Symplectic theory of completely integrable Hamiltonian systems

In memory of Professor J.J. Duistermaat (1942-2010)

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Abstract

This paper explains the recent developments on the symplectic theory of Hamiltonian completely integrable systems on symplectic 4-manifolds, compact or not. One fundamental ingredient of these developments has been the understanding of singular affine structures. These developments make use of results obtained by many authors in the second half of the twentieth century, notably Arnold, Duistermaat and Eliasson, of which we also give a concise survey. As a motivation, we present a collection of remarkable results proven in the early and mid 1980s in the theory of Hamiltonian Lie group actions by Atiyah, Guillemin-Sternberg and Delzant among others, and which inspired many people, including the authors, to work on more general Hamiltonian systems.

1 Introduction

In the mathematical theory of conservative systems of differential equations one finds cases that are solvable in some sense, or *integrable*, which enables one to study their dynamical behavior using differential geometric and Lie[1] theoretic methods, in particular the theory of Lie group actions on symplectic manifolds. Integrable systems are a fundamental class of "explicitly solvable" dynamical systems of current interest in differential and algebraic geometry, representation theory, analysis and physics. Their study usually combines ideas from several areas of mathematics, notably partial differential equations, microlocal analysis, Lie theory, symplectic geometry and representation theory. In this paper we focus on finite dimensional completely integrable Hamiltonian systems (sometimes called "Liouville integrable systems") in the context of symplectic geometry.

Many authors have studied dynamical problems for centuries. Galileo made great advances to the subject in the late XVI and early XVII centuries, and formulated the "laws of falling bodies". An important contribution was made by Huygens in the XVII century, who studied in detail the spherical pendulum, a simple but fundamental example. Galileo's ideas were generalized and reformulated by William Hamilton (1805-1865) using symplectic geometry, who said: "*the theoretical development of the laws of motion of bodies is a problem of such interest and importance that it has engaged the attention of all the eminent mathematicians since the invention of the dynamics as a mathematical science by Galileo, and especially since the wonderful extension which was given to that science by Newton*" (1834, cf. J.R. Taylor [99, p. 237]). Many of the modern notions in the mathematical theory of dynamical systems date back to the late XIX century and the XX century, to the works of Poincaré, Lyapunov, Birkhoff, Siegel and the Russian school in the qualitative theory of ordinary differential equations.

[1] Sophus Lie has been one of the most influential figures in differential geometry. Many modern notions of differential geometry were known to Lie in some form, including the notion of symplectic manifold, symplectic and Hamiltonian vector fields, transformation (=Lie) groups, and (in a particular instance) momentum maps.

A *completely integrable Hamiltonian system* may be given by the following data[2]: (1) a $2n$-dimensional smooth manifold M equipped with a symplectic form, and (2) n smooth functions $f_1, \ldots, f_n \colon M \to \mathbb{R}$ which generate vector fields that are pairwise linearly independent at almost every point, and which Poisson commute. In local symplectic coordinates, this commuting condition amounts to the vanishing of partial differential equations involving the f_i, eg. see Section 6. Many times we will omit the word "Hamiltonian", and refer simply to "completely integrable systems". A completely integrable system has a *singularity* at every point where this linear independence fails to hold. It is precisely at the singularities where the most interesting, and most complicated, dynamical features of the system are displayed. An important class of completely integrable systems, with well-behaved singularities, are those given by Hamiltonian n-torus actions on symplectic $2n$-manifolds. These actions have a momentum map with n components $f_1, \ldots, f_n$, and these components always form a completely integrable system. A remarkable structure theory by Atiyah [6], Guillemin-Sternberg [54] and Delzant [29] exists for these systems, which are usually referred to as *toric systems*.

The study of completely integrable Hamiltonian systems is a vast and active research area. Two motivations to study such systems come from: (i) *Kolmogorov-Arnold-Moser (KAM) theory*: since integrable systems are "solvable" in a precise sense, one expects to find valuable information about the behavior of dynamical systems that are obtained by *small perturbations* of them, and then the powerful KAM theory comes into play (see de la Llave's article [28] for a summary of the main ideas of KAM theory) to deal with the properties of the perturbations (persistence of quasi-periodic motions); (ii) *the theory of singularities of fibrations* $(f_1, \ldots, f_n) \colon M \to \mathbb{R}$ *by the Fomenko school* [11]: the Fomenko school has developed powerful and far reaching methods to study the topology of singularities of integrable systems. It is interesting to notice that there is a relation between these two motivations, which has been explored recently by Dullin-Vũ Ngọc and Zung [38, 39, 112, 115].

In the present article we give an overview of our perspective of the current state of the art of the symplectic geometry of completely integrable systems, with a particular emphasis on the the recent developments on semitoric integrable systems in dimension four. Before this, we briefly review several preceding fundamentals results due to Arnold, Atiyah, Carathéodory, Darboux, Delzant, Duistermaat, Dufour, Eliasson, Guillemin, Liouville, Mineur, Molino, Sternberg, Toulet and Zung, some of which are a key ingredient in the symplectic theory of semitoric integrable systems. This article does not intend to be comprehensive in any way, but rather it is meant to be a fast overview of the current research in the subject; we hope we will convey some of the developments which we consider most representative. Our point of view is that of local phase-space analysis; it advocates for the use of local normal forms, and sheaf theoretic methods, to prove global results by gluing local pieces.

Some of the current activity on integrable systems is concerned with a question of high interest to applied and pure mathematicians and physicists. The question is whether one can reconstruct an integrable system which one does not a priori know, from observing some of its properties. E.g. Kac's famous question: can you hear the shape of a drum? Kac's question in the context of integrable systems can be formulated in the following way: can a completely integrable system be recovered from the semiclassical joint spectrum of the corresponding *quantized* integrable system? In order to study this question one must complement Fomenko's *topological* theory with a *symplectic* theory, which allows one to *quantize* the integrable system. Quantization (the process of assigning Hilbert spaces and operators to symplectic manifolds and smooth real-valued functions) is a motivation of the present article but it is not its goal; instead we lay down the symplectic geometry needed to study the quantization of certain completely integrable systems, and we will stop there; occasionally in the paper, and particularly in the last section, we will make some further

[2] We will explain this definition in detail later, starting with the most basic notions.

comments on quantization. For a basic reference on the so called *geometric quantization*, see for example Kostant-Pelayo [62].

The authors have recently given a global *symplectic* classification of integrable systems with two degrees of freedom which has no hyperbolic singularities and for which one component of the system is a 2π-periodic Hamiltonian [86, 87]; these systems are called *semitoric*. We devote sections 6, 7, 8 this paper to explain this symplectic classification. This symplectic classification of semitoric integrable system described in this paper prepares the ground for answering Kac's question in the context of semitoric completely integrable systems.

Semitoric systems form an important class of integrable systems, commonly found in simple physical models; a semitoric system can be viewed as a Hamiltonian system in the presence of circular symmetry. Perhaps the simplest example of a non-compact non-toric semitoric system is the coupled spin-oscillator model $S^2 \times \mathbb{R}^2$ described in [106, Sec. 6.2], where S^2 is viewed as the unit sphere in $\mathbb{R}^3$ with coordinates (x, y, z), and the second factor $\mathbb{R}^2$ is equipped with coordinates (u, v), equipped with the Hamiltonians $J := (u^2 + v^2)/2 + z$ and $H := \frac{1}{2}(ux + vy)$. Here S^2 and $\mathbb{R}^2$ are equipped with the standard area forms, and $S^2 \times \mathbb{R}^2$ with the product symplectic form. The authors have carried out the quantization of this model in [88].

In the aforementioned papers we combine techniques from classical differential geometry, semiclassical analysis, and Lie theory; these works are representative of our core belief that one can make definite progress in the program to understand the symplectic and spectral theory of integrable systems by combining techniques and ideas from these areas. This symplectic work in turns generalizes the celebrated theory of Hamiltonian Lie group actions by Atiyah, Benoist, Delzant, Guillemin, Kirwan, Sternberg and others, to completely integrable systems. It is also intimately connected with several previous works [31, 32, 33, 34, 82, 106, 105].

While major progress has been made in recent times by many authors, the theory of integrable systems in symplectic geometry is far from complete at the present time, even in the case of integrable systems with two degrees of freedom. For example, it remains to understand the symplectic theory of integrable systems on 4-manifolds when one allows hyperbolic singularities to occur. The presence of hyperbolic singularities has a global effect on the system which makes describing a set of global invariants difficult[3]. We do not know at this time if this is even a feasible problem or whether one can expect to give a reasonable classification extending the case where no hyperbolic singularities occur.

Moreover, the current theory allows us to understand semitoric systems with controlled behavior at infinity; precisely this means that the 2π-periodic Hamiltonian is a proper map, which rules out the classical spherical pendulum (the paper in the works [84] is expected to address this case). The general case is open, however. Shedding light into these two questions would bring us a step closer to understanding the symplectic geometry of *general* completely integrable systems with two degrees of freedom in dimension 4, which we view as one of the major and longstanding unsolved problems in geometry and dynamics (and to which many people have done contributions, several of which are mentioned in the present paper). In addition, answers to these questions constitute another required step towards a quantum theory of integrable systems on symplectic manifolds.

One can find integrable systems in different areas of mathematics, physics. For example, in the context of algebraic geometry a semitoric system naturally gives a toric fibration with singularities, and its base space becomes endowed with a singular integral affine structure. Remarkably, these singular affine structures are

[3]starting with what we will call the "polygon invariant" and which encodes in some precise sense the affine structure induced by the system.

of central importance in various parts of symplectic topology, mirror symmetry, and algebraic geometry – for example they play a central role in the work of Kontsevich and Soibelman [65], cf. Section 9.2 for further details. Interesting semitoric systems also appear as relevant examples in the theory of symplectic quasi-states, see Eliashberg-Polterovich [40, page 3]. Many aspects of the global theory of semitoric integrable systems may be understood in terms of singular affine structures, but we do not know at this time whether all of the invariants may be expressed in terms of singular affine structures (if this were the case, likely involving some asymptotic behavior).

For mathematicians semitoric systems are the next natural class of integrable systems to consider after toric systems. Semitoric systems exhibit a richer, less rigid behavior than toric systems. The mathematical theory of semitoric systems explained in the last few sections of this paper was preceded by a number of interesting works by physicists and chemists working on describing energy-momentum spectra of systems in the context of quantum molecular spectroscopy [43, 91, 18, 5]. Physicists and chemists were the first to become interested in semitoric systems. Semitoric systems appear naturally in the context of quantum chemistry. Many groups have been working on this topic, to name a few: Mark Child's group in Oxford (UK), Jonathan Tennyson's at University College London (UK), Frank De Lucia's at Ohio State University (USA), Boris Zhilinskii's at Dunkerque (France), and Marc Joyeux's at Grenoble (France).

These physicists and chemists have asked whether one can one give a finite collection of invariants characterizing systems of this nature. The theory of semitoric systems described in the present paper was largely motivated by this question, and fits into the broader realization in the physics and chemistry communities that symplectic invariants play a leading role in understanding a number of global questions in molecular spectroscopy — hence any mathematical discovery in this direction will be of interest outside of a pure mathematical context, see Stewart [96].

Direct applications of integrable systems can also be found in the theory of geometric phases, nonholonomic mechanics, rigid body systems, fluid mechanics, elasticity theory and plasma physics, and have been extensively carried out by many authors, including Marsden, Ratiu and their collaborators. The semiclassical aspects of integrable systems have been recently studied in the book [108] and the article [107]. In the book [26] singular Lagrangian fibrations are treated from the point of view of classical mechanics. Finally, we would like to point out Bolsinov-Oshemkov's interesting review article [10], where for instance one can find very interesting information about hyperbolic singularities.

The structure of the paper is as follows. In sections 2, 3, 4 and 5 we summarize some of the most important known results at a local and semiglobal level for Hamiltonian systems, and motivate their study by presenting some influential results from the theory of Hamiltonian Lie group actions due to Atiyah, Guillemin-Sternberg and Delzant. In section 6 we introduce semitoric systems in dimension four, and explain their convexity properties. In sections 7 and 8 we introduce symplectic invariants for these systems and explain the recent global symplectic classification of semitoric systems given by the authors. In Section 9 we briefly discuss some open problems.

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2 Symplectic Dynamics

The unifying topic this paper is *symplectic geometry*, which is the mathematical language to clearly and concisely formulate problems in classical physics and to understand their quantum counterparts (see Marsden--Ratiu classical textbook [74] for a treatment of classical mechanical systems). In the sense of Weinstein's creed, symplectic geometry is of interest as a series of remarkable "transforms" which connect it with several areas of semiclassical analysis, partial differential equations and low-dimensional topology.

One may argue that symplectic manifolds are not the most general, or natural, setting for mechanics. In recent times some efforts have been made to study Poisson structures, largely motivated by the study of coadjoint orbits. However only very few general results on integrable systems are known in the context of Poisson manifolds.

2.1 Symplectic manifolds

A *symplectic form* on a vector space V is a non-degenerate, antisymmetric, bilinear map $V \times V \to \mathbb{R}$. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is symplectic form on M, i.e. smooth collection of symplectic forms ω_p, one for each tangent space $\mathrm{T}_p\, M$, which is globally closed in the sense that the differential equation $\mathrm{d}\omega = 0$ holds.

The simplest example of a symplectic manifold is probably a surface of genus g with an area form. An important example is $\mathbb{R}^{2n}$ with the form $\mathrm{d}x_i \wedge \mathrm{d}y_i$, where $(x_1,\, y_1, \ldots,\, x_n,\, y_n)$ are the coordinates in $\mathbb{R}^{2n}$.

Symplectic manifolds are always even dimensional, so for example S^3, S^1 cannot be symplectic. They are also orientable, where the volume form is given by $\omega \wedge \ldots (n \text{ times}) \ldots \wedge \omega = \omega^n$, if $\dim M = 2n$, so for example the Klein bottle is not a symplectic manifold. Moreover, symplectic manifolds are topologically "non-trivial" in the sense that if M is compact, then the even dimensional de Rham cohomology groups of M are not trivial because $[\omega^k] \in \mathrm{H}^{2k}_{\mathrm{dR}}(M)$ defines a non-vanishing cohomology class if $k \leq n$, i.e. the differential 2-form ω^k is closed but not exact (the proof of this uses Stokes' theorem, and is not completely immediate). Therefore, the spheres S^4, S^6, S^8, $\ldots, S^{2N}$,... cannot be symplectic. Symplectic manifolds were locally classified by Darboux and the end of the XIX century. He proved the following remarkable theorem.

Theorem 2.1 (Darboux [27])**.** *Near each point in $(M,\, \omega)$ there exists coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in which the symplectic form ω has the form $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$.*

It follows from Darboux's theorem that symplectic manifolds have no local invariants other than the dimension. This is a fundamental difference with Riemannian geometry, where the curvature is a local invariant.

2.2 Dynamics of vector fields and torus actions

A smooth vector field $\mathcal{Y}$ on a symplectic manifold $(M,\, \omega)$ is *symplectic* if its flow preserves the symplectic form ω, and it is *Hamiltonian* if the system

$$\omega(\mathcal{Y},\, \cdot) = \mathrm{d}H \quad \text{(Hamilton's PDEs)}$$

has a smooth solution $H\colon M \to \mathbb{R}$. If so, we use the notation $\mathcal{Y} := \mathcal{H}_H$, and call H call the *Hamiltonian* or *Energy Function*.

For instance, the vector field $\frac{\partial}{\partial\theta}$ on $\mathbb{T}^2 := (S^1)^2$ is symplectic but not Hamiltonian (θ is the coordinate on the first copy of S^1 in $\mathbb{T}^2$, for example); on the other hand, the vector field $\frac{\partial}{\partial\theta}$ on S^2 is Hamiltonian: $\frac{\partial}{\partial\theta} = \mathcal{H}_H$ with $H(\theta, h) := h$; here (θ, h) represents a point in the unit sphere S^2 of height h measured from the plane $z = 0$ and angle θ measured about the vertical axes, see Figure 2.1.

Suppose that we have local Darboux coordinates $(x_1, y_1, \ldots, x_n, y_n)$ near a point $m \in M$. Let $\gamma(t) := (x_1(t), y_1(t), \ldots, x_n(t), y_n(t))$ be an integral curve of a smooth vector field $\mathcal{Y}$. Then $\mathcal{Y} = \mathcal{H}_H$ for a local smooth function H if and only if

$$\begin{cases} \frac{\mathrm{d}y_i}{\mathrm{d}t}(t) & = & -\frac{\partial H}{\partial x_i}(\gamma(t)) \\ \frac{\mathrm{d}x_i}{\mathrm{d}t}(t) & = & \frac{\partial H}{\partial y_i}(\gamma(t)) \end{cases}$$

It always holds that $H(\gamma(t)) =$ const, i.e. that energy is conserved by motion (Noether's Principle). If $\mathcal{Y}$ is symplectic, these equations always have a local solution, but in order for the vector field $\mathcal{Y}$ to be Hamiltonian (globally) one must have that $\mathcal{Y} = \mathcal{H}_H$ on M, i.e. the function H has to be the same on each local Darboux chart. From a more abstract point of view, this amounts to saying that the 1-form $\omega(\mathcal{Y}, \cdot)$ is always locally exact, but not necessarily globally exact. So the obstruction to being exact lies in the first de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(M)$ of M; if this group is trivial, then any smooth symplectic vector field on M is Hamiltonian.

Now suppose that we have a torus $T \simeq \mathbb{T}^k := (S^1)^k$, i.e. a compact, connected abelian Lie group. Let $X \in \mathfrak{t} = \mathrm{Lie}(T)$. For X in the Lie algebra $\mathfrak{t}$ of T (i.e. we view X as a tangent vector at the identity element 1 to T), there exists a unique *homomorphism* $\alpha_X \colon \mathbb{R} \to T$ with $\alpha_X(0) = 1, \alpha'_X(0) = X$. Define the so called *exponential map* $\exp\colon \mathfrak{t} \to T$ by $\exp(X) := \alpha_X(1)$. Using the exponential map, one can generate many vector fields on a manifold from a given torus action action. Indeed, for each $X \in \mathfrak{t}$, the vector field $\mathcal{G}(X)$ on M *generated by T-action* from X is defined by

$$\mathcal{G}(X)_p := \text{tangent vector to } \underbrace{t \mapsto \overbrace{\exp(tX)}^{\text{curve in } T} \cdot p}_{\text{curve in } M \text{ through } p} \text{ at } t = 0$$

A T-action on (M, ω) is *symplectic* if all the vector fields that it generates are *symplectic*, i.e. their flows preserve the symplectic form ω. The T-action is *Hamiltonian* if all the vector fields it generates are *Hamiltonian*, i.e. they satisfy Hamilton's PDEs. Any symplectic action on a simply connected manifold is Hamiltonian.

From a given Hamiltonian torus action, one can construct a special type of map, which encodes information about the action – this is the famous *momentum map*. The construction of the momentum map is due to Kostant [60] and Souriau [93] (we refer to Marsden-Ratiu [74, Pages 369, 370] for the history of the momentum map); the momentum map can be defined with great generality for a Hamiltonian Lie group action. The momentum map was a key tool in Kostant's quantization lectures [61] and Souriau discussed it at length in his book [94]. Here we shall only deal with the momentum map in the rather particular case of torus actions, in which the construction is simpler.

Assume that $\dim T = m, \dim M = 2n$. Let $e_1, \ldots, e_m$ be basis of the Lie algebra $\mathfrak{t}$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be the corresponding vector fields. By definition of Hamiltonian action there exists a unique (up to a constant) Hamiltonian H_i such that $\omega(\mathcal{E}_i, \cdot) := \mathrm{d}H_i$, i.e. $\mathcal{E}_i = \mathcal{H}_{H_i}$. Now we define the *momentum map* by

$$\mu := (H_1, \ldots, H_m) \colon M \to \mathbb{R}^m.$$

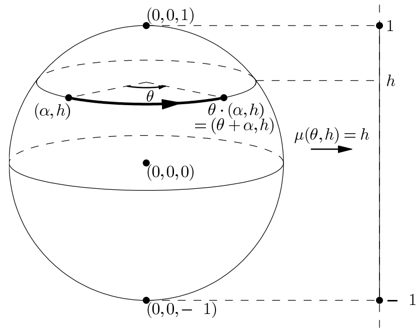

Figure 2.1: The momentum map for the 2-sphere S^2 is the height function $\mu(\theta,\, h) \;=\; h$. The image of S^2 under the momentum map μ is the closed interval $[-1,\, 1]$. Note that as predicted by the Atiyah--Guillemin-Sternberg Theorem (see Theorem 2.2), the interval $[-1,\, 1]$ is equal to the image under μ of the set $\{(0,\, 0,\, -1),\, (0,\, 0,\, 1)\}$ of fixed points of the Hamiltonian S^1-action on S^2 by rotations about the vertical axes.

The map μ is unique up to composition by an element of $\mathrm{GL}(m,\, \mathbb{Z})$ (because our construction depends on the choice of a basis) and translations in $\mathbb{R}^m$ (because the Hamiltonians are defined only up to a constant).

The simplest example of a Hamiltonian torus action is S^2 with the rotational S^1-action depicted in Figure 2.1. It is easy to check that the momentum map for this action is the height function $\mu(\theta,\, h) = h$.

In the US East Coast the momentum map has traditionally been called "moment map", while in the West Coast it has been traditional to use the term "momentum map". In French they both reconcile into the term "application moment".

2.3 Structure theorems for Hamiltonian actions

Much of the authors' intuition on integrable systems was originally guided by some remarkable results proven in the early 80s by Atiyah, Guillemin-Sternberg, and Delzant, in the context of Hamiltonian torus actions. The first of these results was the following influential convexity theorem of Atiyah, Guillemin-Sternberg.

Theorem 2.2 (Atiyah [6], Guillemin-Sternberg [54])**.** *If an m-dimensional torus acts on a compact, connected $2n$-dimensional symplectic manifold $(M,\, \omega)$ in a Hamiltonian fashion, the image $\mu(M)$ under the momentum map $\mu := (H_1, \ldots,\, H_m)\colon\, M \to \mathbb{R}^m$ is a convex polytope.*

See figures 2.1 and 2.2 for an illustration of the theorem. Other remarkable convexity theorems were proven after the theorem above by Kirwan [73] (in the case of compact, non-abelian group actions), Benoist [9] (in the case when the action is not necessarily Hamiltonian but it has some coisotropic orbit) and Giacobbe [48]. Convexity in the case of Poisson actions has been studied by Alekseev, Flaschka-Ratiu, Ortega-Ratiu and Weinstein [3, 44, 81, 117] among others.

Recall that a convex polytope in $\mathbb{R}^n$ is *simple* if there are n edges meeting at each vertex, *rational* if the edges meeting at each vertex have rational slopes, i.e. they are of the form $p + tu_i$, $0 \leq t < \infty$, where $u_i \in \mathbb{Z}^n$, and *smooth* if the vectors $u_1, \ldots, u_n$ may be chosen to be a basis of $\mathbb{Z}^n$ (see Figure 2.2). In the mid 1980s Delzant [29] showed the following classification result, which complements the Atiyah-Guillemin-Sternberg convexity theorem.

Theorem 2.3 (Delzant [29]). *If an n-dimensional torus acts effectively and Hamiltonianly on a compact, connected symplectic $2n$-dimensional manifold (M, ω), the polytope in the Atiyah-Guillemin-Sternberg theorem is simple, rational and smooth, and it determines the symplectic isomorphism type of M, and moreover, M is a toric variety in the sense of complex algebraic geometry. Starting from any simple, rational smooth polytope $\Delta \subset \mathbb{R}^m$ one can construct a compact connected symplectic manifold $(M_\Delta, \omega_\Delta)$ with an effective Hamiltonian action for which its associated polytope is Δ.*

By an isomorphism $\chi\colon (M_1, \omega_1) \to (M_2, \omega_2)$ in Theorem 2.3 we mean an equivariant symplectomorphism such that $\chi^*\mu_2 = \mu_1$, where μ_i is the momentum map of M_i, $i = 1, 2$ (the map χ is an equivariant symplectomorphism in the sense that it is a diffeomorphism which pulls back the symplectic form ω_2 to ω_1 and commutes with the torus actions). The manifolds in Delzant's theorem are called a *symplectic-toric* manifolds or *Delzant manifolds*. See Duistermaat-Pelayo [32] for a detailed study of the relation between Delzant manifolds and toric varieties in algebraic geometry. In the context of symplectic geometry, motivated by Delzant's results usually one refers to simple, rational smooth polytopes as *Delzant polytopes*.

Delzant's theorem tells us that from the point of view of symplectic geometry, complex projective spaces endowed with the standard action by rotations of a torus half the dimension of the corresponding complex projective space are simple polytopes. More precisely, consider the projective space $\mathbb{CP}^n$ equipped with a λ multiple of the Fubini–Study form and the standard rotational action of $\mathbb{T}^n$ (for $\mathbb{CP}^1 = S^2$, we already drew the momentum map in Figure 2.1). The complex projective space $\mathbb{CP}^n$ is a $2n$–dimensional symplectic-toric manifold, and one can check that the momentum map is given by

$$\mu(z) = \Big(\frac{\lambda\ |z_1|^2}{\sum_{i=0}^n\ |z_i|^2}, \ldots, \frac{\lambda\ |z_n|^2}{\sum_{i=0}^n\ |z_i|^2}\Big).$$

It follows that the momentum polytope equals the convex hull in $\mathbb{R}^n$ of 0 and the scaled canonical vectors $\lambda e_1, \ldots, \lambda e_n$, see Figure 2.2.

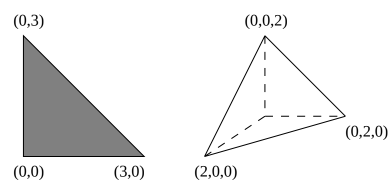

Figure 2.2: Delzant polytopes corresponding to the complex projective spaces $\mathbb{CP}^2$ and $\mathbb{CP}^3$ equipped with scalar multiples of the Fubini-Study symplectic form.

There have been many other contributions to the structure theory of Hamiltonian torus actions, particularly worth noting is Karshon's paper [58] (see also [57]), where she gives a classification of Hamiltonian circle actions on compact connected 4-dimensional symplectic manifolds; we briefly review Karshon's result. To a compact, connected 4-dimensional symplectic manifold equipped with an effective Hamiltonian S^1-action (i.e. a so called *compact 4-dimensional Hamiltonian S^1-space*), we may associate a labelled graph as follows. For each component Σ of the set of fixed points of the S^1-action there is one vertex in the graph, labelled by the real number $\mu(\Sigma)$, where $\mu\colon M \to \mathbb{R}$ is the momentum map of the action. If Σ is a surface, then the corresponding vertex has two additional labels, one being the symplectic area of Σ and the other one being the genus of Σ.

For every finite subgroup F_k of k elements of S^1 and for every connected component C of the set of points fixed by F_k we have an edge in the graph, labeled by the integer $k > 1$. The component C is a

2-sphere, which we call a F_k-*sphere*. The quotient circle S^1/F_k rotates it while fixing two points, and the two vertices in the graph corresponding to the two fixed points are connected in the graph by the edge corresponding to C.

On the other hand, it was proven by Audin, Ahara and Hattori [2, 7, 8] that every compact 4-dimensional Hamiltonian S^1-space is isomorphic (meaning S^1-equivariantly diffeomorphic) to a complex surface with a holomorphic S^1-action which is obtained from $\mathbb{CP}^2$, a Hirzebruch surface or a $\mathbb{CP}^1$-bundle over a Riemann surface (with appropriate circle actions), by a sequence of blow-ups at the fixed points.

Let A and B be connected components of the set of fixed points. The S^1-action extends to a holomorphic action of the group $\mathbb{C}^\times$ of non-zero complex numbers. Consider the time flow given by the action of subgroup $\exp(t)$, $t \in \mathbb{R}$. We say that A *is greater than* B if there is an orbit of the $\mathbb{C}^\times$-action which at time $t = \infty$ approaches a point in A and at time $t = -\infty$ approaches a point in B.

Take any of the complex surfaces with S^1-actions considered by Audin, Ahara and Hattori, and assign a real parameter to every connected component of the set of fixed points such that these parameters are monotonic with respect to the partial ordering we have just described. If the manifold contains two fixed surfaces then assign a positive real number to each of them in such a way that the difference between the numbers is given by a formula involving the previously chosen parameters. Karshon proved [57, Theorem 3] that for every such a choice of parameters there exists an invariant symplectic form and a momentum map on the complex surface such that the values of the momentum map at the fixed points and the symplectic areas of the fixed surfaces are equal to the chosen parameters. Moreover, every two symplectic forms with this property differ by an S^1-equivariant diffeomorphism. Karshon proved the following classification result *à la Delzant*.

Theorem 2.4 (Karshon [58])**.** *If two compact Hamiltonian S^1-spaces*[4] *have the same graph, then they are isomorphic (i.e. S^1-equivariantly symplectomorphic). Moreover, every compact 4-dimensional Hamiltonian S^1-space is isomorphic to one of the spaces listed in the paragraph above.*

Again, in Theorem 2.4, an isomorphism is an equivariant symplectormphism which pulls back the momenum map on one manifold to the momentum map on the other manifold. Theorem 2.4 has useful consequences, for example: every compact Hamiltonian S^1-space admits a S^1-invariant complex structure for which the symplectic form is Kähler.

2.4 Structure theorems for symplectic actions

From the viewpoint of symplectic geometry, the situation described by the momentum polytope is very rigid. It is natural to wonder whether the structure results Theorem 2.2 and Theorem 2.3 for Hamiltonian actions of tori persist in a more general context. There are at least two natural ways to approach this question, which we explain next.

First, one can insist on having a compact group action, but not requiring that the group acts in a Hamiltonian fashion. In other words do the striking theorems above persist if the vector fields generated by action have flows that preserve symplectic form i.e. are symplectic, but Hamilton's PDEs have no solution i.e. the vector fields are not Hamiltonian? Many easy examples fit this criterion, e.g. take the 2-torus $\mathbb{T}^2$ with the standard area form $\mathrm{d}\theta \wedge \mathrm{d}\alpha$ and with the $\mathbb{S}^1$-action on the θ-component; the basic vector field $\frac{\partial}{\partial \theta}$ is symplectic but one can easily check that it is non-Hamiltonian.

[4] I.e. two compact connected 4-dimensional manifolds equipped with an effective Hamiltonian S^1-action.

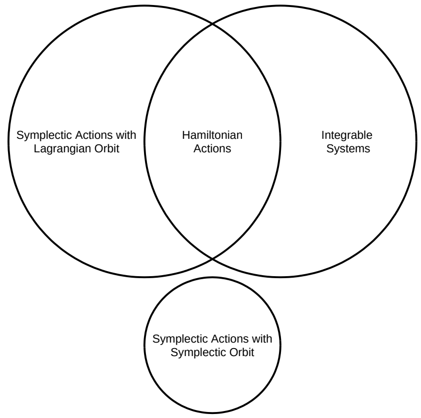

Figure 2.3: Hamiltonian torus actions may be viewed as a subclass of completely integrable Hamiltonian systems, which we study later in this paper, and a subclass of general symplectic actions which have some Lagrangian orbit. If the torus has dimension n and the manifold has dimension $2n$, Hamiltonian torus actions are a subclass of completely integrable systems.

Various works by Giacobbe [48], Benoist [9], Ortega-Ratiu [80], Duistermaat-Pelayo [31], and Pelayo [82] follow this direction. Benoist's paper gives a convexity result for symplectic manifolds with coisotropic orbits, Ortega-Ratiu give a general symplectic local normal form theorem, also studied by Benoist in the case that the orbits are coisotropic. The papers by Duistermaat and Pelayo provide classifications *à la Delzant*. Let us briefly recall these classifications.

Remark 2.5 Hamiltonian torus actions on compact manifolds always have fixed points (equivalently, the Hamiltonian vector fields generated by a Hamiltonian torus actions always have fixed points). Sometimes the condition of being Hamiltonian for vector fields can be detected from the existence of fixed points; this is in general a challenging question.

The first result concerning the relationship between the existence of fixed points and the Hamiltonian character of vector fields generated by a G-action is Frankel's celebrated theorem [46] which says that if the manifold is compact, connected, and Kähler, $G = S^1$, and the symplectic action has fixed points, then it must be Hamiltonian. Frankel's influential work has inspired subsequent research. McDuff [78, Proposition 2] has shown that any symplectic circle action on a compact connected symplectic 4-manifold having fixed points is Hamiltonian. See Tolman-Weitsman [100, Theorem 1], Feldman [47, Theorem 1], [63, Section 8], [71], Giacobbe [48, Theorem 3.13], Duistermaat-Pelayo [31, Corollary 3.9], Ginzburg [49, Proposition 4.2], Pelayo-Tolman [85] for additional results in the case of compact manifolds, and Pelayo-Ratiu [83] for results in the case of non-compact manifolds. $\oslash$

Our next goal is to present a classification *à la Delzant* of symplectic torus actions that have some Lagrangian orbit; this in particular includes all symplectic toric manifolds, because the maximal (in the sense of dimension) orbits of a symplectic-toric manifold are Lagrangian. An n-dimensional submanifold L of a symplectic $2n$-manifold (M, ω) is *Lagrangian* if the symplectic form ω vanishes on L. For example, the orbits of the S^1-action by rotations on S^2 in Figure 2.1 are Lagrangian because an orbit is given by $h =$ const for some constant and the symplectic form is $\mathrm{d}\theta \wedge \mathrm{d}h$, which clearly vanishes when h is constant. So the maximal orbits of the standard symplectic 2-sphere are Lagrangian.

A famous example of a symplectic manifold with a 2-torus action for which all the orbits are Lagrangian is the the *Kodaira-Thurston manifold*. It is constructed as follows: let $(j_1,\, j_2) \in \mathbb{Z}^2$ act on $\mathbb{R}^2$ by the inclusion map on $\mathbb{T}^2$ (i.e. $(j_1,\, j_2) \cdot (x_1,\, y_1) = (j_1 + x_1,\, j_2 + y_1)$), by the 2 by 2 matrix with entries $a_{11} = a_{22} = 1$, $a_{12} = j_2$, $a_{21} = 0$, and on the product $\mathbb{R}^2 \times \mathbb{T}^2$ by the diagonal action. This diagonal action gives rise to a torus bundle over a torus $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$, the total space of which is compact and connected. The product symplectic form $\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$ on $\mathbb{R}^2 \times \mathbb{Z}^2$ descends to a symplectic form on $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$.

Moreover, one can check that $\mathbb{T}^2$ acts symplectically on $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$, where the first circle of $\mathbb{T}^2$ acts on the left most component of $\mathbb{R}^2$, and the second circle acts on the right most component of $\mathbb{T}^2$ (one can check that this is indeed a well-defined, free symplectic action). Because the action is free, it does not have fixed points, and hence it is not Hamiltonian (it follows from the Atiyah-Guillemin-Sternberg theorem that Hamiltonian actions always have some fixed point). All the orbits of this action are Lagrangian submanifolds, because both factors of the symplectic form vanish since each factor has a component which is zero because it is the differential of a constant.

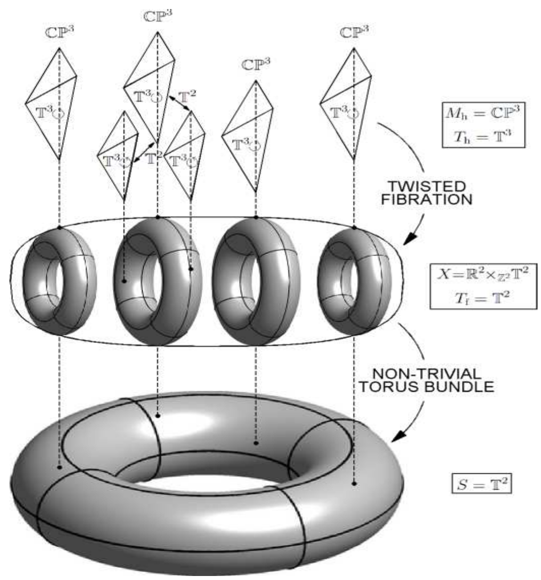

Figure 2.4: 10-dimensional symplectic manifolds with a torus action with Lagrangian orbits. Vector field generated by T-action is "twist" of $\mathcal{Y} = (\mathcal{Y}_{\mathrm{h}}, \mathcal{Y}_{\mathrm{f}})$, where $\mathcal{Y}_{\mathrm{h}}$ Hamiltonian on $\mathbb{CP}^3$, $\mathcal{Y}_{\mathrm{f}}$ symplectic on $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$.

Another example is $\mathbb{T}^2 \times S^2$ equipped with the form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\theta \wedge \mathrm{d}h$, on which the 2-torus $\mathbb{T}^2$ acts symplectically, one circle on each factor. This action has no fixed points, so it is not Hamiltonian. It is also not free. The free orbits are Lagrangian. All of these examples, fit in the following theorem.

Theorem 2.6 (Duistermaat-Pelayo [31])**.** *Assume that a torus $\mathbb{T}^m$ of dimension m acts effectively and symplectically on a compact, connected symplectic $2m$-manifold (M, ω) with some Lagrangian orbit. Then $\mathbb{T}^m$ decomposes as a product of two subtori $\mathbb{T}^m = T_{\mathrm{h}} T_{\mathrm{f}}$, where T_{h} acts Hamiltonianly on M and T_{f} acts freely on M, and there is two-step fibration $M \to X \to S$, where M is the total space of a fibration over X with fibers symplectic-toric manifolds $(M_{\mathrm{h}}, T_{\mathrm{h}})$, and X is a T_{f}-bundle over a torus S of dimension $m - \dim T_{\mathrm{h}}$.*

In this theorem X is a symplectic homogenous space for the twisted group $T \times \mathfrak{t}^*$. The formulation of this theorem in [31] is completely explicit, but it is too involved to be described here. In particular, the formulation contains a complete symplectic classification in terms of six symplectic invariants (e.g. the Chern class of the fibration, the Hamiltonian torus T_{h}, the polytope corresponding to the Hamiltonian action of T_{h} etc). This classification includes Delzant's classification (stated previously in the paper as Theorem 2.3), which corresponds to the case of $T_{\mathrm{h}} = \mathbb{T}^m$ and T_{f} trivial; in this case five of the invariants do no appear, the only invariant is the polytope. Note that the "opposite" situation occurs when $T_{\mathrm{f}} = \mathbb{T}^m$ and T_{h} trivial (eg. the Kodaira-Thurston manifold), and in this case Theorem 2.6 says that M is a torus bundle over a torus with Lagrangian fibers. An example of a manifold which fits in Theorem 2.6 is the family of 10-dimensional twisted examples with Lagrangian orbits illustrated in Figure 2.4.

A classification theorem in the case when there exists a maximal symplectic orbit (i.e. an orbit on which the symplectic form restricts to a symplectic form) was proven in [82]. In this same paper, a classification of symplectic actions of 2-tori on compact connected symplectic 4-manifolds was given building on this result and Theorem 2.6, but the description is involved for the purposes of the present article.

Theorem 2.7 (Pelayo [82], Duistermaat-Pelayo [33])**.** *A compact connected symplectic 4-manifold (M, ω) equipped with an effective symplectic action of a 2-torus is isomorphic (i.e. equivariantly symplectomorphism) to one, and only one, of the spaces in the table:*

SPACE	ACTION	MAXIMAL ORBITS	HAMILTONIAN?	INVARIANT COMPLEX?	KÄHLER?
Toric	*Fixed points*	*Lagrangian*	*Yes*	*Yes*	*Yes*
$M \to \Sigma$	*Locally Free*	*Symplectic*	*No*	*Yes*	*Yes*
$M \to \mathbb{T}^2$	*Free*	*Lagrangian*	*No*	*Yes*	*No*
$\mathbb{T}^2 \times S^2$	*Else*	*Lagrangian*	*No*	*Yes*	*Yes*

The first item is a symplectic-toric manifold with its standard Hamiltonian 2-torus action, the second item is an orbifold 2-torus bundle over a 2-dimensional compact connected orbifold, the third item is a 2-torus bundle over a 2-torus.

The first four columns in Theorem 2.7 were proven in [82], and the last two columns were proven in Duistermaat-Pelayo [31]; the article [31] is based on Kodaira's seminal work [64, Theorem 19] of 1961 on complex analytic surfaces. Moreover, the two middle items above are completely explicit and classified in terms of five symplectic invariants, cf. [82, Theorem 8.2.1].

Another natural generalization of a Hamiltonian torus action is the notion of a completely integrable system, or more generally, of a Hamiltonian system. Probably the most fundamental difference between the theory of Hamiltonian torus actions on compact manifolds and the theory of completely integrable Hamiltonian systems can be seen already at a local level. Completely integrable systems have in general singularities that are quite difficult to understand from a topological, dynamical and analytic view-point. The singularities of Hamiltonian torus actions occur at the lower dimensional orbits only, and are tori of varying dimensions, but case of integrable systems will usually exhibit a wider range of singularities such as pinched tori, eg see Figure 3.1. Indeed, it is only recently that some of these singularities are beginning to be understood in low dimensions. The rest of this paper is focused on the local and global aspects of the symplectic geometry of completely integrable systems.

3 Completely Integrable Systems

3.1 Hamiltonian integrable systems

Let $(M,\,\omega)$ be a $2n$-dimensional symplectic manifold. The pair consisting of the smooth manifold and a classical observable $H\colon\, M \to \mathbb{R}$ in $\mathrm{C}^\infty(M)$ is called a *Hamiltonian system*.

A famous example of a Hamiltonian system is the *spherical pendulum*, which is mathematically described as the symplectic cotangent bundle $(\mathrm{T}^*\,S^2, \omega_{\mathrm{T}^*\,S^2})$ of the unit sphere equipped with the Hamiltonian

$$H(\underbrace{\varphi,\theta}_{\text{sphere}},\underbrace{\xi,\eta}_{\text{fiber}}) = \underbrace{= \frac{1}{2}(v^2+u^2\sin^2\theta)}_{\text{kinetic}} + \underbrace{\sin\theta\sin\varphi}_{\text{potential}}.$$

A classical observable H gives rise to the *Hamiltonian vector field* $\mathcal{H}_H$ on M defined uniquely by $\omega(\mathcal{H}_H,\,\cdot) = \mathrm{d}H$. The algebra $\mathrm{C}^\infty(M)$ of classical observables, which we have been calling *Hamiltonians*, comes naturally endowed with the Poisson bracket: $\{J,\,H\} := \omega(\mathcal{H}_J,\,\mathcal{H}_H)$. As a derivation, $\mathcal{H}_H$ is just the Poisson bracket by H; in other words the evolution of a function f under the flow of $\mathcal{H}_H$ is given by the equation $\dot{f} = \{H,\,f\}$.

An *integral* of the Hamiltonian H is a function which is invariant under the flow of $\mathcal{H}_H$, i.e. a function f such that $\{H,\,f\} = 0$. The Hamiltonian H is said to be *completely integrable* if there exists $n-1$ independent functions $f_2,\ldots,f_n$ (*independent* in the sense that the differentials $\mathrm{d}_m f_2,\,\ldots,\,\mathrm{d}_m f_n$ are linearly independent at almost every point $m \in M$) that are integrals of H and that pairwise Poisson commute, i.e.[5] $\{H,\,f_i\} = 0$ and $\{f_i,\,f_j\} = 0$. For example, the spherical pendulum $(\mathrm{T}^*S^2, \omega_{\mathrm{T}^*S^2}, H)$ is integrable by considering the function $f_2(\varphi,\theta,\eta,\xi) = \frac{1}{2}v^2 + \frac{1}{2}u^2\sin^2\theta + \sin\theta\sin\varphi$. It is clear that H does not play a distinguished role among the functions $f_2,\ldots,f_n$. The point of view in this paper will always be to consider, as a whole, a collection of such functions.

Definition 3.1 A *completely integrable system* on the $2n$-dimensional symplectic manifold M, compact or not, is a collection of n Poisson commuting functions $f_1,\ldots,f_n \in \mathrm{C}^\infty(M)$ which are independent.

An important class of completely integrable systems are those given by Hamiltonian n-torus actions on symplectic $2n$-manifolds (i.e. symplectic-toric manifolds). These actions have a momentum map with n components $f_1,\ldots,f_n$, and these components always form a completely integrable system in the sense of the definition above.

3.2 Singularities and regular points

From a topological, analytical and dynamical view-point, the most interesting features of the completely integrable system on a symplectic manifold are encoded in the "singular" fibers of the momentum map $F = (f_1,\,\ldots,\,f_n) : M \to \mathbb{R}^n$, and in their surrounding neighborhoods.

A point $m \in M$ is called a *regular point* if $\mathrm{d}_m\,F$ has rank n. A point $c \in \mathbb{R}^n$ is a *regular value* if the fiber $F^{-1}(c)$ contains only regular points. If c is a regular value, the fiber $F^{-1}(c)$ is called a *regular fiber*. A point $m \in M$ is a *critical point*, or a *singularity*, if $\mathrm{d}_m\,F$ has rank strictly less than n. Geometrically this means that the Hamiltonian vector fields generated by the components of F are linearly dependent at m, see Figure 3.1. A fiber $F^{-1}(c)$ is a *singular fiber* if it contains at least one critical point, see figure 5.4.

[5]This condition is a consequence of the former for "generic" H. However, when studying particular models or *normal forms*, which are not generic, this condition is crucial.

It follows from the definition of a completely integrable system (simply follow the flows of the Hamiltonian vector fields) that if X is a connected component of a regular fiber $F^{-1}(c)$, where we write

$$F := (f_1, \ldots, f_n) \colon M \to \mathbb{R}^n,$$

and if the vector fields $\mathcal{H}_{f_1}, \ldots, \mathcal{H}_{f_n}$ are complete on $F^{-1}(c)$, then X is diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$. Moreover, if the regular fiber $F^{-1}(c)$ is compact, then $\mathcal{H}_{f_1}, \ldots, \mathcal{H}_{f_n}$ are complete, and thus the component X is diffeomorphic to $\mathbb{T}^n$; this is always true of for example some component f_i is proper.

The study of singularities of integrable systems is fundamental for various reasons. On the one hand, because of the way an integrable system is defined in terms n smooth functions on a manifold, it is expected (apart from exceptional cases) that singularities will necessarily occur. On the other hand these functions define a dynamical system such that their singularities correspond to fixed points and relative equilibria of the system, which are of course one of the main characteristics of the dynamics.

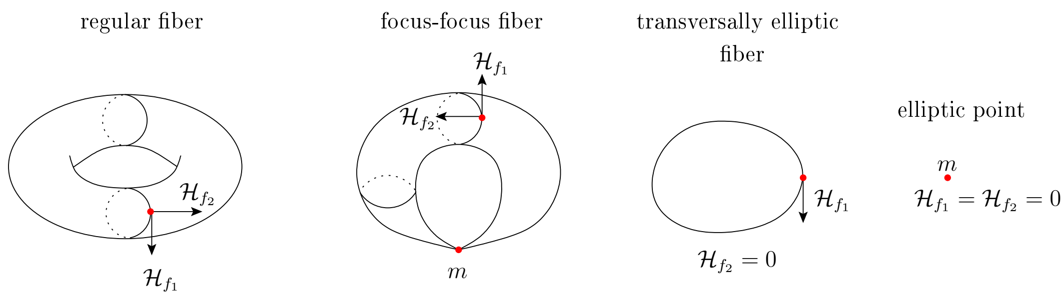

Figure 3.1: The figures show some possible singularities of a completely integrable system. On the left most figure m is a regular point (rank 2); on the second figure m is a focus-focus point (rank 0); on the third one m is a transversally elliptic singularity (rank 1); on the right most figure m is a elliptic elliptic point.

As a remark for those interested in semiclassical analysis we note that from a semiclassical viewpoint, we know furthermore that important wave functions such as eigenfunctions of the quantized system have a microsupport which is invariant under the classical dynamics; therefore, in a sense that we shall not present here (one should talk about semiclassical measures), they concentrate near certain singularities[6] (see for instance [21] and the work of Toth [101]). This concentration entails not only the growth in norm of eigenfunctions (see for instance [102]) but also a higher local density of eigenvalues [22, 104, 25]).

Let $f_1, \ldots, f_n$ define an integrable system on a symplectic manifold M, and let F be the associated momentum map. Suppose that F is a proper map so that the regular fibers of F are n-dimensional tori. Indeed, Liouville proved in 1855 [70] that, locally, the equations of motion defined by any of the functions f_i are integrable by quadratures. This holds in a neighborhood of any point where the differentials $\mathrm{d}f_j$ are linearly independent.

A pleasant formulation of Liouville's result, due to Darboux and Carathéodory, says that there exist canonical coordinates (x, ξ) in which the functions f_j are merely the "momentum coordinates" ξ_j. In 1935 Henri Mineur [72] stated[7] in the special case of $\mathbb{R}^n \times \mathbb{R}^n$ that if Λ is a compact level set of the momentum map $F = (f_1, \ldots, f_n)$, then Λ is a torus. Moreover, there exist symplectic coordinates (x, ξ), where x varies in the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and ξ varies in a neighborhood of the origin in $\mathbb{R}^n$, in which the functions f_j depend

[6] those called hyperbolic.

[7] J.J. Duistermaat has pointed out [35] some gaps in Mineur's proof.

only on the ξ-variables. In geometric terms, the system is symplectically equivalent to a neighborhood of the zero section of the cotangent bundle $\mathrm{T}^*(\mathbb{T}^n)$ equipped with the integrable system $(\xi_1, \ldots, \xi_n)$. This result, proved in the general case in 1963 by Arnold [4] (Arnold was not aware of Mineur's work), is known as the *action-angle theorem* or the *Liouville-Arnold theorem*. The tori are the famous *Liouville tori*. Although Liouiville's theorem has been originally attached to this theorem for some time, we are not aware of Liouville having contributed to this result; we thank J.J. Duistermaat for pointing this out to us [35].

We will study these and other results in more detail in the following two sections.

4 Local theory of completely integrable systems

4.1 Local model at regular points

Let $(f_1, \ldots, f_n)$ be a completely integrable system on a $2n$-symplectic manifold M, with momentum map F. By the local submersion theorem, the fibers $F^{-1}(c)$ for c close to $F(m)$ are locally n-dimensional submanifolds near a regular point m. The local structure of regular points of completely integrable systems is simple:

Theorem 4.1 (Darboux-Carathéodory)**.** *Let $(f_1, \ldots, f_n)$ be a completely integrable system on a $2n$-symplectic manifold M, with momentum map F. If m is regular, F is symplectically conjugate near m to the linear fibration $(\xi_1, \ldots, \xi_n)$ on the symplectic space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and symplectic form $\sum_i \mathrm{d}\xi_i \wedge \mathrm{d}x_i$.*

In other words, the Darboux-Carathéodory theorem says that there exists smooth functions $\phi_1, \ldots, \phi_n$ on M such that $(\phi_1, \ldots, \phi_n, f_1, \ldots, f_n)$ is a system of canonical coordinates in a neighborhood of m. In principle the name of Liouville should be associated with this theorem, since well before Darboux and Carathéodory, Liouville gave a nice explicit formula for the functions ϕ_j. This result published in 1855 [70] explains the local integration of the flow of any completely integrable Hamiltonian (possibly depending on time) near a regular point of the foliation in terms of the *Louville 1-form* $\sum_i \xi_i \mathrm{d}x_i$. In this respect it implies the Darboux-Carathéodory theorem, even if Liouville's formulation is more complicated.

4.2 Local models at singular points

One can approach the study of the singularities of Hamiltonian systems in two different ways: one can analize the flow of the vector fields — this is the "dynamical systems" viewpoint — or one can study of the Hamiltonian functions themselves — this is the "foliation" perspective.

In the case of *completely* integrable systems, the dynamical and foliation points of view are equivalent because the vector fields of the n functions $f_1, \ldots, f_n$ form a basis of the tangent spaces of the leaves of the foliation $f_i = \mathrm{const}_i$, at least for regular points. The foliation perspective usually displays better the geometry of the problem, and we will frequently use this view-point. However, the foliations we are interested in are *singular*, and the notion of a singular foliation is already delicate. Generally speaking these foliations are of Stefan-Süßmann type [95] : the leaves are defined by an integrable distribution of vector fields. But they are more than that: they are Hamiltonian, and they are *almost regular* in the sense that the singular leaves cannot fill up a domain of positive measure.

4.2.1 Non-degenerate critical points

In singularity theory for differentiable functions, "generic" singularities are Morse singularities. In the theory of completely integrable systems there exists a natural analogue of the notion of Morse singularities (or more generally of Morse-Bott singularities if one allows critical submanifolds). These so-called *non-degenerate* singularities are now well defined and exemplified in the literature, so we will only recall briefly the definition in Vey's paper [103].

Let $F = (f_1, \ldots, f_n)$ be a completely integrable system on M. A fixed point $m \in M$ is called *non-degenerate* if the Hessians $\mathrm{d}_m^2 f_j$ span a Cartan subalgebra of the Lie algebra of quadratic forms on the tangent space $\mathrm{T}_m M$ equipped with the linearized Poisson bracket. This definition applies to a fixed point; more generally: if $\mathrm{d}_m F$ has corank r one can assume that the differentials $\mathrm{d}_m f_1, \ldots, \mathrm{d}_m f_{n-r}$ are linearly independent; then we consider the restriction of $f_{n-r+1}, \ldots, f_n$ to the symplectic manifold Σ obtained by local symplectic reduction under the action of $f_1, \ldots, f_{n-r}$. We shall say that m is *non-degenerate* (or *transversally non-degenerate*) whenever m is a non-degenerate fixed point for this restriction of the system to Σ.

In order to understand the following theorem one has to know the linear classification of Cartan subalgebras of $\mathrm{sp}(2n, \mathbb{R})$. This follows from the work of Williamson [111], which shows that any such Cartan subalgebra has a basis build with three type of blocks: two uni-dimensional ones (the elliptic block: $q = x^2 + \xi^2$ and the real hyperbolic one: $q = x\xi$) and a two-dimensional block called focus-focus: $q_1 = x\eta - y\xi, \quad q_2 = x\xi + y\eta$. If $k_{\mathrm{e}}, k_{\mathrm{h}}, k_{\mathrm{f}}$ respectively denote the number of elliptic, hyperbolic and focus-focus components, we may associate the triple $(k_{\mathrm{e}}, k_{\mathrm{h}}, k_{\mathrm{f}})$ to a singularity. The triple is called by Zung the *Williamson type of the singularity*.

Theorem 4.2 (Eliasson [41, 42, 110])**.** *The non-degenerate critical points of a completely integrable system $F\colon M \to \mathbb{R}^n$ are linearizable, i.e. if $m \in M$ is a non-degenerate critical point of the completely integrable system $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ then there exist local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about m, in which m is represented as $(0, \ldots, 0)$, such that $\{f_i, q_j\} = 0$, for all indices i, j, where we have the following possibilities for the components $q_1, \ldots, q_n$, each of which is defined on a small neighborhood of $(0, \ldots, 0)$ in $\mathbb{R}^n$:*

(i) *Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$, where j may take any value $1 \leq j \leq n$.*

(ii) *Hyperbolic component: $q_j = x_j \xi_j$, where j may take any value $1 \leq j \leq n$.*

(iii) *Focus-focus component: $q_{j-1} = x_{j-1}\,\xi_j - x_j\,\xi_{j-1}$ and $q_j = x_{j-1}\,\xi_{j-1} + x_j\,\xi_j$ where j may take any value $2 \leq j \leq n-1$ (note that this component appears as "pairs").*

(iv) *Non-singular component: $q_j = \xi_j$, where j may take any value $1 \leq j \leq n$.*

Moreover if m does not have any hyperbolic block, then the system of commuting equations $\{f_i, q_j\} = 0$, for all indices i, j, may be replaced by the single equation

$$(F - F(m)) \circ \varphi = g \circ (q_1, \ldots, q_n),$$

where $\varphi = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)^{-1}$ and g is a local diffeomorphism of $\mathbb{R}^n$ such that $g(0, \ldots, 0) = (0, \ldots, 0)$.

If the dimension of M is 4 and F has no hyperbolic singularities – which is the case which is most important to us in this paper – we have the following possibilities for the map (q_1, q_2), depending on the rank of the critical point:

(1) if m is a critical point of F of rank zero, then q_j is one of

  (i) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = (x_2^2 + \xi_2^2)/2$.

  (ii) $q_1 = x_1\xi_2 - x_2\xi_1$ and $q_2 = x_1\xi_1 + x_2\xi_2$; on the other hand,

(2) if m is a critical point of F of rank one, then

  (iii) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = \xi_2$.

In this case, a non-degenerate critical point is respectively called *elliptic-elliptic*, *transversally-elliptic* or *focus-focus* if both components $q_1,\ q_2$ are of elliptic type, one component is of elliptic type and the other component is ξ, or $q_1,\ q_2$ together correspond to a focus-focus component.

The analytic case of Eliasson's theorem was proved by Rüßmann [90] for two degrees of freedom systems and by Vey [103] in any dimension. In the C^∞ category the *lemme de Morse isochore* of Colin de Verdière and Vey [23] implies Eliasson's result for one degree of freedom systems. Eliasson's proof of the general case was somewhat loose at a crucial step, but recently this has been clarified [76].

4.2.2 Degenerate critical points

Degenerate critical points appear in many applications, i.e. rigid body dynamics. The study of non--degenerate critical points of integrable systems is difficult, and little is known in general. A few particular situations are relatively understood. For analytic systems with one degree of freedom a more concrete method is presented in [24]. A general linearization result in the analytic category is given in [116]. Another result may be found in [114]. Further study of degenerate singularities may be found in Kalashnikov [56] where a semiglobal topological classification of stable degenerate singularities of corank 1 for systems with two degrees of freedom is given, Bolsinov-Fomenko-Richter [12] where it was shown how semiglobal topological invariants of degenerate singularities can be used to describe global topological invariants of integrable systems with two degrees of freedom and Nekhoroshev-Sadovskii-Zhilinskii [79], where the so called fractional monodromy phenomenon is explained via topological properties of degenerate singularities corresponding to higher order resonances. The best context to approach these singularities is probably algebraic geometry, and we hope that this article may bring some additional interactions between algebraic geometers and specialists on integrable systems; interactions have began to develop in the context of mirror symmetry, where the study of singular Lagrangian fibrations is relevant [52, 53, 51, 50, 13, 14, 15].

Throughout this paper, and unless otherwise stated, we assume that all singularities are non-degenerate.

5 Semiglobal theory of completely integrable systems

If one aims at understanding the geometry of a completely integrable foliation or its microlocal analysis, the *semiglobal* aspect is probably the most fundamental. The terminology semiglobal refers to anything that deals with an invariant neighborhood of a leaf of the foliation. This semiglobal study is what allows for instance the construction of quasimodes associated to a Lagrangian submanifold. Sometimes semiglobal merely reduces to local, when the leaf under consideration is a critical point with only elliptic blocks.

5.1 Regular fibers

The analysis of neighborhoods of regular fibers, based on the so called Liouville-Arnold-Mineur theorem (also known as the action-angle theorem) is now routine and fully illustrated in the literature, for classical aspects as well as for quantum ones. It is the foundation of the whole modern theory of completely integrable systems in the spirit of Duistermaat's seminal article [30], but also of KAM-type perturbation theorems.

The microlocal analysis of action-angle variables starts with the work of Colin de Verdière [20], followed in the $\hbar$ semiclassical theory by Charbonnel [16] and more recently by the second author and various articles by Zelditch, Toth, Popov, Sjöstrand and many others. The case of compact symplectic manifolds has recently started, using the theory of Toeplitz operators [19].

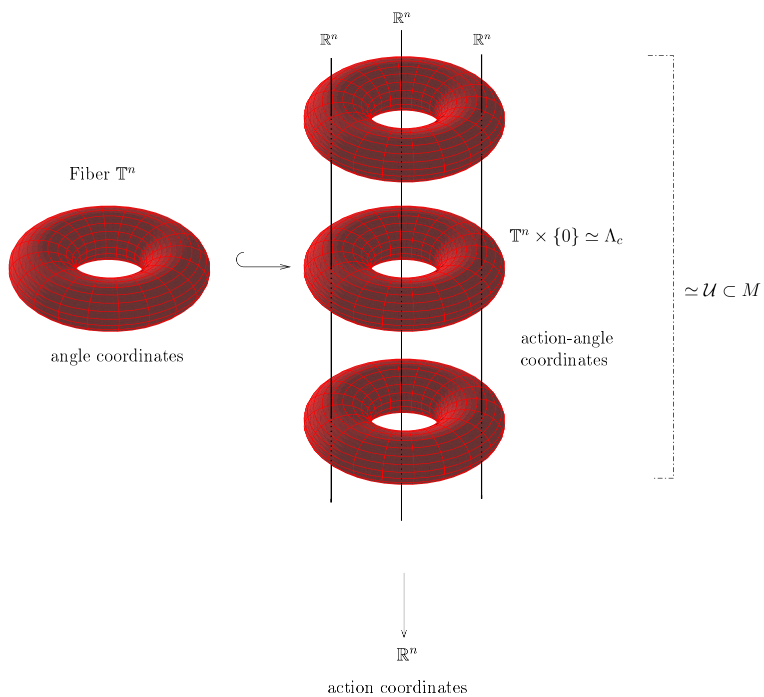

Figure 5.1: According to the Arnold-Liouville-Mineur theorem a tubular neighborhood $\mathcal{U}$ of a regular fiber Λ_c embeds symplectically into $\mathrm{T}^*(\mathbb{T}^n) \simeq \mathbb{T}^n \times \mathbb{R}^n$.

Let $(f_1, \ldots, f_n)$ be an integrable system on a symplectic manifold M. For the remainder of this article we shall assume the momentum map F to be *proper*, in which case all fibers are compact. Let c be a regular value of F. If we restrict to an adequate invariant open set, we can always assume that the fibers of F are connected. Let $\Lambda_c := F^{-1}(c)$. The fibers being compact and parallelizable (by means of the vector fields $\mathcal{H}_{f_i}$), they are tori. In what follows we identify $\mathrm{T}^*\,\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq S^1$, equipped with coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ such that the canonical Liouville 1-form is $\sum_i \xi_i \, \mathrm{d}x_i$.

Theorem 5.1 (Liouville-Arnold-Mineur [72, 4])**.** *If Λ_c is regular, there exists a local symplectomorphism χ from the cotangent bundle $\mathrm{T}^*\,\mathbb{T}^n$ of $\mathbb{T}^n$ into M sending the zero section onto the regular fiber Λ_c in such a way that that $F \circ \chi = \varphi \circ (\xi_1, \ldots, \xi_n)$ for φ a local diffeomorphism of $\mathbb{R}^n$.*

It is important to remark that $\mathrm{d}\varphi$ is an invariant of the system since it is determined by periods of periodic trajectories of the initial system. Regarded as functions on M the ξ_j's are called *action variables* of the system for one can find a primitive α of ω in a neighborhood of Λ_c such that the ξ_j's are integrals of α on a basis of cycles of Λ_c depending smoothly on c. The coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ are known as *action-angle variables*. See [66] for a version of the action-angle theorem in the case of Poisson manifolds.

5.2 Singular fibers

This section is devoted to the semi-global structure of fibers with non-degenerate singularities. We are only aware of a very small number of semi-global results for degenerate singularities as mentioned in Section 4.2.2.

The topological analysis of non-degenerate singular fibers was mainly initiated by Fomenko [45] and was successfully expanded by a number of his students cf. [11]. As far as we know Lerman and Umanskii [67, 68] were certainly among the first authors who systematically studied critical points of Poisson actions on symplectic 4-manifolds; their paper [67] is an English translation of their original paper which was written in Russian and published in a local journal. This paper is probably the first where focus-focus singularities are treated in detail. These works by Lerman and Umanskii appear to have had an essential influence on the Fomenko school. We would also like to mention that M. Kharlamov appears to be the first author who systematically did a topological analysis of integrable systems in rigid body dynamics. His results and methods [59] were precursors of various aspects of the mathematical theory of Hamiltonian systems that we discuss in the present paper (unfortunately [59] has not been translated into English yet, but many of the references therein refer to English papers of the author where his original results may be found).

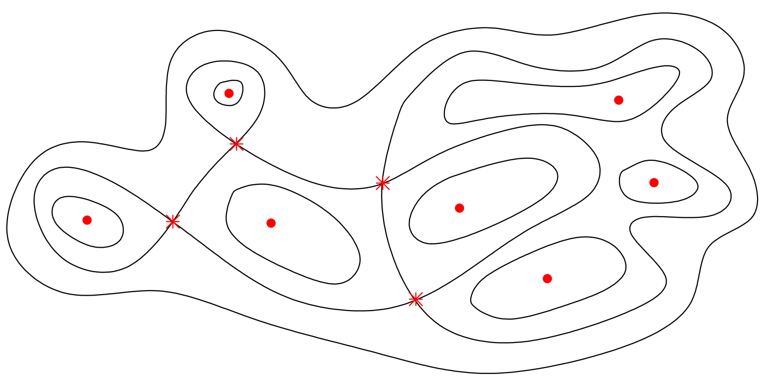

Figure 5.2: Elliptic and hyperbolic singularities on a surface (the figure shows how in the same leaf there may be several singularities, though we make the generic assumption that there is at most one singularity per fiber). Hyperbolic singularities are represented by a red star, elliptic singularities are represented by a red dot. Note how nearby a hyperbolic singularity the local model looks as in Figure 5.3. Around an elliptic singularity the leaves are concentric circles around the singularity.

5.2.1 Elliptic case

Near an elliptic fixed point, the fibers are small tori and are entirely described by the local normal form, for classical systems as well as for semiclassical ones (the system is reduced to a set of uncoupled harmonic oscillators). Therefore we shall not talk about this type of singularity any further, even if strictly speaking the semi-global semiclassical study has not been fully carried out for *transversally elliptic* singularities. But no particular difficulties are expected in that case.

5.2.2 Hyperbolic case

Just as elliptic blocks, hyperbolic blocks have 1 degree of freedom (normal form $q_i = x_i\xi_i$); but they turn out to be more complicated. However, in the particular case that M is a surface there is a classification due to Dufour, Molino and Toulet, which we present next. In addition to the result, this classification is interesting to us because it introduces a way to construct symplectic invariants which is similar to the way symplectic invariants are constructed for focus-focus singularities (which will be key to study semitoric integrable systems in sections 6, 7, 8, of this paper). Moreover, using this classification as a stepping stone, Dufour, Molino and Toulet gave a global symplectic classification of completely integrable systems on surfaces, which serves as an introduction to the recent classification of semitoric integrable systems on symplectic 4-manifolds given later in the paper.

Two one-degree of freedom completely integrable systems $(M,\,\omega,\,f_1)$ and $(M,\,\omega',\,f_1')$ are *isomorphic* if there exists a symplectomorphism $\chi\colon M \to M'$ and a smooth map g such that $\chi^* f_1' = g \circ f_1$. A (non-degenerate) critical point p of $(M,\,\omega,\,f_1)$ is either elliptic or hyperbollic (there cannot be focus-focus points).

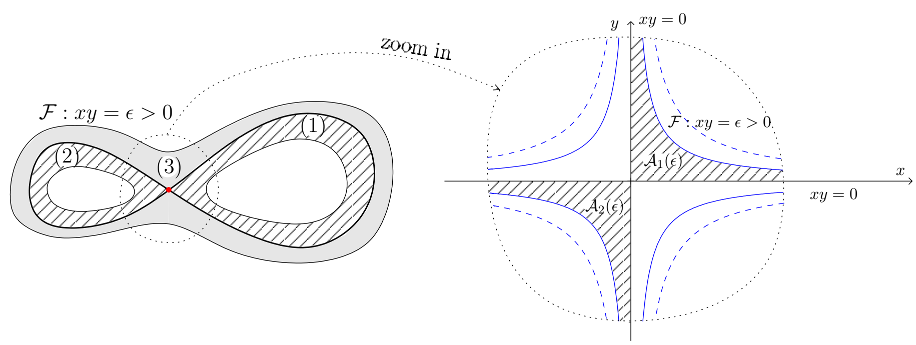

Figure 5.3: Zoom in around a hyperbolic singularity at the intersection of the x and y axes.

If p is elliptic, there exist local coordinates $(x,\,y)$ and a function g in a neighborhood V of p such that $f_1 = g(x^2+y^2)$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$, so geometrically the integral curves of the Hamiltonian vector field $\mathcal{H}_{f_1}$ generated by f_1 are concentric circles centered at p. If p is hyperbolic, there exist local coordinates $(x,\,y)$ in a neighborhood U of p and a function h such that $f_1 = h(xy)$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$. In this case the integral curves of $\mathcal{H}_{f_1}$ are hyperboloid branches $xy =$ constant. One usually calls these integral curves the *leaves* of the foliation induced by f_1. We make the generic assumption that our systems may have at most one singularity per leaf of the induced foliation.

The saturation of U by the foliation has the appearance of an enlarged figure eight with three components, (1) and (2) corresponding to $xy > 0$, and (3) corresponding to $xy < 0$, see Figure 5.3 and Figure 5.2, where therein $\mathcal{F}$ denotes the entire leaf of the foliation generated by f_1 defined in local coordinates near p by $xy = \epsilon > 0$, for some $\epsilon > 0$. The area in region (1) between $\mathcal{F}$ and the figure eight defined locally by $xy = 0$ is given by $\mathcal{A}_1(\epsilon) = -\epsilon\ln(\epsilon) + h_1(\epsilon)$ for some smooth function $h_1 = h_1(\epsilon)$. Similarly the area in region (2) between $\mathcal{F}$ and the figure eight $xy = 0$ is given by $\mathcal{A}_2(\epsilon) = -\epsilon\ln(\epsilon) + h_2(\epsilon)$ for some smooth function $h_2 = h_2(\epsilon)$, and the area in region (3) between $\mathcal{F}$ and the figure eight is given by $\mathcal{A}_3(\epsilon) = 2\epsilon \ln|\epsilon| + h(\epsilon)$ for some smooth function $h = h(\epsilon)$. In addition, for each q,

$$h^{(q)} = -h_1^{(q)}(0) + h_2^{(q)}(0).$$

The Taylor series at 0 of h_1 and h_2 are symplectic invariants of $(M,\, \omega,\, f_1)$, cf. [37, Proposition 1].

Let $\mathcal{G}$ be the topological quotient of M by the relation $a \sim b$ if and only if a and b are in the same leaf of the foliation induced by f_1. The space $\mathcal{G}$ is called the *Reeb graph of* M. Let $\pi\colon M \to \mathcal{G}$ be the canonical projection map. In this context we call a *regular point* the image by π of a regular leaf, a *bout* the image of an elliptic point, and a *bifurcation point* the image of a figure eight leaf. The *edges* are the parts of $\mathcal{G}$ contained between two singular points. If s is a bifurcation point, then it has three edges, one of which corresponds to the leaves extending along the separatrix. We say that this edge is the *trunk of* s. The two other edges are the *branches of* s. The graph $\mathcal{G}$ is provided with the measure μ, the image by π of the measure defined by the symplectic form ω on M.

Let $p \in M$ be a hyperbolic point and let $(x,\, y)$ be the aforementioned local coordinates in a neighborhood of p. We define a function $\epsilon = xy$ in a neighborhood of the corresponding bifurcation point s, such that $xy > 0$ on the branches of s and $xy < 0$ on the trunk of s. The expression of μ in this neighborhood is:

$$\left\{\begin{array}{rcl} \mathrm{d}\mu_i(\epsilon) & = & (\ln(\epsilon) + g_i(\epsilon))\,\mathrm{d}\epsilon \;\text{ on each branch } i = 1,\, 2 \\ \mathrm{d}\mu(\epsilon) & = & (2\ln|\epsilon| + g(\epsilon))\,\mathrm{d}\epsilon \;\text{ on the trunk, with } g,\, g_1,\, g_2 \\ & & \text{smooth functions satisfying, for each } q, \\ g^{(q)}(0) & = & ((g_1^{(q)}(0) + g_2^{(q)}(0)). \end{array}\right. \tag{5.1}$$

It follows from [37, Proposition 1] stated above that the Taylor series at 0 of $g_1,\, g_2$ are invariants of $(\mathcal{G},\, \mu)$.

Definition 5.2 [Définition 1 in [37]] Let $\mathcal{G}$ be a topological 1-complex whose vertices have degrees 1 or 3. For each degree 3 vertex s, which one calls a *bifurcation point*, one distinguishes an edge and calls it the *trunk of* s; the two others are the *branches of* s. We provide $\mathcal{G}$ with an atlas of the following type:

- Outside of the bifurcation points it is a classical atlas of a manifold with boundary of dimension 1.
- In a neighborhood of each bifurcation point s, there exists an open set V and a continuous map $\varphi\colon V \to (-\epsilon,\, \epsilon)$, $\epsilon > 0$, with $\varphi(s) = 0$ and such that, if T is the trunk of s and $B_1,\, B_2$ are the branches of s, $\varphi|_{B_i}$ is bijective on $[0,\, \epsilon)$, $i = 1,\, 2$ and $\varphi|_T$ is bijective on $(-\epsilon,\, 0]$. We require that the changes of charts are smooth on each part $T \cup B_i$, $i = 1,\, 2$.

The topological 1-complex $\mathcal{G}$ is provided with a measure given by a non-zero density, smooth on each edge, and such that for each vertex s of degree 3, there exists a chart φ at s in which the measure μ is written as in 5.1. We denote by $(\mathcal{G},\, \mathcal{D},\, \mu)$ such a graph provided with its smooth structure and its measure, and we call it an *affine Reeb graph*. An *isomorphism* of such a graph is a bijection preserving the corresponding smooth structure and measure.

One can show [37, Lemme 2] that if the measure μ is written in another chart $\tilde{\varphi}$ in a neighborhood of s,

$$\begin{array}{rcl} \mathrm{d}\mu_i(\epsilon) & = & (\ln(\epsilon) + \tilde{g}_i(\epsilon))\,\mathrm{d}\epsilon \quad \text{on each branch } B_i \text{ of } s \\ \mathrm{d}\mu(\epsilon) & = & (2\ln|\epsilon| + \tilde{g}(\epsilon))\,\mathrm{d}\epsilon \quad \text{on the trunk of } s, \end{array}$$

then the functions g_i and $\tilde{g}_i$ have the same Taylor series at the origin (hence the Taylor series of g and $\tilde{g}$ are equal), which shows that the Taylor series of the functions g_i give invariants for the bifurcation points.

Let $\mathcal{G}$ be a combinatorial graph with vertices of degree 1 of of degree 3. For each vertex s of degree 3, one distinguishes in the same fashion as in Definition 5.2 the trunk and the branches of s, and one associates to each a sequence of real numbers. In addition, to each edge one associates a positive real number, called its *length*. Such a graph is called a *weighted Reeb graph*. To each affine Reeb graph one naturally associated a weighted Reeb graph, the sequence of numbers associated to the branches corresponding to coefficients of the Taylor series of the functions g_i. The lengths of the edges are given by their measure.

These considerations tell us the first part of the the following beautiful classification theorem.

Theorem 5.3 (Dufour-Molino-Toulet [37]). *One can associate to a triplet $(M,\,\omega,\,f_1)$ an affine Reeb graph $(\mathcal{G},\,\mathcal{D},\,\mu)$, which is unique up to isomorphisms, and to such an affine Reeb graph a weighted Reeb graph, unique up to isomorphisms. Conversely, every weighted Reeb graph is the graph associated to an affine Reeb graph, unique up to isomorphisms, and every affine Reeb graph is the Reeb graph associated to a triplet $(M,\,\omega,\,f_1)$, unique up to isomorphisms.*

The higher dimensional case will not be treated in general in the present paper, as will assume our systems do not have hyperbolic singularities.

5.2.3 Focus-focus case

Unless otherwise stated, for the remaining of the paper we focus on the case when the symplectic manifold M is 4 dimensional. Eliasson's theorem gives the local structure of focus-focus singularities. Several people noticed in the years 1996-1997 that this was enough to determine the *monodromy* of the foliation around the singular fiber. Actually this local structure is a starting point for understanding much more: the semiglobal classification of a singular fiber of focus-focus type. Unlike monodromy which is a topological invariant, already observed in torus fibrations without Hamiltonian structure, the semiglobal classification involves purely symplectic invariants. We proceed to describe this semiglobal classification, which complements Eliasson's theorem, in two steps.

(a) *Application of Eliasson's theorem.* Let $F=(f_1,\,f_2)$ be a completely integrable system with two degrees of freedom on a 4-dimensional symplectic manifold M. Let $\mathcal{F}$ be the *associated singular foliation* to the completely integrable system $F=(f_1,\,f_2)$, the leaves of which are by definition the connected components of the fibers $F^{-1}(c)$ of $F\colon\, M\to\mathbb{R}^2$. Let m be a critical point of focus-focus type. We assume for simplicity that $F(m)=0$, and that the (compact, connected) fiber $\Lambda_0:=F^{-1}(0)$ does not contain other critical points. One can show that Λ_0 is a "pinched" torus[8] surrounded by regular fibers which are standard 2-tori, see Figure 5.4. What are the semi-global invariants associated to this singular fibration?

One of the major characteristics of focus-focus singularities is the existence of a *Hamiltonian action of* S^1 that commutes with the flow of the system, in a neighborhood of the singular fiber that contains m. By Eliasson's theorem [42] there exist symplectic coordinates $(x,\,y,\,\xi,\,\eta)$ in a neighborhood U around m in which $(q_1,\,q_2)$, given by

$$q_1=x\eta-y\xi,\;\; q_2=x\xi+y\eta \tag{5.2}$$

is a momentum map for the foliation $\mathcal{F}$; here the critical point m corresponds to coordinates $(0,\,0,\,0,\,0)$. Fix $A'\in\mathcal{F}_m\cap(U\setminus\{m\})$ and let Σ denote a small 2-dimensional surface transversal to $\mathcal{F}$ at the point A', and let Ω be the open neighborhood of $\mathcal{F}_m$ which consists of the leaves which intersect the surface Σ.

[8] Lagrangian immersion of a sphere S^2 with a transversal double point

Since the Liouville foliation in a small neighborhood of Σ is regular for both F and $q = (q_1,\, q_2)$, there is a local diffeomorphism φ of $\mathbb{R}^2$ such that $q = \varphi \circ F$, and we can define a global momentum map $\Phi = \varphi \circ F$ for the foliation, which agrees with q on U.

Write $\Phi := (H_1,\, H_2)$ and $\Lambda_z := \Phi^{-1}(c)$. Note that $\Lambda_0 = \mathcal{F}_m$. It follows from (5.2) that near m the H_1-orbits must be periodic of primitive period 2π.

(b) *Symplectic semi-global classification of focus-focus point.* Suppose that $A \in \Lambda_c$ for some regular value c. Let $\tau_2(c) > 0$ be the time it takes the Hamiltonian flow associated with H_2 leaving from A to meet the Hamiltonian flow associated with H_1 which passes through A.

Let $\tau_1(z) \in \mathbb{R}/2\pi\mathbb{Z}$ the time that it takes to go from this intersection point back to A, closing the trajectory.

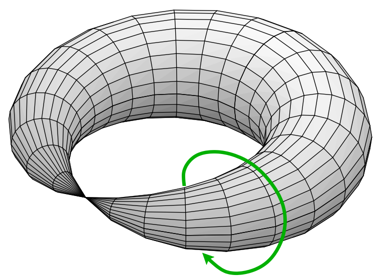

Figure 5.4: Focus-focus singularity and vanishing cycle for the pinched torus

Write $c = (c_1,\, c_2) = c_1 + \mathrm{i}\, c_2$ ($c_1,\, c_2 \in \mathbb{R}$), and let $\ln c$ be a fixed determination of the logarithmic function on the complex plane. Let

$$\left\{ \begin{array}{rcl} \sigma_1(c) & = & \tau_1(c) - \Im(\ln c) \\ \sigma_2(c) & = & \tau_2(c) + \Re(\ln c), \end{array} \right. \tag{5.3}$$

where $\Re$ and $\Im$ respectively stand for the real and imaginary parts of a complex number. Vũ Ngọc proved in [105, Prop. 3.1] that σ_1 and σ_2 extend to smooth and single-valued functions in a neighborhood of 0 and that the differential 1-form $\sigma := \sigma_1\, \mathrm{d}c_1 + \sigma_2\, \mathrm{d}c_2$ is closed. Notice that if follows from the smoothness of σ_1 that one may choose the lift of τ_1 to $\mathbb{R}$ such that $\sigma_1(0) \in [0,\, 2\pi)$. This is the convention used throughout.

Following [105, Def. 3.1] , let S be the unique smooth function defined around $0 \in \mathbb{R}^2$ such that

$$\mathrm{d}S = \sigma, \quad S(0) = 0. \tag{5.4}$$

The Taylor expansion of S at $(0,\, 0)$ is denoted by $(S)^\infty$.

Loosely speaking, one of the components of the system is indeed 2π-periodic, but the other one generates an arbitrary flow which turns indefinitely around the focus-focus singularity, deviating from periodic behavior in a logarithmic fashion, up to a certain error term; this deviation from being logarithmic is the symplectic invariant $(S)^\infty$.

Theorem 5.4 (Vũ Ngọc [105])**.** *The Taylor series expansion $(S)^\infty$ is well-defined (it does not depend on the choice of Eliasson's local chart) and it classifies the singular foliation in a neighborhood of Λ_0 in the sense that another system has the same Taylor series invariant near a focus-focus singularity if and only if there is* *a symplectomorphism which takes foliated a neighborhood of the singular fiber to a foliated neighborhood of the singular fiber preserving the leaves of the foliation and sending the singular fiber to the singular fiber.*

Moreover, if S is any formal series in $\mathbb{R}[\![X,Y]\!]$ with X-coefficient in $[0,\,2\pi)$ and without constant term, then there exists a singular foliation of focus-focus type whose Taylor series expansion is S.

The fact that two focus-focus fibrations are always semiglobally *topologically* conjugate was already proved by Lerman-Umanskii and Zung [67, 113], who introduced various topological notions of equivalence.

S can be interpreted as a *regularized* (or *desingularized*) *action*. Indeed if γ_z is the loop on Λ_z defined just as in the description of τ_j above, and if α is a semiglobal primitive of the symplectic form ω, let $\mathcal{A}(z) = \int_{\gamma_c} \alpha$; then

$$S(z) = \mathcal{A}(z) - \mathcal{A}(0) + \Re(z \ln z - z).$$

5.3 Example

One can check that the singularities of the coupled spin–oscillator $S^2 \times \mathbb{R}^2$ model mentioned in the Section 1 (equipped with the product symplectic form or the standard area forms) are non-degenerate and of elliptic-elliptic, transversally-elliptic or focus-focus type.

It has exactly one focus-focus singularity at the “North Pole” $((0,\,0,\,1),\,(0,\,0)) \in S^2 \times \mathbb{R}^2$ and one elliptic-elliptic singularity at the “South Pole” $((0,\,0,\,-1),\,(0,\,0))$. Let us parametriza the singular fiber $\Lambda_0 := F^{-1}(1,\,0)$. This singular fiber Λ_0 corresponds to the system of equations $J = 1$ and $H = 0$, which explicitly is given by system of two nonlinear equations $J = (u^2+v^2)/2+z = 0$ and $H = \frac{1}{2}(ux+vy) = 0$ on the coordinates $(x,\,y,\,z,\,u,\,v)$.

In order to solve this system of equations one introduces polar coordinates $u + \mathrm{i}\,v = r\mathrm{e}^{\mathrm{i}\,t}$ and $x + \mathrm{i}\,y = \rho\mathrm{e}^{\mathrm{i}\,\theta}$ where recall that the 2-sphere $S^2 \subset \mathbb{R}^3$ is equipped with coordinates $(x,\,y,\,z)$, and $\mathbb{R}^2$ is equipped with coordinates $(u,\,v)$. For $\epsilon = \pm 1$, we consider the mapping

$$S_\epsilon : [-1,\,1] \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \times S^2$$

given by the formula

$$S_\epsilon(p) = (r(p)\,\mathrm{e}^{\mathrm{i}\,t(p)},\,(\rho(p)\,\mathrm{e}^{\mathrm{i}\,\theta(p),\,z(p)}))$$

where $p = (\tilde{z},\,\tilde{\theta}) \in [-1,1] \times [0,2\pi)$ and

$$\begin{cases} r(p) = \sqrt{2(1-\tilde{z})} \\ t(p) = \tilde{\theta} + \epsilon\frac{\pi}{2} \\ \rho(p) = \sqrt{1-\tilde{z}^2} \\ \theta(p) = \tilde{\theta} \\ z(p) = \tilde{z}. \end{cases}$$

Then the map S_ϵ, where $\epsilon = \pm 1$, is continuous and S_ϵ restricted to $(-1,\,1)\times\mathbb{R}/2\pi\mathbb{Z}$ is a diffeomorphism onto its image. If we let $\Lambda_0^\epsilon := S_\epsilon([-1,\,1] \times \mathbb{R}/2\pi\mathbb{Z})$, then $\Lambda_0^1 \cup \Lambda_0^2 = \Lambda_0$ and

$$\Lambda_0^1 \cap \Lambda_0^2 = \Big(\{(0,\,0)\} \times \{(1,\,0,\,0)\}\Big) \cup \Big(C_2 \times \{(0,\,0,\,-1)\}\Big),$$

where C_2 denotes the circle of radius 2 centered at $(0, 0)$ in $\mathbb{R}^2$. Moreover, S_ϵ restricted to $(-1, 1)\times\mathbb{R}/2\pi\mathbb{Z}$ is a smooth Lagrangian embedding into $\mathbb{R}^2 \times S^2$. The singular fiber Λ_0 consists of two sheets glued along a point and a circle; topologically Λ_0 is a pinched torus, i.e. a 2-dimensional torus $S^1 \times S^1$ in which one circle $\{p\} \times S^1$ is contracted to a point (which is of course not a a smooth manifold at the point which comes from the contracting circle). The statements correspond to [88, Proposition 2.8].

It was proven in [88, Theorem 1.1] that the linear deviation from exhibiting logarithmic in a saturated neighborhood of the focus-focus singularity is given by the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ with expression

$$L(X, Y) = 5\ \ln 2\, X + \frac{\pi}{2}\, Y.$$

In other words, we have an equality

$$(S(X, Y))^\infty = L(X, Y) + \mathcal{O}((X, Y)^2).$$

This computation is involved and uses some deep formulas from microlocal analysis proven in the late 1990s. At the time of writing this paper we do not have an strategy to compute the higher order terms of the Taylor series invariant.

5.4 Applications

Theorem 5.4 leads to a number of applications, which although are outside the scope of this paper, we briefly note. One can for instance exploit the fact that the set of symplectic equivalence classes of these foliations acquires a vector space structure. That is what Symington does in [97] to show that neighborhoods of focus-focus fibers are always symplectomorphic (after forgetting the foliation, of course). For this one introduces functions S_0 and S_1 whose Taylor expansions give the invariants of the two foliations, and constructs a "path of foliations" by interpolating between S_0 and S_1. Then a Moser type argument yields the result (since the symplectic forms are cohomologous).

The theorem is also useful for doing calculations in a neighborhood of the fiber. For instance it is possible in this way to determine the validity of non-degeneracy conditions that appear in KAM type theorems[9], for a perturbation of a completely integrable system with a focus-focus singularity (see also [112]).

Theorem 5.5 (Dullin-Vũ Ngọc [38])**.** *Let H be a completely integrable Hamiltonian with a loxodromic singularity at the origin (i.e. H admits a singular Lagrangian foliation of focus-focus type at the origin). Then Kolmogorov's non-degeneracy condition is fulfilled on all tori close to the critical fiber, and the "isoenergetic turning frequencies" condition is fulfilled except on a 1-parameter family of tori corresponding to a curve through the origin in the image of the momentum map which is transversal to the lines of constant energy H.*

5.5 A topological classification

The present paper is devoted to the symplectic theory of Hamiltonian integrable systems. A large number authors have made contributions to the topological theory of Hamiltonian integrable systems, in particular Fomenko and his students, and Zung. In this section we briefly present a classification result due to Zung, which holds in any dimension, and for all so called topologically stable (non-degenerate) singularities. For precise statements we refer to Zung [113, Section 7].

[9] A nice discussion of these various conditions can be found in [89]

Let $\mathcal{F}$ be a singular leaf (fiber corresponding to a non-degenerate singularity) of an integrable system. In what follows, a *tubular neighborhood* $\mathcal{U}(\mathcal{F})$ of $\mathcal{F}$ means an appropriately chosen sufficiently small saturated tubular neighborhood. We denote by $(\mathcal{U}(\mathcal{F}),\, \mathcal{L})$ the Lagrangian foliation in a tubular neighborhood $\mathcal{U}(\mathcal{F})$ of $\mathcal{F}$. The leaf $\mathcal{F}$ is a deformation retract of $\mathcal{F}$.

Let $\mathcal{F}_1,\, \mathcal{F}_2$ be (non-degenerate) singular leaves of two integrable systems, of codimensions $k_1,\, k_2$, respectively, and let $(\mathcal{U}(\mathcal{F}_1),\, \mathcal{L}_1)$ and $(\mathcal{U}(\mathcal{F}_2),\, \mathcal{L}_2)$ be the corresponding Lagrangian foliations. Here a *singular leaf of codimension* k is a singular leaf whose "regular" part is $n-k$ dimensional. The *direct product* of these singularities is the singular leaf $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ of codimension $k_1 + k_2$, with the associated Lagrangian foliation

$$(\mathcal{U}(\mathcal{F}),\, \mathcal{L}) := (\mathcal{U}(\mathcal{F}_1),\, \mathcal{L}_1) \times (\mathcal{U}(\mathcal{F}_2),\, \mathcal{L}_2).$$

A (non-degenerate) singularity (or singular pair) $(\mathcal{U}(\mathcal{F}),\, \mathcal{L})$ of codimension k and Williamson type $(k_{\mathrm{e}},\, k_{\mathrm{h}},\, k_{\mathrm{f}})$ of an integrable system with n degrees of freedom is called *of direct product type topologically* if it is homeomorphic, together with the Lagrangian foliation, to the direct product

$$\begin{aligned}(\mathcal{U}(\mathbb{T}^{n-k}),\, \mathcal{L}_{\mathrm{r}}) \times (\mathrm{P}^2(\mathcal{F}_1),\, \mathcal{L}_1) \times \ldots \times (\mathrm{P}^2(\mathcal{F}_{k_{\mathrm{e}}+k_{\mathrm{h}}}),\, \mathcal{L}_{k_{\mathrm{e}}+k_{\mathrm{h}}}) \times \\ (\mathrm{P}^4(\mathcal{F}'_1),\, \mathcal{L}'_1) \times \ldots \times (\mathrm{P}^4(\mathcal{F}'_{k_{\mathrm{f}}}),\, \mathcal{L}'_{k_{\mathrm{f}}})\end{aligned} \tag{5.5}$$

where:

- the tuple $(\mathcal{U}(\mathbb{T}^{n-k},\, \mathcal{L}_{\mathrm{r}})$ denotes the Lagrangian foliation in a tubular neighborhood of a regular $(n-k)$-dimensional torus of an integrable system with $n-k$ degrees of freedom;
- the tuple $(\mathrm{P}^2(\mathcal{F}_i),\, \mathcal{L}_i)$, $1 \leq i \leq k_{\mathrm{e}}+k_{\mathrm{h}}$, denotes a codimension 1 (non-degenerate) surface singularity (i.e. a singularity of an integrable system with one degree of freedom);
- the tuple $(\mathrm{P}^4(\mathcal{F}_i),\, \mathcal{L}'_i)$, $1 \leq i \leq k_{\mathrm{f}}$, denotes a focus-focus singularity of an integrable system with two degrees of freedom.

A (non-degenerate) singularity of an integrable system is *of almost direct product type topologically* if a finite covering of it is homeomorphic, together with the Lagrangian foliation, to a direct product singularity. Zung proved [113, Theorem 7.3] the following classification result.

Theorem 5.6 (Zung [113])**.** *If $(\mathcal{U}(\mathcal{F}),\, \mathcal{L})$ is a (non-degenerate) topologically stable singularity of Williamson type $(k_{\mathrm{e}},\, k_{\mathrm{h}},\, k_{\mathrm{f}})$ and codimension k of an integrable system with n degrees of freedom, then it can be written homeomorphically in the form of a quotient of a direct product singularity as in (5.5) by the free action of a finite group Γ which acts component-wise on the product, and acts trivially on the elliptic components.*

As Zung points out, the decomposition in Theorem 5.6 is *not* symplectic.

6 Introduction to semitoric completely integrable systems

For the remainder of this paper we are now going to focus exclusively on semitoric completely integrable systems with 2-degrees of freedom on 4-manifolds; for brevity we will call these simply "semitoric systems". Essentially this means that the system is half-toric, and half completely general, see Definition 6.1 for a precise definition.

Semitoric systems form an important class of integrable systems, commonly found in simple physical models. Indeed, a semitoric system can be viewed as a Hamiltonian system in the presence of an S^1-symmetry [92]. In our personal opinion, it is much simpler to understand the integrable system on its whole rather than writing a theory of Hamiltonian systems on Hamiltonian S^1-manifolds.

6.1 Meaning of the integrability condition

Let us recall what the general definition of an integrable system in Section 1 means in dimension 4. In this case an integrable system on M is a pair of real-valued smooth functions J and H on M, for which the Poisson bracket $\{J, H\} := \omega(\mathcal{H}_J, \mathcal{H}_H)$ identically vanishes on M, and the differentials $\mathrm{d}J, \mathrm{d}H$ are almost-everywhere linearly independendent.

Of course, here $(J, H)\colon M \to \mathbb{R}^2$ is the analogue of the momentum map in the case of a torus action. In some local Darboux coordinates of M, (x, y, ξ, η), the symplectic form ω is given by $\mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y$, and the vanishing of the Poisson brackets $\{J, H\}$ amounts to the partial differential equation

$$\frac{\partial J}{\partial \xi}\frac{\partial H}{\partial x} - \frac{\partial J}{\partial x}\frac{\partial H}{\partial \xi} + \frac{\partial J}{\partial \eta}\frac{\partial H}{\partial y} - \frac{\partial J}{\partial y}\frac{\partial H}{\partial \eta} = 0.$$

This condition is equivalent to J being constant along the integral curves of $\mathcal{H}_H$ (or H being constant along the integral curves of $\mathcal{H}_J$).

6.2 Singularities

We introduce the main object of the remaining part of this paper, semitoric systems, and explain what singularities can occur in these systems.

Definition 6.1 A *semitoric integrable system* on M is an integrable system for which the component J is a proper momentum map for a Hamiltonian circle action on M, and the associated map $F := (J, H) : M \to \mathbb{R}^2$ has only non-degenerate singularities in the sense of Williamson, without real-hyperbolic blocks.

Let us spell this definition concretely. The properness of J means that the preimage by J of a compact set is compact in M (which is immediate if M is compact). The non-degeneracy hypothesis for F means that, if m is a critical point of F, then there exists a 2 by 2 matrix B such that, if we write $\tilde{F} = B \circ F$, one of the situations described in the following table holds in some local symplectic coordinates (x, y, ξ, η) near m in which $m = (0, 0, 0, 0)$ and $\omega = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y$.

TYPE	$F := (H, J)\colon M \to \mathbb{R}^2$ IN COORDINATES (x, y, ξ, η)
Transversally elliptic	$F = (\eta + \mathcal{O}(\eta^2), \frac{1}{2}(x^2 + \xi^2) + \mathcal{O}((x, \xi)^3)$
Elliptic-elliptic	$F = \frac{1}{2}(x^2 + \xi^2, y^2 + \eta^2) + \mathcal{O}((x, \xi, y, \eta)^3)$
Focus-focus	$F = (x\eta - y\xi, x\xi + y\eta) + \mathcal{O}((x, \xi, y, \eta)^3)$

In the case of semitoric systems the Williamson of the singularities are of the form $(k_{\mathrm{e}}, 0, \mathrm{k_f})$, i.e. $k_{\mathrm{h}} = 0$.

Note that this is not a result in a neighborhood of a fiber.

Again, perhaps the simplest non-compact semitoric integrable systems is the coupled spin-oscillator $S^2 \times \mathbb{R}^2$. The component J is the momentum map for the Hamiltonian circle action on M which rotates simultaneously about the vertical axis of S^2 and about the origin of $\mathbb{R}^2$. By the action-angle theorem of Arnold-Liouville-Mineur the regular fibers are 2-tori. We saw that this system has a unique focus-focus singularity at $(0, 0, 1, 0, 0)$ with fiber a pinched torus. The other singular fibers are either circles or points. The authors have studied the symplectic and spectral theory of this system in [88].

6.3 Convexity properties: the polygon invariant

This section analyzes to what extent the convexity theorem of Atiyah and Guillemin-Sternberg holds in the context of semitoric completely integrable systems. The second author proved in [106] that one can meaningfully associate a convex polygonal region such a system.

6.3.1 Bifurcation diagrams

It is well established in the integrable systems community that the most simple and natural object, which tells much about the structure of the integrable system under study, is the so-called bifurcation diagram. As a matter of fact, bifurcations diagrams may be defined in great generality as follows. Let M and N be smooth manifolds. Recall that a smooth map $f : M \to N$ is locally trivial at $n_0 \in f(M)$ if there is an open neighborhood $U \subset N$ of n_0 such that $f^{-1}(n)$ is a smooth submanifold of M for each $n \in U$ and there is a smooth map $h : f^{-1}(U) \to f^{-1}(y_0)$ such that $f \times h : f^{-1}(U) \to U \times f^{-1}(n_0)$ is a diffeomorphism. The *bifurcation set or bifurcation diagram* Σ_f consists of all the points of N where f is not locally trivial. Note, in particular, that $h|_{f^{-1}(n)} : f^{-1}(n) \to f^{-1}(n_0)$ is a diffeomorphism for every $n \in U$. Also, the set of points where f is locally trivial is open in N. Note that Σ_f is a closed subset of N. It is well known that the set of critical values of f is included in the bifurcation set (see [1, Proposition 4.5.1]). In general, the bifurcation set strictly includes the set of critical values. This is the case for the momentum-energy map for the two-body problem [1, §9.8]. It is well known [1, Page 340] that if $f \colon M \to N$ is a smooth proper map, the bifurcation set of f is equal to the set of critical values of f.

It follows that when the map $F = (J, H) \colon M \to \mathbb{R}^2$ that defines the integrable system is a proper map the bifurcation diagram is equal to the set of critical values of F inside of the image $F(M)$ of F. This is the case for semitoric integrable systems, since the properness of the component J implies the properness of J. As it turns out, the arrangement of such critical values is indeed important, but other crucial invariants that are more subtle and cannot be detected from the bifurcation diagram itself are needed to understand a semitoric system F; we deal with these ones in Section 7. The authors proved [86, 87] that these invariants are enough to completely determine a semitoric system up to isomorphisms.

The proof relies on a number of remarkable results by other authors on integrable systems, including Arnold [4], Atiyah [6], Dufour-Molino [36], Eliasson [41], Duistermaat [30], Guillemin-Sternberg [54], Miranda-Zung [75] and Vũ Ngọc [105, 106]. In this section we explain the so called polygon invariant, which was originally introduced by the second author in [106], and can be considered an analogue (for completely integrable semitoric systems) of the convex polytope that appears in the Atiyah-Guillemin-Sternberg convexity theorem (in the context of symplectic torus actions on compact manifolds).

6.3.2 Affine structures

The plane $\mathbb{R}^2$ is equipped with its standard affine structure with origin at $(0, 0)$, and its standard orientation. Let $\mathrm{Aff}(2, \mathbb{R}) := \mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ be the group of affine transformations of $\mathbb{R}^2$. Let $\mathrm{Aff}(2, \mathbb{Z}) := \mathrm{GL}(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ be the subgroup of *integral-affine* transformations. It was proven in [106] that a semitoric system $(M, \omega, F := (J, H))$ has finitely many focus-focus critical values $c_1, \ldots, c_{m_f}$, that if we write $B := F(M)$ then the set of regular values of F is $\mathrm{Int}(B) \setminus \{c_1, \ldots, c_{m_f}\}$, that the boundary of B consists of all images of elliptic singularities, and that the fibers of F are connected. The integer m_f was the first invariant that we associated with such a system.

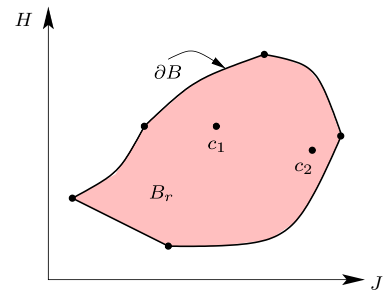

Figure 6.1: Image under $(J,\, H)$ of M.

Let $\mathfrak{I}$ be the subgroup of $\mathrm{Aff}(2,\, \mathbb{Z})$ of those transformations which leave a vertical line invariant, or equivalently, an element of $\mathfrak{I}$ is a vertical translation composed with a matrix T^k, where $k \in \mathbb{Z}$ and

$$T^k := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{GL}(2,\, \mathbb{Z}). \tag{6.1}$$

Let $\ell \subset \mathbb{R}^2$ be a vertical line in the plane, not necessarily through the origin, which splits it into two half-spaces, and let $n \in \mathbb{Z}$. Fix an origin in ℓ. Let $t_\ell^n \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the identity on the left half-space, and T^n on the right half-space. By definition t_ℓ^n is piecewise affine. Let ℓ_i be a vertical line through the focus-focus value $c_i = (x_i,\, y_i)$, where $1 \leq i \leq m_f$, and for any tuple $\vec{n} := (n_1,\, \ldots,\, n_{m_f}) \in \mathbb{Z}^{m_f}$ we set $t_{\vec{n}} := t_{\ell_1}^{n_1} \circ \cdots \circ t_{\ell_{m_f}}^{n_{m_f}}$. The map $t_{\vec{n}}$ is piecewise affine.

A *convex polygonal set* Δ is the intersection in $\mathbb{R}^2$ of (finitely or infinitely many) closed half-planes such that on each compact subset of the intersection there is at most a finite number of corner points. We say that Δ is *rational* if each edge is directed along a vector with rational coefficients. For brevity, in this paper we usually write "*polygon*" (or "convex polygon") instead of "*convex polygonal set*". Note that the word "polygon" is commonly used to refer to the convex hull of a finite set of points in $\mathbb{R}^2$ which is a compact set (this is not necessarily the case in algebraic geometry, e.g. Newton polygons). Notice (see Ziegler [118, Thm. 1.2]) that a convex polygonal set Δ which is not a half-space has exactly two edges of infinite Euclidean length if and only if it is non-compact, and Δ has all of its edges of finite length if and only if it is compact.

6.3.3 The polygon invariant

Let $B_\mathrm{r} := \mathrm{Int}(B) \setminus \{c_1,\, \ldots,\, c_{m_f}\}$, which is precisely the set of regular values of F. Given a sign $\epsilon_i \in \{-1, +1\}$, let $\ell_i^{\epsilon_i} \subset \ell_i$ be the vertical half line starting at c_i and extending in the direction of ϵ_i : upwards if $\epsilon_i = 1$, downwards if $\epsilon_i = -1$. Let $\ell^{\vec{\epsilon}} := \bigcup_{i=1}^{m_f} \ell_i^{\epsilon_i}$. In Th. 3.8 in [106] it was shown that:

Theorem 6.2. *For $\vec{\epsilon} \in \{-1, +1\}^{m_f}$ there exists a homeomorphism $f = f_\epsilon \colon B \to \mathbb{R}^2$, modulo a left composition by a transformation in $\mathfrak{I}$, such that $f|_{(B \setminus \ell^{\vec{\epsilon}})}$ is a diffeomorphism into the image of f, $\Delta := f(B)$, which is a* rational convex polygon, *$f|_{(B_r \setminus \ell^{\vec{\epsilon}})}$ is affine (it sends the integral affine structure of B_r to the standard structure of $\mathbb{R}^2$) and f preserves J: i.e. $f(x,\, y) = (x,\, f^{(2)}(x,\, y))$.*

The map f satisfies further properties [86], which are relevant for the uniqueness proof. In order to arrive at Δ one cuts $(J,\, H)(M) \subset \mathbb{R}^2$ along each of the vertical half-lines $\ell_i^{\epsilon_i}$. Then the resulting image becomes simply connected and thus there exists a global 2-torus action on the preimage of this set. The polygon Δ is just the closure of the image of a toric momentum map corresponding to this torus action.

We can see that this polygon is not unique. The choice of the "cut direction" is encoded in the signs ϵ_j, and there remains some freedom for choosing the toric momentum map. Precisely, the choices and the corresponding homeomorphisms f are the following :

(a) *an initial set of action variables* f_0 *of the form* $(J,\, K)$ near a regular Liouville torus in [106, Step 2, pf. of Th. 3.8]. If we choose f_1 instead of f_0, we get a polygon Δ' obtained by left composition with an element of $\mathfrak{I}$. Similarly, if we choose f_1 instead of f_0, we obtain f composed on the left with an element of $\mathfrak{I}$;

(b) *a tuple* $\vec{\epsilon}$ *of* 1 *and* -1. If we choose $\vec{\epsilon'}$ instead of $\vec{\epsilon}$ we get $\Delta' = t_{\vec{u}}(\Delta)$ with $u_i = (\epsilon_i - \epsilon_i')/2$, by [106, Prop. 4.1, expr. (11)]. Similarly instead of f we obtain $f' = t_{\vec{u}} \circ f$.

Once f_0 and $\vec{\epsilon}$ have been fixed as in (a) and (b), respectively, then there exists a unique toric momentum map μ on $M_r := F^{-1}(\mathrm{Int}B \setminus (\bigcup \ell_j^{\epsilon_j}))$ which preserves the foliation $\mathcal{F}$, and coincides with $f_0 \circ F$ where they are both defined. Then, necessarily, the first component of μ is J, and we have $\overline{\mu(M_r)} = \Delta$.

We need now for our purposes to formalize choices (a) and (b) in a single geometric object. The details of how to do this have appeared in [87]. We will simply say that essentially this object consists of the convex polygon itself together with a collection of oriented cuts as in the figure below. We call this object a *weighted polygon*, and denote it by Δ_{w}. The cuts are a collection of vertical lines that go through the singularities. The actual object is actually more complex, as it is defined as an equivalence class of such polygon considered as a part of a larger space of polygons on which several groups act non-trivially. These groups are $\{-1,\, 1\}^{m_f}$ and $\mathfrak{I}$. The actual invariant is then denoted by $[\Delta_{\mathrm{w}}]$.

7 More symplectic invariants of semitoric systems

In [86, Th. 6.2] the authors constructed, starting from a given semitoric integrable system on a 4-manifold, a collection of five symplectic invariants associated with it and proved that these completely determine the integrable system up to global isomorphisms of semitoric systems. Let M_1, M_2 be symplectic 4-manifolds equipped with semitoric integrable systems $(J_1,\, H_1)$ and $(J_2,\, H_2)$. An *isomorphism* between these integrable systems is a symplectomorphism $\varphi\colon\, M_1 \to M_2$ such that $\varphi^*(J_2,\, H_2) = (J_1,\, f(J_1,\, H_1))$ for some smooth function f such that $\frac{\partial f}{\partial H_1}$ nowhere vanishes.

We recall the definition of the invariants that we assigned to a semitoric integrable system in our previous paper [86], to which we refer to further details. Then we state the uniqueness theorem proved therein.

7.1 Taylor series invariant

We assume that the critical fiber $\mathcal{F}_m := F^{-1}(c_i)$ contains only one critical point m, which according to Zung [113] is a generic condition, and let $\mathcal{F}$ denote the associated singular foliation. Moreover, we will make for simplicity an even stronger generic assumption : if m is a focus-focus critical point for F, then m is the unique critical point of the level set $J^{-1}(J(m))$. A semitoric system is *simple* if this generic assumption is satisfied.

These conditions imply that the values $J(m_1),\, \ldots,\, J(m_{m_f})$ are pairwise distinct. We assume throughout the article that the critical values c_i's are *ordered* by their J-values : $J(m_1) < J(m_2) < \cdots < J(m_{m_f})$.

Let $(S_i)^\infty$ be the a formal power series expansion (in two variables with vanishing constant term) corresponding to the integrable system given by F at the critical focus-focus point c_i, see Theorem 5.4. We say that $(S_i)^\infty$ is the *Taylor series invariant of* $(M,\,\omega,\,(J,\,H))$ *at the focus-focus point* c_i.

7.2 The Volume Invariant

Consider a focus-focus critical point m_i whose image by $(J,\,H)$ is c_i, and let Δ be a rational convex polygon corresponding to the system $(M,\,\omega,\,(J,\,H))$. If μ is a toric momentum map for the system $(M,\,\omega,\,(J,\,H))$ corresponding to Δ, then the image $\mu(m_i)$ is a point in the interior of Δ, along the line ℓ_i. We proved in [86] that the vertical distance

$$h_i := \mu(m_i) - \min_{s\in\ell_i\cap\Delta} \pi_2(s) > 0 \tag{7.1}$$

is independent of the choice of momentum map μ. Here $\pi_2\colon \mathbb{R}^2 \to \mathbb{R}$ is $\pi_2(x,\,y) = y$. The reasoning behind writing the word "volume" in the name of this invariant is that it has the following geometric interpretation: the singular manifold $Y_i = J^{-1}(c_i)$ splits into $Y_i \cap \{H > H(m_i)\}$ and $Y_i \cap \{H < H(m_i)\}$, and h_i is the Liouville volume of $Y_i \cap \{H < H(m_i)\}$.

7.3 The Twisting-Index Invariant

This is a subtle invariant of semitoric systems; it quantifies the dynamical complexity of the system at a global level, while involving the behavior near all of the focus-focus singularities of the system simultaneously.

The twisting-index expresses the fact that there is, in a neighborhood of any focus-focus critical point c_i, a *privileged toric momentum map* ν. This momentum map, in turn, is due to the existence of a unique hyperbolic radial vector field in a neighborhood of the focus-focus fiber. Therefore, one can view the twisting-index as a dynamical invariant. Since any semitoric polygon defines a (generalized) toric momentum map μ, we will be able to define the twisting-index as the integer $k_i \in \mathbb{Z}$ such that

$$\mathrm{d}\mu = T^{k_i}\mathrm{d}\nu.$$

(Recall formula (6.1) for the formula of T^{k_i}.) We could have equivalently defined the twisting-indices by comparing the privileged momentum maps at different focus-focus points.

The precise definition of k_i requires some care, which we explain now. Let Δ_{w} be the weighted polygon associated to M, which recall consists of the polygon Δ plus a collection of oriented vertical lines ℓ_j, where the orientation of each line is given by ± 1 signs ϵ_j, $j = 1,\,\ldots,\,m_f$.

Let $\ell := \ell_i^{\epsilon_i} \subset \mathbb{R}^2$ be the vertical *half-line* starting at c_i and pointing in the direction of $\epsilon_i\, e_2$, where $e_1,\,e_2$ are the canonical basis vectors of $\mathbb{R}^2$.

By Eliasson's theorem, there is a neighbourhood $W = W_i$ of the focus-focus critical point $m_i = F^{-1}(c_i)$, a local symplectomorphism $\phi : (\mathbb{R}^4,\,0) \to W$, and a local diffeomorphism g of $(\mathbb{R}^2,\,0)$ such that $F \circ \phi = g \circ q$, where q is given by (5.2).

Since $q_2 \circ \phi^{-1}$ has a 2π-periodic Hamiltonian flow, it is equal to J in W, up to a sign. Composing if necessary ϕ by $(x,\xi) \mapsto (-x,-\xi)$ one can assume that $q_1 = J \circ \phi$ in W, i.e. g is of the form $g(q_1,\,q_2) = (q_1,\,g_2(q_1,\,q_2))$. Upon composing ϕ with $(x,\,y,\,\xi,\,\eta) \mapsto (-\xi,\,-\eta,\,x,\,y)$, which changes $(q_1,\,q_2)$ into $(-q_1,\,q_2)$, one can assume that $\frac{\partial g_2}{\partial q_2}(0) > 0$. In particular, near the origin ℓ is transformed by g^{-1} into the positive imaginary axis if $\epsilon_i = 1$, or the negative imaginary axis if $\epsilon_i = -1$.

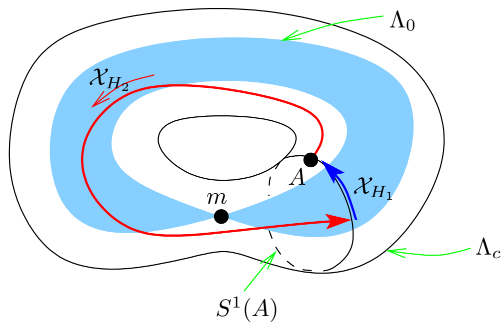

Figure 7.1: Singular foliation near the leaf $\mathcal{F}_m$, where $S^1(A)$ denotes the S^1-orbit generated by $H_1 = J$.

Let us now fix the origin of angular polar coordinates in $\mathbb{R}^2$ on the *positive* imaginary axis, let $V = F(W)$ and define $\tilde{F} = (H_1,\, H_2) = g^{-1} \circ F$ on $F^{-1}(V)$ (notice that $H_1 = J$).

Recall that near any regular torus there exists a Hamiltonian vector field $\mathcal{H}_p$, whose flow is 2π-periodic, defined by

$$2\pi\mathcal{H}_p = (\tau_1 \circ \tilde{F})\mathcal{H}_{H_1} + (\tau_2 \circ \tilde{F})\mathcal{H}_J,$$

where τ_1 and τ_2 are functions on $\mathbb{R}^2 \setminus \{0\}$ satisfying (5.3), with $\sigma_2(0) > 0$. In fact τ_1 is multivalued, but we determine it completely in polar coordinates with angle in $[0,\, 2\pi)$ by requiring continuity in the angle variable and $\sigma_1(0) \in [0,\, 2\pi)$. In case $\epsilon_i = 1$, this defines $\mathcal{H}_p$ as a smooth vector field on $F^{-1}(V \setminus \ell)$.

In case $\epsilon_i = -1$ we keep the same τ_1-value on the negative imaginary axis, but extend it by continuity in the angular interval $[\pi,\, 3\pi)$. In this way $\mathcal{H}_p$ is again a smooth vector field on $F^{-1}(V \setminus \ell)$.

Let μ be the generalized toric momentum map associated to Δ. On $F^{-1}(V \setminus \ell)$, μ is smooth, and its components $(\mu_1,\, \mu_2) = (J,\, \mu_2)$ are smooth Hamiltonians whose vector fields $(\mathcal{H}_J, \mathcal{H}_{\mu_2})$ are tangent to the foliation, have a 2π-periodic flow, and are *a.e.* independent. Since the couple $(\mathcal{H}_J, \mathcal{H}_p)$ shares the same properties, there must be a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $(\mathcal{H}_J, \mathcal{H}_{\mu_2}) = A(\mathcal{H}_J, \mathcal{H}_p)$. This is equivalent to saying that there exists an integer $k_i \in \mathbb{Z}$ such that $\mathcal{H}_{\mu_2} = k_i\mathcal{H}_J + \mathcal{H}_p$.

It was shown in [86, Prop. 5.4] that k_i is well defined, i.e. does not depend on choices. The integer k_i is called the *twisting index of Δ_{w} at the focus-focus critical value c_i*.

It was shown in [86, Lem. 5.6] that there exists a unique smooth function H_p on $F^{-1}(V \setminus \ell)$ with Hamiltonian vector field $\mathcal{H}_p$ and such that $\lim_{m \to m_i} H_p = 0$. The toric momentum map $\nu := (J,\, H_p)$ is called *the privileged momentum map for* $(J,\, H)$ around the focus-focus value c_i. If k_i is the twisting index of c_i, one has $\mathrm{d}\mu = T^{k_i}\mathrm{d}\nu$ on $F^{-1}(V)$. However, the twisting index does depend on the polygon Δ. Thus, since we want to define an invariant of the initial semitoric system, we need to quotient out by the natural action of groups $G_{m_f} \times \mathfrak{I}$; because this is a rather technical task and we refer to [86, p. 580] for details.

It was shown in [86, Prop. 5.8] that if two weighted polygons Δ_{w} and $\Delta'_{\mathrm{weight}}$ lie in the same G_{m_f}-orbit, then the twisting indices $k_i,\, k'_i$ associated to Δ_{w} and $\Delta'_{\mathrm{weight}}$ at their respective focus-focus critical values $c_i,\, c'_i$ are equal.

To a semitoric system we associate what we call the *twisting-index* invariant, which is nothing but the tuple $(\Delta_{\mathrm{w}},\, \mathbf{k})$ consisting of the polygon Δ labeled by the tuple twisting indices $\mathbf{k} = (k_j)_{j=1}^{m_f}$. Actually, as explained above, one needs to take into consideration the group actions of G_{m_f} and $\mathfrak{I}$, so the twisting index invariant associated to the semitoric system is an equivalence class $[(\Delta_{\mathrm{w}},\, \mathbf{k})]$ under a twisted action of G_{m_f} and $\mathfrak{I}$. The formula for this action is long and we chose to not write it here, but details appear in [86].

7.4 Example

In the case of the coupled spin oscillator the twisting index invariant does not appear because there is only one focus-focus point. So in addition to the Taylor series invariant (of which as we said one can compute its linear approximation), the height invariant and the polygon invariant are easy to compute. They are explicitly given in Figure 8.2 in the next section.

8 Global symplectic theory of semitoric systems

8.1 First global result: uniqueness

The symplectic invariants constructed in [86], for a given 4-dimensional semitoric integrable system, are the following: (i) *the number of singularities invariant*: an integer m_f counting the number of isolated singularities; (ii) *the singularity type invariant*: a collection of m_f infinite formal Taylor series on two variables which classifies locally the type of each (focus-focus) singularity; (iii) *the polygon invariant*: the equivalence class of a weighted rational convex Δ_{w} consisting of a convex polygon Δ and the collection of vertical lines ℓ_j crossing it, where ℓ_j is oriented upwards or downwards depending on the sign of ϵ_j, $j = 1, \ldots, m_f$; (iv) *the volume invariant*: m_f numbers measuring volumes of certain submanifolds at

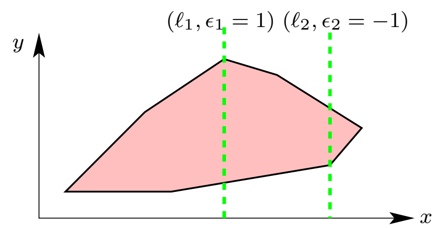

Figure 8.1: Weighted polygon $(\Delta, (\ell_1,\, \ell_2),\, (1,\, -1))$.

the singularities; (v) *the twisting index invariant*: m_f integers measuring how twisted the system is around singularities. This is a subtle invariant, which depends on the representative chosen in (iii). Here, we write m_f to emphasize that the singularities that m_f counts are focus-focus singularities. We then proved:

Theorem 8.1 (Pelayo-Vũ Ngọc [86]). *Two semitoric systems* $(M_1,\, \omega_1,\, (J_1,\, H_1))$ *and* $(M_2,\, \omega_2,\, (J_2,\, H_2))$ *are isomorphic if and only if they have the same invariants (i)–(v), where an* isomorphism *is a symplectomorphism* $\varphi\colon M_1 \to M_2$ *such that* $\varphi^*(J_2,\, H_2) = (J_1,\, f(J_1,\, H_1))$ *for some smooth function* f *such that* $\frac{\partial f}{\partial H_1}$ *nowhere vanishes.*

8.2 Second global result: existence

We have found that some restrictions on the symplectic invariants we have just defined must be imposed [87]. Indeed, we call a “semitoric list of ingredients” the following collection of items:

(i) An non-negative integer m_f.

(ii) An m_f-tuple of formal Taylor series with vanishing constant term $((S_i)^\infty)_{i=1}^{m_f} \in (\mathbb{R}[[X,\, Y]]_0)^{m_f}$.

(iii) A Delzant semitoric polygon $[\Delta_{\rm w}]$ of complexity m_f consisting of a polygon Δ and vertical lines ℓ_j intersecting Δ, each of which is oriented according to a sign $\epsilon_j = \pm 1$;

(iv) An m_f-tuple of numbers $\mathbf{h} = (h_j)_{j=1}^{m_f}$ such that $0 < h_j < \text{length}(\Delta \cap \ell_i)$.

(v) An equivalence class $[(\Delta_{\rm w}, \mathbf{k})]$, where $\mathbf{k} = (k_j)_{j=1}^{m_f}$ is a collection of integers.

In the definition the term $\mathbb{R}[[X,\, Y]]$ refers to the algebra of real formal power series in two variables, and $\mathbb{R}[[X,\, Y]]_0$ is the subspace of such series with vanishing constant term, and first term $\sigma_1\, X + \sigma_2\, Y$ with $\sigma_2 \in [0,\, 2\,\pi)$. For the definition of Delzant semitoric polygon, which is somewhat involved, see [87, Sec. 4.2]. The main result of [87] is the following existence theorem:

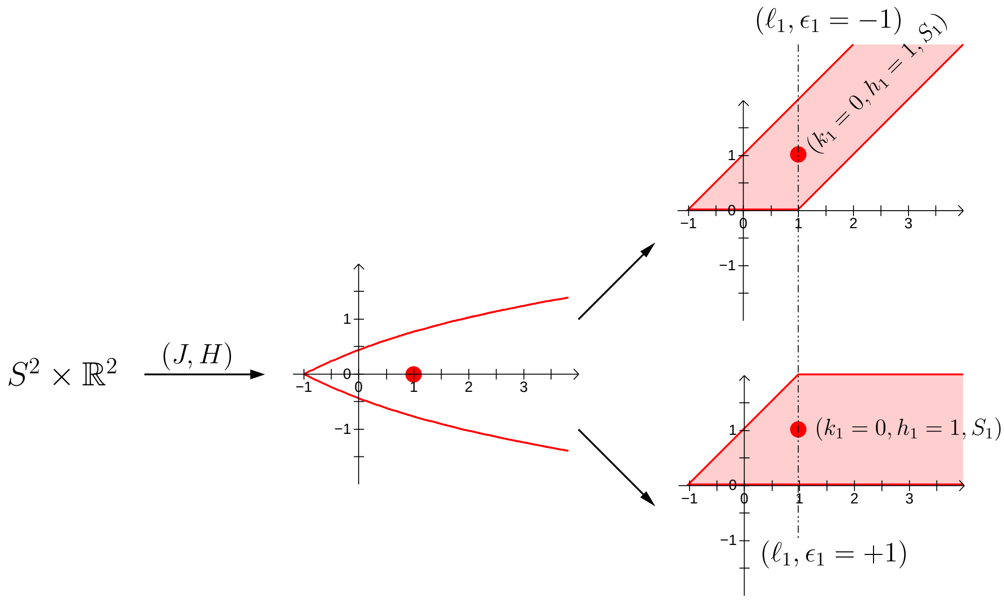

Figure 8.2: The coupled spin-oscillator example. The middle figure shows the image of the initial moment map $F = (J,\, H)$. Its boundary is the parameterized curve $(j(s) = \frac{s^2-3}{2s}, h(s) = \pm\frac{s^2-1}{2s^{3/2}}),\ s \in [1, \infty)$. The image is the connected component of the origin. The system is a simple semitoric system with one focus-focus point whose image is $(1,\, 0)$. The invariants are depicted on the right hand-side. Since $m_f = 1$, the class of generalized polygons for this system consists of two polygons.

Theorem 8.2 (Pelayo-Vũ Ngọc [87])**.** *For each semitoric list of ingredients there exists a* 4*-dimensional simple semitoric integrable system with list of invariants equal to this list of ingredients.*

The proof is involved, but the main idea of proof is simple. We start with a representative of $[\Delta_{\rm w}]$ with all ϵ_j's equal to $+1$. The strategy is to construct the system locally and semiglobally around the singularities and around the regular parts, to then perform symplectic glueing in order to obtain a semitoric system by constructing a suitable singular torus fibration above $\Delta \subset \mathbb{R}^2$.

Rather subtle analytical issues appear when glueing, and one eventually ends up with a system given by a momentum map which is not smooth along the cuts ℓ_j. More concretely, first we construct a "semitoric system" over the part of the polygon away from the sets in the covering that contain the cuts ℓ_j^+; then we attach to this "semitoric system" the focus-focus fibrations i.e. the models for the systems in a small neighborhood of the nodes (singularities). We use a symplectic gluing theorem to do this glueing (cf. [87] for a statement/proof).

Third, we continue to glue the local models in a small neighborhood of the cuts. The “semitoric system” is given by a proper toric map only in the preimage of the polygon away from the cuts. There is a analytically rather subtle issue near the cuts and one has to change the momentum map carefully to make it smooth while preserving the structure of the system up to isomorphisms.

In the last step we prove that the system we have constructed has the right invariants. Here we have to appeal to the uniqueness theorem, as the equivalence class of the invariants may have shifted in the construction.

9 Some open problems

9.1 Inverse spectral theory

Finding out how information from quantum completely integrable systems leads to information about classical systems is a fascinating “inverse” problem with very few precise results at this time.

The *symplectic* classification in terms of concrete invariants described in sections 6, 7, 8 serves as a tool to quantize semitoric systems. In Delzant’s theory, the image of the momentum map, for a toric completely integrable action, completely determines the system. In the quantum theory, the image of the momentum map is replaced by the joint spectrum. Can one determine the underlying classical system from *one* of the joint spectra of the associated quantum system? In this vast, essentially unexplored program, one can ask the less ambitious but still spectacular question: does the semi-classical joint spectrum determine the underlying classical system?

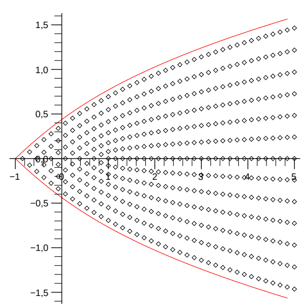

Figure 9.1: Sections 4, 5 of the author’s article [88] are devoted to the spectral theory of the quantum coupled spin-oscillator, and they are a first step towards proving this conjecture for spin-oscillators. The unbounded operators $\hat{J} := \mathrm{Id} \otimes \Big(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\,u^2} + \frac{u^2}{2} \Big) + (\hat{z} \otimes \mathrm{Id})$ and $\hat{H} = \frac{1}{2}(\hat{x} \otimes u + \hat{y} \otimes (\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u})$ on the Hilbert space $\mathcal{H} \otimes \mathrm{L}^2(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R}^2) \otimes \mathrm{L}^2(\mathbb{R})$ are self-adjoint and commute, and they define the quantum spin-oscillator. Their joint spectrum is depicted in the figure.

Conjecture 9.1. *A semitoric system J, $H$is determined up to symplectic equivalence by its semiclassical joint spectrum as $\hbar \to 0$. From any such spectrum one can construct explicitly the associated semitoric system, i.e. the set of points in $\mathbb{R}^2$ where on the x-basis we have the eigenvalues of $\hat{J}$, and on the vertical axes the eigenvalues of $\hat{H}$ restricted to the λ-eigenspace of $\hat{J}$.*

The strategy to prove this is clear: given the joint spectrum, detect in it the symplectic invariants. Once we have computed the symplectic invariants, we can symplectically recover the integrable system by [86, 87], and hence the quantum system. The authors have done this for the coupled-spin oscillator [88]. The method to recover the symplectic invariants from the joint spectrum combines microlocal analysis and Lie theory. Recovering the polygon invariant is probably the easiest and most pictorial procedure, as long as one stays on a heuristic level. Making the heuristic rigorous should be possible along the lines of the toric case explained in [108] and [109].

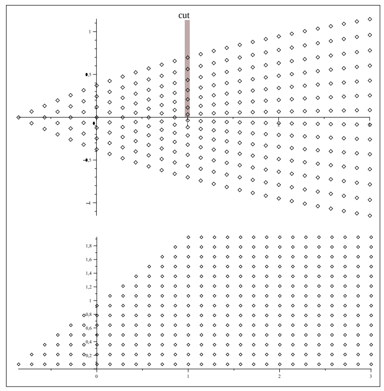

Figure 9.2: Recovering the polygon invariant. The top picture is the joint spectrum of $(\hat{J}, \hat{H})$. In the bottom picture, we have developed the joint eigenvalues into a regular lattice. One can easily check on this illustration that the number of eigenvalues in each vertical line in the same in both pictures.

The convex hull of the resulting set is a rational, convex polygonal set, depending on $\hbar$. Since the semiclassical affine structure is an $\hbar$-deformation of the classical affine structure, we see that, as $\hbar \to 0$, this polygonal set converges to the semitoric polygon invariant.

9.2 Mirror symmetry

When dealing with semitoric systems we are in a situation where the moment map (J, H) is a "torus fibration" with singularities, and its base space becomes endowed with a singular integral affine structure. These same affine structures appear as a central ingredient in the work of Kontsevich and Soibelman [65]. These structures have been studied in the context of integrable systems (in particular by Zung [113]), but also became a central concept in the works by Symington [98] and Symington-Leung [69] in the context of symplectic geometry and topology, and by Gross-Siebert, Castaño-Bernard, Castano-Bernard-Matessi [52, 53, 51, 50, 13, 14, 15], among others, in the context of mirror symmetry and algebraic geometry. In fact the polygon invariant could have been expressed in terms of this affine structure. It will be interesting to interpret the results of this paper in the context of mirror symmetry; at the least the classification of semitoric systems would give a large class of interesting examples. We hope to explore these ideas in the future.

9.3 Higher dimensions

It is natural to want to extend our 4-dimensional results to higher dimensions. Physically, there is no reason for dimension 4 to be more or less relevant than higher dimensions. The fact is that some of the results that our classification uses (primarily those of Vũ Ngọc, but not exclusively) are 4-dimensional; but in principle there should be extensions to higher dimensions, as the proofs do not involve *tools* that are specific to dimension 4.

9.4 Lagrange Top equations

The heavy top equations in body representation are known to be Hamiltonian on $\mathfrak{se}(3)^*$. These equations describe a classical Hamiltonian system with 2 degrees of freedom on the magnetic cotangent bundle $\mathrm{T}\, S^2_{\|\Gamma_0\|}$. This two degrees of freedom system has a conserved integral but it does not have, generically, additional integrals. However, in the Lagrange case, one can find one additional integral which makes the system completely integrable. It is classically known that the Lagrange heavy top is integrable. Moreover, one can check that it is semitoric, but it is given by a non-proper momentum map.

Problem 9.2. *Develop the theory of semitoric systems $F = (J, H)\colon M \to \mathbb{R}^2$ when the component J generating a Hamiltonian circle action is not proper (one may require at first that F is proper too).*

The authors' general theory does not cover the case stated in Problem 9.2, but many of techniques do extend at least to the case when F is a proper map; we have been exploring this case in [84].

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