This examination contains 4 problems. This is a closed book exam.

Show all your computations and justify/explain your answers.
Calculators and electronic devices are NOT allowed.

(1) (10 points) Let $f$ be function on $[a, b]$. Assume that $f'$ is continuous on $[a, b]$ and $f''$ exists on $(a, b)$. Prove that there exists some $x \in (a, b)$ so that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(x)}{2}(b - a)^2.$$  

(2) (a) (8 points) Let $f : [0, 1] \rightarrow [0, 1]$ be a differentiable function which satisfies $|f'(t)| < c < 1$ for all $t \in [0, 1]$. Prove that for every $x \in [0, 1]$ the sequence

$$f(x), f(f(x)), \ldots$$

is a Cauchy sequence.

(b) (7 points) Let $f$ be as in part (a). Prove that there exists a unique $x \in [0, 1]$ so that $f(x) = x$.

(3) (a) (5 points) Define what it means for a bounded function to be Riemann integrable on an interval $[a, b]$.

(b) (5 points) Let $f$ be a bounded non-negative function which is Riemann integrable on $[0, 3]$. Assume further that $f \geq 1$ on $[1, 2]$. Using the definition, prove that

$$\int_{0}^{3} f(x)dx \geq 1.$$  

(4) (7 points) Let $f : [0, \infty] \rightarrow \mathbb{R}$ be twice differentiable. Assume that

$$f + f'' = -f'.$$

Prove that $f$ is bounded (from above and below) on $[0, \infty]$.

(Hint: Prove that $f^2 + (f')^2$ is decreasing.)