Convergence of Adaptive Methods in the FEEC Framework, case k = n

Adam Mihalik
Advisor: Michael Holst

UCSD Mathematics

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Introduction

Finite element exterior calculus (FEEC) is a framework which applies general results on differential complexes to a large class of numerical methods, in particular, mixed finite element problems. In this talk I will discuss show how this framework can be used to prove a convergence and optimality result for a class of adaptive mixed finite element problems.

Outline

Hilbert Complexes [1]

Definition

A Hilbert complex (W, d) is a sequence of Hilbert spaces W^k equipped with closed, densely-defined linear maps d^k, with domain V^k, such that:

$$\mathrm{d}^k \colon V^k \subset W^k \to V^{k+1} \subset W^{k+1}$$
$$\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$$

Definition

The dual complex (W^*, d^*) of a Hilbert complex (W, d) consists of the spaces $W_k^* = W^k$ and adjoint operators $\mathrm{d}_k^* = (d^{k-1})^*$.

$$\cdots \to V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \to \cdots$$

$$\cdots \leftarrow V_{k-1}^* \xleftarrow{d_k^*} V_k^* \xleftarrow{d_{k+1}^*} V_{k+1}^* \leftarrow \cdots$$

Hilbert Complexes

Definition

The Domain complex (V,d) of a Hilbert complex (W,d) is the complex consisting of the domains V^k defined above, with inner product:

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle d^k u, d^k v \rangle_{W^{k+1}}$$

Definition

Given two Hilbert complexes, (W, d) and (W', d'), a morphism of Hilbert complexes is a sequence of bounded linear maps $f^k : W^k \to W'^k$, such that $d'^k f^k = f^{k+1} d^k$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & V^k & \xrightarrow{d^k} & V^{k+1} & \longrightarrow & \cdots \\
 & & \downarrow f^k & & \downarrow f^{k+1} & & \\
\cdots & \longrightarrow & V'^k & \xrightarrow{d'^k} & V'^{k+1} & \longrightarrow & \cdots
\end{array}$$

Hodge Decomposition

Lemma

A Hilbert complex can be decomposed into a direct sum of exact, co-exact and harmonic components. For a closed Hilbert complex, we have the strong Hodge decomposition:

$$W^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp_W} = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W} = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}^*_k$$

$$V^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp_V} = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}$$

co-boundaries: $\mathfrak{B}^k = \mathrm{d}^{k-1} V^{k-1}$

co-cycles: $\mathfrak{Z}^k = \mathrm{Ker}\, \mathrm{d}^k$

harmonic space: $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp}$

Abstract Poincaré Inequality

Lemma

If (V,d) is a bounded, closed Hilbert complex, then there exists a constant c_P such that:

$$\|v\|_V \leq c_P \|d^k v\|_V, \forall v \in \mathfrak{Z}^{k\perp}$$

The abstract Hodge Laplacian/Variational Problem

Definition

The abstract Hodge Laplacian is the operator:

$$L = dd^* + d^*d, \qquad W^k \to W^k$$

with domain: $D_L = \{u \in V^k \cap V_k^* \mid du \in V_{k+1}^*, d^*u \in V^{k-1}\}$

Variational Problem

If $u \in D_L$ is a solution of Lu = f, then it also satisfies the following variational principle:

$$\langle du, dv \rangle + \langle d^*u, d^*v \rangle = \langle f, v \rangle, \qquad \forall v \in V^k \cap V_k^*$$

Minimization Problem

$\inf_{v \in V^k \cap V_k^*} \frac{1}{2}\langle dv, dv \rangle + \frac{1}{2}\langle d^*v, d^*v \rangle - \langle f, v \rangle$

Mixed Variation Problem

Mixed Variational Problem

Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle = 0, \qquad \forall \tau \in V^{k-1}$$

$$\langle \mathrm{d}\sigma, v \rangle + \langle\, \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle = \langle f, v \rangle, \qquad \forall v \in V^k$$

$$\langle u, q \rangle = 0, \qquad \forall q \in \mathfrak{H}^k$$

Saddle Point Problem

$\inf_{v \in V^k} \sup_{q \in V^{k-1}} \langle\, \mathrm{d}q,\ v \rangle - \frac{1}{2}\langle q, q \rangle + \frac{1}{2}\langle\, \mathrm{d}v, \mathrm{d}v\, \rangle - \langle\, f, v\, \rangle$

Approximation by a Subcomplex

Mixed Variational Problem

Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that:

$$\langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle = 0, \qquad \forall \tau \in V_h^{k-1}$$

$$\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle = \langle f, v \rangle, \qquad \forall v \in V_h^k$$

$$\langle u_h, q \rangle = 0, \qquad \forall q \in \mathfrak{H}_h^k$$

Assumptions: i_h, π_h

$i_h : V_h \hookrightarrow V$ is an inclusion mapping, morphism of Hilbert complexes.
$\pi_h : V \to V_h$ is V-bounded, surjective, and idempotent.

Error Bounds on Approximate Solution

T 3.9 AFW

Let (V_h, d) be a family of subcomplexes of the domain complex (V, d) of a closed Hilbert complex, parameterized by h and admitting V-bounded cochain projection, and let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ be the solution to the MVP, and let $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ be the solution to discrete MVP. Then

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \leq C(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\| + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|)$$

$$\mu = \sup_{r \in h^{\mathfrak{H}}, \|r\|=1} \|(I - \pi_h^k) r\|$$

Proof Sketch:

1) Relate bilinear form of discrete and continuous solutions with discrete test function

2) Inf-Sup condition holds for discrete complex.

3) Bound from above the $B(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h)$ in terms of continuous solution. Where τ, v, q are orthogonal projections of σ, u and p into discrete space.

4) Factor out test function, account for harmonic term, and use triangle inequality.

Assume Ω is a bounded domain in R^n with a piecewise smooth Lipschitz boundary, and let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω.

$$\delta : \Lambda^k \to \Lambda^{k-1}, \text{ defined by } \star\delta\omega = (-1)^k d \star \omega$$

Let $H\Lambda^k$ be the space of forms in $L^2\Lambda^k$ with exterior derivative in $L^2\Lambda^{k+1}$. The exterior derivative operating on $L^2\Lambda(\Omega)$ forms a Hilbert Complex with the associated domain complex:

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

Hodge-de Rham Problems

$(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ is a solution to the Abstract Hodge-Laplacian on the de Rham complex if and only if:

$$\begin{gathered} \sigma = \delta u, d\sigma + \delta du = f - p \quad \text{in } \Omega \\ \text{tr} \star u = 0, \text{ tr} \star du = 0 \text{ on } \partial\Omega \\ u \perp \mathfrak{H}^k \end{gathered}$$

de Rham Complex in $\Re^3$:

$$0 \rightarrow H^1(\Omega) \xrightarrow{grad} H(curl;\Omega) \xrightarrow{curl} H(div;\Omega) \xrightarrow{div} L^2(\Omega) \rightarrow 0$$

Hodge-de Rham Problems, case k = n

$(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ is a solution to the Abstract Hodge-Laplacian on the de Rham complex if and only if:

$$\sigma = \delta u, d\sigma = f \quad \text{in } \Omega$$
$$\text{tr} \star u = 0 \text{ on } \partial\Omega$$

de Rham Complex in $\mathfrak{R}^n$:

$$0 \to H^1(\Omega) \xrightarrow{d} \cdots \to H(div;\Omega) \xrightarrow{div} L^2(\Omega) \xrightarrow{d} 0, \tag{1}$$

Outline

Basic Ideas for Adaptive FEM

In order to improve the convergence rate, refine the mesh non-uniformly, depending on the behavior of the solution.

SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE

1) Solve the discrete problem on the current mesh.
2) Use computable *a posteriror* error indicators based on discrete solution(s) and problem data to estimate where the error is large.
3) Strategically mark elements for refinement, using the element error indicators and problem data.
4) Refine the mesh such that it is still conforming and shape regular.

Adaptive FEM for Mixed Problem [2]

Chen/Holst/Xu(06) established the convergence and optimality of an adaptive mixed FEM for the Poisson equation for simply connected polygonal domains in 2 dimensions.

Given $f \in L^2(\Omega)$, find $(\sigma, u) \in \Sigma \times U$ satisfying:

$$\begin{aligned} (\sigma, \tau) - (\text{div}\ \tau, u) &= 0 \qquad \forall \tau \in \Sigma \\ (\text{div}\ \sigma, v) &= (f, v) \qquad \forall v \in U_H \end{aligned}$$

$$\Sigma = H(\text{div}, \Omega) \quad U = L^2(\Omega)$$

-One main issue is lack of orthogonality. Quasi-orthogonality is proven using the orthogonality of the error to the divergence free subspace of Σ_H, and a discrete stablity result that bounds the remaining error by data oscillation.

$$(1-\delta)\|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\delta}\text{osc}^2(f_h, T_H)$$

-No need to deal with harmonics, given the choice of domain.

A Posteriori Error indicator in FEEC

-Demlow/Hirani (2012) have proven an a posteriori error estimator for Hodge-de Rham problems in the FEEC Framework.

$$\begin{aligned}
&\|e_\sigma\|_V + \|e_u\|_V + \|e_p\| \\
&\qquad \leq \frac{1}{\gamma} \sup{}_{(\tau,v,q)\in V^{k-1}\times V^k\times\mathfrak{H}} \left(\langle \sigma_h, \tau - \pi_h\tau\rangle - \langle d(\tau-\pi_h\tau), u_h\rangle \right. \\
&\qquad + \langle f - d\sigma_h - p_h, v - \pi_h v\rangle - \langle du_h, d(v-\pi_h v)\rangle + \langle e_u, q\rangle \left. \right) \\
&\qquad + (1+\frac{1}{\gamma})\|P_{\mathfrak{H}} p_h - p_h\|
\end{aligned}$$

A Posteriori Error indicator in FEEC

Notable Points:

1) Harmonic terms are treated a bit differently than other terms since $\mathfrak{H}_h^k$ isn't necessarily a subspace of $\mathfrak{H}^k$.

2) Estimator is proven to be both reliable and efficient.

3) Integration by parts was needed multiple times when transforming the above inequality for analysis of error indicators. Since we don't necessarily know the discrete harmonic component of f, this forces $f \in H^1\Lambda^k(K)$.

4) Important properties and analysis of the commuting quasi-interpolant were needed to bound the terms of the above inequality. This new interpolant will be useful in proving the upper-bound in being presented in the next section.

Outline

Quasi-Orthogonality- FEEC framework case k = n

Given a triangulation $\mathcal{T}_{\mathcal{H}}$ of the domain Ω, we will define oscillation as follows:

$$\operatorname{osc}(f, \mathcal{T}_H) := \|h(f - f_H)\|_{0,\mathcal{T}_{\mathcal{H}}} := \Big(\sum_{T \in \mathcal{T}_h} \|h_T(f - f_H)\|_T^2\Big)^{1/2}$$

Theorem

Let k = n. Given $f \in L^2\Lambda^k(\Omega)$ and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \leq \sqrt{C_0}\|\sigma - \sigma_h\| osc(f_h, \mathcal{T}_H)$$

and for any $\delta > 0$,

$$(1 - \delta)\|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\delta} osc^2(f_h, \mathcal{T}_H) \qquad (2)$$

-Uses a discrete stability result

Discrete Stability

Theorem

(Discrete Stability Result) Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be two nested conforming triangulations. Let $\tilde{\sigma}_h = L_h^{-1} f_H$ and $\sigma_h = L_h^{-1} f_h$. Then there exists a constant C_0 s.t.

$$\|\sigma_h - \tilde{\sigma}_h\| \leq \sqrt{C_0} osc(f_h, \mathcal{T}_H) \tag{3}$$

Error Indicator

Definition

Element Error Estimator

$$\eta_T^2(\sigma_H) = h_T \| [[\mathrm{tr} \star \sigma_H]] \|_{0,\partial T}^2 + h_T^2 \| \delta \sigma_H \|_{0,T}^2 + h_T^2 \| d(\sigma - \sigma_H) \|_{0,T}^2$$

For a subset $\tilde{\mathcal{T}}_H \subset \mathcal{T}_H$, define

$$\eta^2(\sigma_H, \tilde{\mathcal{T}}_H) := \sum_{T \in \tilde{\mathcal{T}}_H} \eta_T^2(\sigma_H)$$

Continuity of the Error Indicator

Theorem

Continuity of the Error Estimator: Suppose $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ and σ_h, σ_H are the solutions to the respective discrete problems on these meshes. Then we have:

$$\beta(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)) \leq \|\sigma_h - \sigma_H\|^2 + osc^2(\sigma_h, \mathcal{T}_H) \quad (4)$$

Upper Bound

Theorem

Continuous Upper-Bound

$$\|\sigma - \sigma_H\|^2 \leq C_1 \eta^2(\sigma_H, \mathcal{T}_H) \quad (5)$$

Analyze each component of the Hodge-decomposition separately:

$$\sigma - \sigma_H = (\sigma - P_{\mathfrak{Z}^\perp}\sigma_H) - P_{\mathfrak{B}^{k-1}}\sigma_H - P_{\mathfrak{H}^{k-1}}\sigma_H,$$

1)Use continuous stability and Galerkin orthogonality.

$$\|(\sigma - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 \leq C\eta^2(\sigma_H, F_H), \quad (6)$$

2) Integration by parts, interpolation properties:

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 \leq C\eta^2(\sigma_H, F_H) \quad (7)$$

3) $\sigma_H \perp \mathfrak{H}_H, \sigma \perp \mathfrak{H}$, bound $\delta(\mathfrak{H}, \mathfrak{H}_H)$

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \leq C\|\sigma - \sigma_H\|^2, C < 1 \quad (8)$$

Convergence of AMFEM

In order to make the next few arguments more readable, we make the following notation:

$$e_k = \|\sigma - \sigma_k\|^2, E_k = \|\sigma_{k+1} - \sigma_k\|^2, o_k = \text{osc}^2(f, \mathcal{T}_k),$$
$$\hat{o}_k = \text{osc}^2(f_{k+1}, \mathcal{T}_k) \text{ and } \eta_k = \eta^2(\sigma_k, \mathcal{T}_k)$$

Combining the continuity of the error indicator result with the marking strategy yields:

Lemma

$$\beta\eta_{k+1} \leq \beta(1 - \lambda\theta)\eta_k + E_k + \hat{o}_k \tag{9}$$

Convergence of AMFEM

$$(1-\delta)e_{k+1} \leq e_k - E_k + \frac{C_0}{\delta}\hat{o}_k, \text{ for any } \delta > 0, \tag{10}$$

$$\beta\eta_{k+1} \leq \beta(1-\lambda\theta)\eta_k + E_k + \hat{o}_k \tag{11}$$

$$e_k \leq C_1\eta_k \tag{12}$$

Basic algebra using these three results leads to:

Theorem

When

$$0 < \delta < min\{\frac{\beta}{2C_1}\theta, 1\},$$

there exists $\alpha \in$ *(0,1) and* C_δ *such that*

$$(1-\delta)e_{k+1} + \beta\eta_{k+1} \leq \alpha[(1-\delta)e_k + \beta\eta_k] + C_\delta\hat{o}_k \tag{13}$$

Convergence of AMFEM

Theorem

Termination in Finite Steps: Let σ_k *be the solution obtained in the kth loop in the algorithm AMFEM, then for any* $0 < \delta < min\{\frac{\beta}{2C_1}\theta, 1\}$*, there exists positive constants* C_δ *and* $0 < \gamma_\delta < 1$ *depending only on given data and the initial grid such that,*

$$(1-\delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k) + \zeta osc^2(f, \mathcal{T}_k) \leq C_q\gamma_\delta^k,$$

and the algorithm will terminate in finite steps.

Outline

-This takes CHX convergence analysis from 2-D connected domains to arbitrary dimensions and arbitrary topology.
-Algorithm implementation is more efficient than CHX since now it doesn't mark separately for oscillation.
-In addition to the implementation benefits, it is also more efficient when oscillation is negligible or very large; i.e. you don't waste computational effort on wasteful marking.
-Discrete version of bounds also proved, and these can be used to show optimality of the method.

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus: from Hodge theory to numerical stability.
Bull. Amer. Math. Soc. (N.S.), 47(2):281–354, 2010.

Long Chen, Michael Holst, and Jinchao Xu.
Convergence and optimality of adaptive mixed finite element methods.
Math. Comp., 78(265):35–53, 2009.

Alan Demlow and Gerhard Dziuk.
An adaptive finite element method for the Laplace-Beltrami operator on implicitly defined surfaces.
SIAM J. Numer. Anal., 45(1):421–442 (electronic), 2007.

Gerhard Dziuk.
Finite elements for the Beltrami operator on arbitrary surfaces.
In *Partial differential equations and calculus of variations*, volume 1357 of *Lecture Notes in Math.*, pages 142–155. Springer, Berlin, 1988.

Michael Holst and Ari Stern.
Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces.
Found. Comput. Math., 12(3):263–293, 2012.

Michael Holst and Ari Stern.
Semilinear mixed problems on Hilbert complexes and their numerical approximation.
Found. Comput. Math., 12(3):363–387, 2012.

Pedro Morin, Ricardo H. Nochetto, and Kunibert G. Siebert.
Convergence of adaptive finite element methods.
SIAM Rev., 44(4):631–658 (electronic) (2003), 2002.
Revised reprint of “Data oscillation and convergence of adaptive FEM” [SIAM J. Numer. Anal. **3**8 (2000), no. 2, 466–488 (electronic); MR1770058 (2001g:65157)].