

# Algebra Qual Prep: Summer, 2008.

## Ring Theory Problems

August 10, 2008

Unless otherwise stated,  $R$  is a ring with unity.

- Which of the following pairs of rings are isomorphic? (Prove your assertions.)
  - $2\mathbb{Z}$  and  $3\mathbb{Z}$  (thought of as subrings of  $\mathbb{Z}$ ).
  - $\mathbb{Q}[\frac{5+\sqrt{2}}{4}]$  and  $\mathbb{Q}[x]/(x^2 - 2)$
- Prove or disprove: an ideal of a ring  $R$  contains a unit if and only if it is the whole ring.
- Let  $I_1, \dots, I_n$  be ideals in an integral domain. Prove that if  $I_1 \cap \dots \cap I_n = 0$  then at least one of the  $I_i$  must be  $(0)$ . Give an example where the conclusion fails for an intersection of infinitely many ideals. Then show that if the intersection of infinitely many nonzero ideals  $\{I_1, I_2, I_3, \dots\}$  is, in fact,  $(0)$  then  $I_k \cap I_{k+1} \cap I_{k+2} \dots$ , is  $(0)$  as well.
- Prove that the characteristic of an integral domain is either 0 or a prime.
- \*Give examples of the following (and prove that they are examples):
  - a non-commutative ring;
  - a commutative ring that is not an integral domain;
  - an integral domain that is not a unique factorization domain;
  - a UFD that is not a principal ideal domain;
  - a PID that is not a Euclidean domain;
  - a finite dimensional field extension of  $\mathbb{Q}$  that is not Galois.
  - a finite dimensional field extension of  $\mathbb{F}_p$  that is Galois.
  - a finite dimensional field extension of  $\mathbb{F}_p(x)$  that is not Galois.
- \* Prove that every integral domain can be embedded in a field.
- \* Let  $R$  be an integral domain and prove that if  $R[x]$  is a PID then  $R$  is a field. (Hint: let  $a \in R$  with  $a \neq 0$  and  $a \notin R^*$  and consider  $(a, x) < R[x]$ .)

8. Let  $R$  be a commutative ring and let  $f = r_n x^n + \dots + r_0$  be a zero divisor in  $R[x]$ . Prove that there is a nonzero  $s \in R$  such that  $r_n s = r_{n-1} s = \dots = r_0 s = 0$ .
9. Let  $R$  be a commutative ring and  $I$  an ideal of  $R$ . The radical of an ideal  $I$  is  $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$ .  $I$  is said to be radical if  $I = \sqrt{I}$ . Prove that if  $I$  is maximal then  $I$  is radical.
10. \* Let  $R$  be a local ring (ie,  $R$  has a unique maximal ideal,  $M$ ). Prove that every element of  $R \setminus M$  is a unit. Prove conversely that if  $R$  is a ring in which the units form an ideal,  $R$  must be local.
11. Suppose  $R$  is an integral domain. Prove that if the following two conditions hold then  $R$  is a PID:
  - (a) any two nonzero elements  $a, b \in R$  have a gcd that can be written  $ax + yb$  for some  $x, y \in R$ .
  - (b) if  $a_1, a_2, \dots, a_n$  are nonzero element of  $R$  satisfying  $a_{i+1} \mid a_i$  for all  $i$  then there is a positive  $N$  for which  $a_n$  is a unit times  $a_N$  for all  $n \geq N$ .
12. Prove that if  $R$  is a PID and  $D$  is a multiplicatively closed subset of  $R$  then  $D^{-1}R$  is a PID.
13. Let  $R$  be a ring and denote by  $R[[t]]$  the formal power series over  $R$ . Prove that an element  $f(t) \in R[[t]]$ , say  $f(t) = a_0 + a_1 t + a_2 t^2 + \dots$  is invertible if and only if  $a_0$  is a unit in  $R$ . Determine all ideals of  $R[[t]]$ .
14. \* Let  $k$  be a field and let  $P$  be a prime ideal in  $k[x, y]$ . Show that  $P$  is generated by two elements, by considering cases:  $P \cap k[x] = (0)$  and  $P \cap k[x] \neq (0)$ .
15. \* Calculate  $\text{Spec} \mathbb{C}[z]$ .
16. \* Let  $A$  be an integral domain and let  $S$  be a multiplicative subset of  $A$ , so  $S^{-1}A$  is the localization of  $A$  with respect to  $S$ .
  - (a) Prove that any ideal in  $S^{-1}A$  is of the form  $S^{-1}I$  for some ideal  $I \subseteq A$ .
  - (b) Prove that if  $P \subseteq A$  is a prime ideal of  $A$  such that  $P \cap S = \emptyset$  then  $S^{-1}P \subseteq S^{-1}A$  is a prime ideal.
  - (c) Prove that the natural map  $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$  is injective or give a counterexample.
17. Let  $R$  be a Noetherian ring and  $S$  a multiplicative subset. Prove that  $S^{-1}R$  is Noetherian.
18. Let  $R$  be a finite dimensional  $F$ -algebra. We say that  $R$  is *split* if  $R$  is isomorphic to a direct product of matrices of the form  $M_n(F)$ . A field extension  $K$  of  $F$  *splits*  $R$  if  $R \otimes_F K$  is split as a  $K$ -algebra.
  - (a) Let  $R$  be a simple algebra over its center  $C$ . Show that every ideal of  $R \otimes_C K$  contains an element of the form  $1 \otimes k \neq 0$  for some  $k \in K$ . In particular, if  $K$  is simple, then  $R \otimes_C K$  is also simple.

- (b) Show a finite dimensional simple algebra over an algebraically closed field is simple Artinian, and thus isomorphic to matrices over a division ring.
- (c) Use the previous parts to show that if  $R$  is a simple algebra with center  $C$  and  $K$  is the algebraic closure of  $C$ , then  $R \otimes_C K \cong M_n(K)$ .
- (d) In particular, count dimensions to conclude that  $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$ .