This examination contains 4 problems. This is a closed book exam.

Show all your computations and justify/explain your answers.
Calculators and electronic devices are NOT allowed.

(1) (10 points) Let $f$ be differentiable on $[a, b]$. Let $c \in \mathbb{R}$ be so that $f'(a) < c < f'(b)$. Prove that there exists some $x \in (a, b)$ so that $f''(x) = c$.

(2) (a) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function which satisfies
\[
f = f' \quad f(0) = 0
\]
Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function which satisfies
\[
f = f' \quad f(0) = c
\]
for some $c \in \mathbb{R}$. Prove that $f(x) = ce^x$. (You may use without a proof the fact that $(e^x)' = e^x$.)

(3) (a) (5 points) Define what it means for a bounded function to be Riemann integrable on an interval $[a, b]$.

(b) (5 points) Using the definition, prove that
\[
f(x) = \begin{cases} 0 & x \neq 1 \\ 3 & x = 1 \end{cases}
\]
is Riemann integrable on $[0, 2]$.

(4) (7 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that there are $\ell \in \mathbb{R}$ and $A \in \mathbb{R}$ so that
\[
\lim_{x \to \infty} f(x) = f'(x)^2 = \ell \quad \text{and} \quad \lim_{x \to \infty} f(x) = A
\]
Prove that $A = \ell$. 