Algebra Qual Prep: Summer, 2oo8.
Field Theory and Galois Theory Problems

August 24, 2008

1. Let F be a field and $f(x) = x^4 + bx^2 + c \in F[x]$. If K is the splitting field of $f(x)$, prove that the Galois group of K/F is contained in the dihedral group D_4 of order 8.

2. *(i) Let E/F be a field extension. Define the term *transcendence basis* for E/F. Define the term *transcendence degree* for E/F, and state carefully the theorem required for the definition to make sense.
(ii) Let $E = F(x)$ be a simple transcendental extension and let $f, g \in E$. Prove that there exists a $\phi \in F[y, z]$ such that $\phi(f, g) = 0$.

3. *(a) Prove that if $p(x)$ is an irreducible polynomial over a field K then there exists an extension $K(\alpha)$ of K generated by an element α such that $p(\alpha) = 0$.
(b) Let E/K be a Galois extension of degree p^2q where p and q are primes, $q < p$ and q does not divide $p^2 - 1$. Prove that there exist intermediate fields L and M such that $[L : K] = p^2$ and $[M : K] = q$; that such fields L and M must be Galois over K; and that the Galois group of E/K must be Abelian.

4. Let E denote the splitting field of $x^7 - 1$ over $\mathbb{Q}$. Determine all subfields of E, expressing the results in the form $\mathbb{Q}(\alpha)$ for various $\alpha \in \mathbb{Q}$. Show that one of the subfields is $\mathbb{Q}(\sqrt{m})$ for some square-free integer m, and find m. Find the maximal subfield of E that can be embedded in $\mathbb{R}$.

5. *Let K be a field and $f(x) \in K[x]$ a polynomial with no multiple roots in any extension of K. Prove that $f(x)$ is irreducible in $K[x]$ if and only if the Galois group of f over K acts transitively on the roots of f.

6. *Let E be a splitting field of $(x^3 - 2)(x^2 - 3)$ over $\mathbb{Q}$.
   (a) Find a $\mathbb{Q}$-vector space basis for E.
   (b) Determine the structure of the Galois group of E over $\mathbb{Q}$.
   (c) Which subfields of E are normal over $\mathbb{Q}$?

7. *Let F be a field of characteristic $p > 0$, and set $F^p = \{x^p | x \in F\}$.
   (a) Show that F^p is a field.

(b) Say a field is perfect if every irreducible polynomial over K is separable. Show that F is perfect if and only if $F^p = F$.

(c) Show that every finite field is perfect.

(d) Give an example of an imperfect field.

8. Let q be a power of a prime. Prove that the extension $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ is Galois with cyclic Galois group.

9. Let K be the splitting field over $\mathbb{Q}$ of an irreducible cubic $f(x)$. Prove that if $f(x)$ has exactly one real root, then $[K : \mathbb{Q}] = 6$ and that the Galois group of $f(x)$ is isomorphic to S_3.

10. Find the Galois group of the splitting field of $x^5 - 7x^4 + 3$ over $\mathbb{Q}$.

11. *Let k be an algebraically closed field of characteristic $p > 0$ and let $K = k(t)$ be a purely transcendental extension. Let L be the splitting field of the polynomial $X^n - t$ over K, and let $G = Aut_K(L)$. Find the degree of the field extension $[L^G : K]$. (Hint: the answer depends upon the relationship between n and p.)

12. Let K be a splitting field over $\mathbb{Q}$ of the polynomial $(x^3 - 2)(x^2 - 2x - 1)$.

    (a) Find the Galois group of $K/\mathbb{Q}$.

    (b) Let $k = \mathbb{Q}(1 + \sqrt{2}) \subseteq K$. Prove that K/k is Galois and find its Galois group.

    (c) Write down, without repetition, all intermediate fields $k \subseteq L \subseteq K$.

13. *Let p and q be distinct primes. Consider the field extension $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ over $\mathbb{Q}$.

    (a) Prove that the Galois group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

    (b) Prove that every degree two subfield of K is of the form $\mathbb{Q}(\sqrt{m})$ for $m \in \{p, q, pq\}$.

    (c) Show that there is an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

14. Determine the degree of a splitting field over $\mathbb{Q}$ of the polynomial $x^5 - 3$. Describe in purely group-theoretic terms the Galois group of $x^5 - 3$ over $\mathbb{Q}$. Is G Abelian, or nilpotent or solvable?

15. Explicitly construct a field F of sixteen elements and exhibit a polynomial $f(x)$ with with coefficients in $\mathbb{F}_2$ such that the multiplicative group of F is generated by a root of $f(x)$.

16. Factor the polynomial $x^3 - x + 1$ and find the Galois group of its splitting field over $\mathbb{R}$, over $\mathbb{Q}$, and over $\mathbb{Z}_2$.

17. *Let $K = k(x, y)$ where $x^p \in k$ and $y^p \in k$. Then K is a field extension of degree p^2 of k. Show directly that K is not of the form $k(z)$ for any $z \in K$. Show further that there are infinitely many intermediate fields E such that $k \subseteq E \subseteq K$.