, and the 0-dimensional faces are the vertices of the polytope. If the convex polytope is full-dimensional, there is a minimal and unique description in terms of the half-spaces containing the facets.

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