Algebra Qual Prep: Summer, 2oo8.
Group Theory Problems

August 3, 2008

1. If G is a group of even order, prove it contains an element of order 2.
2. If G is a group and H is a subgroup of index two then H is normal in G.
3. All subgroups of the quaternion group are normal. s
4. * Find the center of the following groups:
   - $\mathbb{Z}_n$
   - S_n
   - D_n
   - $Gl_n(\mathbb{C})$
5. * If G is a finite group and $G/Z(G)$ is cyclic then G is Abelian.
6. Let G and H be cyclic groups of order m and n, respectively. Categorize the homomorphisms from G to H.
7. Prove that $\mathbb{Q} \times \mathbb{Q}$ is not cyclic.
8. * Prove that a group of order 15 is cyclic.
9. * Let H and K be subgroup of a group G. Prove that HK is a subgroup if and only if $HK = KH$.
10. Let M and N be normal subgroups of a group G with $G = MN$. Prove that $G/M \cap N \cong G/M \times G/N$.
11. If p is prime, prove every group of order p^2 is Abelian.
12. Let G be a group and N a normal subgroup. Show that G is solvable if and only if G/N and N are both solvable.
13. * Let p and q be primes. Show that groups of size p^n, pq, and p^2q are solvable.
14. * If G is a group of order 231 then show that the 11-Sylow subgroup is normal.
15. How many Abelian groups of order $2^4 3^6$ are there?
16. Show that $\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$.