MATH 20F PRACTICE FINAL - 28 JULY, 2OO8

Name: ____

(1) True or False? Circle one.

T **F** (a) The row space of A is the column space of A^T

T **F** (b) The inverse of an invertible $n \times n$ matrix A can be found by row reducing the augmented matrix $[A|I_n]$.

T **F** (c) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a finite dimensional vector space V then $\mathcal{B}$ and $\mathcal{C}$ have the same number of vectors in them.

T **F** (d) Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal if and only if their dot product, $\vec{u} \cdot \vec{v}$, is greater than or equal to $\vec{0}$.

T **F** (e) If $\mathcal{B}$ is a basis for a finite dimensional vector space V and $\mathcal{C}$ is a basis for a subspace of V then each vector in $\mathcal{C}$ can be written as a linear combination of the vectors in $\mathcal{B}$.

T **F** (f) The area of the parallelogram whose corners have coordinates $(0,0)$, $(1,0)$, $(2,3)$, and $(3,3)$ is 9.

T **F** (g) If $m < n$ then the columns of an $m \times n$ matrix A could span $\mathbb{R}^n$.

T **F** (h) If A and B are row equivalent matrices then $\text{Row}(A) = \text{Row}(B)$.

T **F** (i) If the second column in a matrix is a pivot column then x_2 is not a free variable.

T **F** (j) The determinant of a matrix is equal to the determinant of its transpose.

T **F** (k) A matrix with n distinct eigenvalues is diagonalizable.

T **F** (l) If A and B are row equivalent matrices then the column space of A is the same as the column space of B.

T **F** (m) A linearly independent set of vectors in $\mathbb{R}^n$ containing n vectors is a basis for $\mathbb{R}^n$.

T **F** (n) A set of more than n vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

T **F** (o) A set of vectors than spans a vector space V contains a basis for V.

T **F** (p) If V is a k-dimensional vector space and S is a set of $k+1$ vectors in V then S contains a basis of V.

T **F** (q) The Gram-Schmidt algorithm returns an orthogonal basis for a given subspace.

T **F** (r) If $\det(A) = d$ then $\det(kA) = k^n d$.

T **F** (s) If A is an 4×4 matrix and the null space of A is a plane in $\mathbb{R}^4$ then the column space of A has a basis consisting of 2 vectors.

T **F** (t) If $\vec{v}$ is a nonzero vector in a vector space V then $\frac{1}{||\vec{v}||}\vec{v}$ is a unit vector in the direction of $\vec{v}$.

(2) Let $A = \begin{bmatrix} 5 & -1 & 1/2 \\ 4 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(a) Calculate $\det(A)$.
(b) Find A^{-1}.
(c) Determine the span of the columns of A.
(d) Find the characteristic polynomial of A.
(e) Find the eigenvalue(s) of A.
(f) For each eigenvalue λ you found in () find a basis for the associated eigenspace E_λ.
(g) If possible, find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If not possible, explain why not.

(3) Let $B = \begin{bmatrix} 1 & 2 & -3 & -4 & 5 & 6 \\ 0 & 3 & -4 & 1 & 9 & 10 \\ 0 & 2 & -1 & 0 & 8 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & 0 & 8 & -3 \\ 0 & 3 & 0 & 0 & -4 & 2 \end{bmatrix}$.

(a) Calculate the determinant of B by choosing clever rows and columns along which to expand.

(b) What is the dimension of the row space of B? Justify your answer.

(c) What is the dimension of the null space of B? Justify your answer.

(4) Let $\mathcal{B} = \left\{ b_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ c_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, c_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ and $\mathcal{D} = \left\{ \vec{d_1}, \vec{d_2} \right\}$ be three bases for $\mathbb{R}^2$. Suppose further that $\vec{d_1} = \vec{c_1} + 3\vec{c_2}$ and $\vec{d_2} = 2\vec{c_1} - \vec{c_2}$.

(a) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.
(b) Find the change of basis matrix from $\mathcal{D}$ to $\mathcal{C}$.
(c) Now use (a) and (b) to find the change of basis matrix from $\mathcal{B}$ to $\mathcal{D}$.
(d) If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ find $[\vec{x}]_{\mathcal{C}}$.
(e) Find $[\vec{y}]_{\mathcal{C}}$ for $\vec{y} = 5\vec{d_1} - 2\vec{d_2}$

(5) Let $U = \begin{bmatrix} 0 & -1 & 3 & 0 & 2 \\ 5 & 2 & 4 & 1 & 1 \\ -1 & 1 & -2 & 7 & 3 \end{bmatrix}$

(a) Find U^T.
(b) Calculate $U^T U$. Does U have orthonormal columns?
(c) Find a basis for the null space of U.
(d) Find a basis for the column space of U.
(e) Find a basis for the row space of U.
(f) Find a basis for the null space of U^T.

(6) Complete the following definitions for an $n \times n$ matrix A
   (a) A is *invertible* if
   (b) The nonzero vector $\vec{v}$ is an *eigenvector* of A if
(7) Please prove two of the following. Indicate clearly which two you would like graded.
   (a) Assuming that A is an $n \times n$ invertible matrix, show that the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution without appealing to the Invertible Matrix Theorem. Then show that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$.
   (b) If V is a vector space and $\vec{v_1}, \ldots, \vec{v}_n$ are vectors in V prove that $\text{Span}\{v_1, \ldots, \vec{v}_n\}$ is a subspace of V.
   (c) Let A be an $m \times n$ matrix. Prove that every vector in $\text{Nul}(A)$ is in the orthogonal complement of $\text{Row}(A)$. (Hint: recall that it is enough to show that any vector in $\text{Nul}(A)$ is orthogonal to the rows of A.)