Groups of Special Units

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1 Introduction

Inspired by his observations that every field extension of the rational numbers with abelian Galois group is contained in a pure cyclotomic field and that a single holomorphic function, $f(z) = e^{2\pi i z}$, evaluated at rational numbers is responsible for every such extension, Leopold Kronecker (1823-1891) dreamt that such a function might be found to thus classify abelian extensions of imaginary quadratic fields. His dream came true in the form of complex multiplication, hailed by David Hilbert (1862-1943) as "not only the most beautiful part of mathematics but also of all science"[9]. In fact, the natural generalization, namely to find a function or family of functions whose special values generate the maximal abelian extension of any number field, became Hilbert's twelfth problem.

In [10] Harold Stark put forth a conjectural program about the special values of L-functions that, in some settings, offers a solution to Hilbert's twelfth problem. I will motivate and describe the abelian conjectures where the order of vanishing at $s = 0$ is one, but there are conjectures about the first non-vanishing coefficient in the higher order of vanishing setting as well. I will then map out the related work of Anderson, Lang, Kubert, and Sinnott, whose work has inspired my own.

The mingling of algebra and analysis that Hilbert so revered is typified by the analytic class number formula, which relates the first nonzero Taylor series coefficient of the Dedekind zeta function to algebraic invariants of the number field K. Let K/k be a Galois extension of number fields. The Dedekind zeta function of K is defined by its Euler product in the right half-plane $\Re(s) > 1$ and there it factors as the product of its constituent Artin L-functions

$$\zeta_K(s) = \prod_p (1 - \mathcal{N}p^{-s})^{-1} = \prod_\chi L_{K/k}(s, \chi)$$

where $\mathcal{N}(p)$ is the absolute norm of p, the first product is taken over prime ideals p of the ring of integers of K, and the second is the product over irreducible characters of the Galois group of K/k. Dirichlet's analytic class number formula then says that the Dedekind zeta function has a Taylor expansion about $s = 0$ that looks like

$$\zeta_K(s) = -\frac{hR}{W} s^{r_1+r_2-1} + O(s^{r_1+r_2}),$$

where r_1 and r_2 are the number of real and half the number of complex embeddings of K, respectively, h is the class number, R the regulator, and W the number of roots of unity in K. Stark wanted to split the regulator matrix into character components in correspondence with the L-function decomposition of the zeta function. Furthermore, he wanted to see whether the leading terms of the L-functions factor into a transcendental part corresponding to the regulator and a rational (or at worst algebraic) part corresponding to $-h/W$ in the class number formula.

To give an account of the Stark conjectures we need to set the stage a bit. Let K/k be an abelian extension of number fields. Let S be a finite set of primes in k containing all infinite and ramified primes of k, and at least one prime, say v, that splits completely from k to K, and at least two primes overall. Let $L_S(s, \chi)$ be the Dirichlet L-function associated to $\chi \in \widehat{G}$ with Euler-factors associated to primes in S removed. Then there exists an S-unit $\epsilon \in K$, unique up to roots of unity, such that

$$L'_S(0, \chi) = -\frac{1}{W} \sum_{\sigma} \overline{\chi}(\sigma) \log |\epsilon|_{w^\sigma}, \quad \forall \chi \in \widehat{G}.$$

The sum on the right is over σ in the Galois group of K/k and w is a fixed prime above v. In this setting, Stark further conjectures that the W^{th} root of ϵ generates an abelian extension not only over K, but over k, and experimental data confirms this.

In [1], Anderson defines an *almost abelian group* to be one such that every commutator is central and squares to the identity. He defines $G^{ab+\epsilon}$ to be the quotient of $G(\overline{Q}/Q)$ universal for continuous homomorphisms to almost abelian profinite groups. He shows that the corresponding Galois extension of $\mathbb{Q}$ is, in fact, the maximal abelian extension of $\mathbb{Q}$ with the fourth roots of rational primes and certain gamma-monomials adjoined. As he remarks,

"The relations standing between the Main Formula, the index formulas of Sinnott, Deligne reciprocity, the theory of Fröhlich, the theory of Das, the theory of the group cohomology of the universal ordinary distribution and Stark's conjecture and its variants deserve to be thoroughly investigated. We have only scratched the surface here. Stark's conjecture is relevant in view of the well known expansion

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$

of the Hurwitz zeta function at $s = 0$" [1].

2 Thesis Research

2.1 Units in the field of modular functions

Kubert and Lang prove in [4] that if U is the group of units in the field of modular functions on congruence subgroups of $SL_2(\mathbb{Z})$, and S is the subgroup generated by the Siegel units, then U/S has exponent two. Their work begs the question of how to construct explicitly elements in S whose square roots are in $U \setminus S$. This relates to Anderson's construction over $\mathbb{Q}^{ab}$ because the special values of these functions and their square roots might together generate almost abelian extensions of an imaginary quadratic extension of $\mathbb{Q}$. Anderson says

"Perhaps there is an analogue... over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing. "

I was able to find combinations of Siegel units analogous to Anderson's gamma-monomials that are expected to have a square root in U. I can show, using Shoeneberg's theorem on the level of a modular function given its width at the cusps [6], that if the square root of an element of S of level N has a level then that level is $2N$. In order to better understand the behavior of these square roots I decided a finer study of Anderson's work over $\mathbb{Q}$ was in order. What follows is a discussion of that study.

2.2 Explicit squares in cyclotomic fields

Anderson's result discussed above includes the construction of gamma-monomials that, together with roots of unity, generate almost abelian extensions of $\mathbb{Q}$. Let k be the cyclotomic field $\mathbb{Q}(e^{2\pi i/pq})$ and let $\alpha \in k$ be a unit such that $k(\sqrt{\alpha})/\mathbb{Q}$ is almost abelian. Let σ be an element of the Galois group of $k/\mathbb{Q}$. Then $\frac{\alpha}{\alpha^\sigma}$ is the square of a unit in k and, thus, has a square root in k. By considering the group of relations satisfied by gamma-monomials, I can find this square root explicitly. Thus, finding a multiplicative basis for the units of index class number extending these squares would prove certain class numbers are even.

My work in this area led to two results. First, I found a family of new trigonometric identities indexed by products of distinct primes pq, exemplified by $pq = 15$:

$$\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)} = 1.$$

Second, in light of the expansion of the Hurwitz zeta function, and the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

these identities show that certain sums of Hurwitz zeta functions vanish to second order at $s = 0$. For example,

$$\zeta(s,\frac{3}{15})-\zeta(s,\frac{1}{15})-\zeta(s,\frac{4}{15})-\zeta(s,\frac{9}{15})+\zeta(s,\frac{12}{15})-\zeta(s,\frac{14}{15})-\zeta(s,\frac{11}{15})-\zeta(s,\frac{6}{15})=$$

$$s\log\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)}+O(s^2)$$

vanishes to second order at $s=0$. Therefore, the coefficient of s^2 is of interest as we shall see below.

2.3 L-functions attached to characters of conductor p

The vanishing of these Hurwitz zeta functions is, in turn, connected to the known first-order vanishing of L-functions associated to even characters of conductor p. Let χ_p be a non-trivial even character of conductor p, and χ_{pq} its inflation to the group $(\mathbb{Z}/pq\mathbb{Z})^\times$. Suppose $S=\{p,q\}$, and recall that $L_S(s,\chi)$ has Euler factors associated to primes in S removed. Then the second order vanishing of $(pq)^sL(s,\chi_p)(1-q^{-s})$ at $s=0$ can be expressed in terms of the second order vanishing of Hurwitz zeta functions as in the following example for $S=\{3,5\}$

$$15^sL(s,\chi_5)(1-3^{-s})=$$

$$15^sL(s,\chi_5)(1-\chi_5(3)3^{-s})-35^sL(s,\chi_5)[3^{-s}(1-\chi_5(3))]=$$

$$15^sL_S(s,\chi_{15})-5^sL(s,\chi_5)+5^s\chi_5(3)L(s,\chi_5)=$$

$$\sum_{\substack{a=1\\(a,15)=1}}^{14}\zeta(s,\frac{a}{15})\chi_{15}(a)-\sum_{b=1}^{4}\zeta(s,\frac{3b}{15})\chi_5(b)+\sum_{c=1}^{4}\zeta(s,\frac{3c}{15})\chi_5(3c).$$

If we now use the character orthogonality relations to isolate the $a=1,4,11,14$, $b=2,3$, and $c=1,4$ terms then we see that the first non-vanishing coefficient from the previous section is, in fact, the lead term of $-2\cdot 15^sL(s,\chi_5)(1-3^{-s})$, which is known (because the Stark conjectures are proved in this setting) to be

$$-2\log(3)L'(0,\chi_5)=-\log(3)[\log|(1-\zeta_5)(1-\zeta_5^{-1})|+\log|(1-\zeta_5^2)(1-\zeta_5^{-2})|]=$$

$$-2\log(3)\log|(1-\zeta_5)(1-\zeta_5^2)^{-1}|,$$

where ζ_5 is a primitive fifth root of unity.

3 Future Directions

My research currently has three branches:

1. Further investigate my proposed square root.
2. Find a multiplicative basis for the subgroup of cyclotomic units with index class number in the full unit group of $\mathbb{Q}(\zeta_m)^+$.
3. Let $k = \mathbb{Q}(\zeta_m)^+$ and $G = G(k/\mathbb{Q})$. Let S be the set containing the infinite prime of $\mathbb{Q}$ and all the finite primes dividing m. Find a $\mathbb{Z}[G]$-submodule of U_S, the S-units in k, call it C, that contains the cyclotomic numbers and is of index class number in U_S. Show that $\widehat{H}^{q-2}(G, X) \cong \widehat{H}^q(G, C)$, where X is the $\mathbb{Z}[G]$-module of degree zero divisors of primes of k supported at the primes dividing m.

To accomplish (1), I will compute the transformation of my proposed square root using software I am developing. There are three possibilities for the outcome. First, the square root could fail to be in U. Second, the square root could transform under $\Gamma(N)$ for some N, but fail to have cyclotomic coefficients. Third, the square root could transform under $\Gamma(N)$ for some N and have cyclotomic coefficients. Let the imaginary quadratic base field be k. In the third case, special values of the function would generate an abelian extension K/k and its square root would generate a quadratic extension of K that is abelian over k. In this case, there would exist units that are squares. Together with (2), this could be used to show certain class numbers are even. In the second case, the square root would generate quadratic extensions of the maximal abelian extension of k in analogy with Anderson's construction. In this case, the next step would be to investigate to what extent these square roots generate the maximal almost abelian extension of k.

For (2), I will start with $m = pq$, for p and q distinct odd primes. Let $k = \mathbb{Q}(\zeta_{pq})^+$. Sinnott proves in [9] that the cyclotomic units have index in the full group of units equal to the class number of k. I have been working on finding a multiplicative basis for these units with an eye towards proving certain class number are even using results of Anderson. I have used Stark's conjecture in conjunction with explicit group-ring determinants and successfully accomplished this in examples. There are two avenues that I plan to explore more thoroughly: using Tate's representation-theoretic reformulation of Stark's conjectures [11] to break up the group determinant into character components; and using Sinnott's proof to break the construction of a basis into steps correlated to his intermediate index calculations.

(3) is a natural question that arises in the theory of Tate sequences and was pointed out to me by Popescu. In the case that m is a prime power, the cyclotomic units, call

them C_S, are already index class number in U_S. In this setting, C_S is isomorphic to $\mathbb{Z}[G]\epsilon$, where ϵ is the Stark unit in k. Furthermore, the group of divisors on primes above p and infinity, which we shall call Y_S, is simply $Y_S \cong \mathbb{Z} \oplus \mathbb{Z}[G]$. Thus, the short exact sequence

$$0 \to X_S \to Y_S \overset{deg}{\to} \mathbb{Z} \to 0$$

splits and we see that X_S is isomorphic to $\mathbb{Z}[G]$. Hence, both C_S and X_S are cohomologically trivial. I am currently working on the next case, namely $m = pq$.

4 Bibliography

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