EXAM I: PRACTICE SHEET

Please go over all the HW problems as well as notes from class.

(1) Let $S$ and $T$ be two nonempty, bounded subsets of the real numbers. Prove that
$$\sup\left(T \cup S\right) = \max\left\{\sup T, \sup S\right\}.$$ 

(2) Let $a, b, d$ be nonnegative real numbers and let $c$ be a positive real number. 
Find with justification the infimum and the supremum of the following set.
$$\left\{\frac{an + b}{cn + d} : n = 1, 2, \ldots\right\}.$$ 

(3) Let $b > 1$ be a real number. Define the sequence $\{a_n\}$ as follows: $a_1 = 1$ and
$$a_{n+1} = \frac{a_n + b}{2} \text{ for all integers } n.$$
(a) Show that $0 < a_n < b$ for all $n$.
(b) Find $\sup\{a_n : n = 1, 2, \ldots\}$. Justify your answer.
(c) Find $\inf\{a_n : n = 1, 2, \ldots\}$. Justify your answer.

(4) Let $n \in \mathbb{N}$ and let $a > 0$. Prove that
$$\inf\{x : x \in \mathbb{R}, x > 0, x^n > a\} = a^{1/n}.$$ 


(6) Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces. Let $X = X_1 \times X_2$.
(a) Define
$$\rho_\infty((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$ 
Prove that $(X, \rho_\infty)$ is a metric space.
(b) Define
$$\rho_1((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$ 
Prove that $(X, \rho_1)$ is a metric space.
(c) Define
$$\rho_2((x_1, x_2), (y_1, y_2)) = \left(d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2\right)^{1/2}.$$ 
Prove that $(X, \rho_2)$ is a metric space.
(d) Let $(X_i, d_i)$ be $\mathbb{R}$ with respect to the standard metric for $i = 1, 2$. 
Describe $N_1(0)$ in $X = \mathbb{R}^2$ with respect to the metrics $\rho_\infty, \rho_1$ and $\rho_2$.

(7) Determine all the limit points of the following sets in $\mathbb{R}$ and determine whether the sets are open or closed (or neither).
(a) All integers.
(b) The interval $(a, b]$. 

(c) All numbers of the form $\frac{1}{n}$, where $n = 1, 2, 3, \ldots$
(d) All rational numbers.
(e) All numbers of the form $(-1)^n + \frac{1}{m}$, where $m, n = 1, 2, \ldots$
(f) All numbers of the form $\frac{1}{n} + \frac{1}{m}$ where $m, n = 1, 2, \ldots$
(g) All numbers of the form $(-1)^n \frac{n}{1+(1/n)}$, where $n = 1, 2, \ldots$

(8) Let $S$ be a subset of $\mathbb{R}^n$. Show that
(a) $S'$ is a closed set.
(b) If $S \subset T$, then $S' \subset T'$.
(c) $(S \cup T)' = S' \cup T'$.

(9) In this problem you may use without a proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ with the usual metric.
(a) Show that $A = \{a + bi : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$ with the usual metric on $\mathbb{C}$.
(b) Show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ with the usual metric on $\mathbb{R}^n$.
(c) Let $A$ be as in part (a). Show that $A^n$ is dense in $\mathbb{C}^n$ with the usual metric on $\mathbb{C}^n$.

(10) Let
$$
\ell^\infty = \left\{(a_1, a_2, \ldots) : a_n \in \mathbb{R} \& \sup\{|a_n| : n \in \mathbb{N}\} < \infty\right\}.
$$
We use componentwise addition and scalar multiplication on $\ell^\infty$. For any $(a_n) \in \ell^\infty$, define
$$
\|(a_n)\|_\infty = \sup\{|a_n| : n \in \mathbb{N}\}.
$$
Define
$$
d((a_n), (b_n)) = \|(a_n - b_n)\|_\infty
$$
for any two sequences $(a_n)$ and $(b_n)$.
(a) Show that $d((a_n), (b_n))$ is a finite number for any two $(a_n)$ and $(b_n)$ in $\ell^\infty$.
(b) Show that $(\ell^\infty, d)$ is a metric space.
(c) (Bonus Problem) Is $(\ell^\infty, d)$ a separable space? (A metric space $X$ is called separable if there is a countable and dense subset $A \subset X$, e.g. $\mathbb{R}$ with the usual metric is separable.)

(11) For any $m \in \mathbb{N} \cup \{0\}$ let
$$
X_m = \{p(z) = \sum_{n=0}^m a_n z^n : z \in \mathbb{C}, |z| = 1, a_n \in \mathbb{C}\},
$$
and let $X = \bigcup_{m=0}^\infty X_m$.
(a) For any $p(z) \in X$ show that the set $\{|p(z)| : |z| = 1\}$ is a bounded set and conclude that
$$
\sup\{|p(z)| : |z| = 1\}
$$
exists.
(b) For any \( p(z), q(z) \in X \) define \( d(p(z), q(z)) = \|p(z) - q(z)\|_\infty \) where 
\[ \|r(z)\|_\infty = \sup\{|r(z)| : |z| = 1\} \] for any \( r(z) \in X \).

Show that \((X, d)\) is a metric space.

(c) Is \((X_m, d)\) a separable metric space? How about \((X, d)\)? (Hint: Problem 8(c) is useful.)

(12) For any \( m \in \mathbb{N} \), let 
\[ X_m = \{(z_n) : z_n \in \mathbb{C} \text{ and } z_n = 0 \forall \ n > m\}, \]
and let \( X = \bigcup_{m=1}^{\infty} X_m \). For any \( (z_n) \in X \) define \( \|(z_n)\|_1 = \sum |z_n| \), note that this is a finite sum.

(a) For any \( (z_n), (w_n) \in X \) define \( d((z_n), (w_n)) = \|(z_n) - (w_n)\|_1 \). Show that \((X, d)\) is a metric space.

(b) Is \((X, d)\) separable.

(iii) For each \( m \in \mathbb{N} \) define \( z(m) \in X \) as follows
\[ (z(m))_n = \begin{cases} \frac{1}{2^n} & \text{if } 1 \leq n \leq m \\ 0 & \text{otherwise} \end{cases}. \]

Show that \( \|z(m)\|_1 \leq 1 \) for all \( m \in \mathbb{N} \).

(c) Let \( z(m) \) be defined as above, show that \( d\left(z(m), z(m')\right) \leq \frac{1}{2^{\min(m, m')}} \).

(d) Does \( \{z(m) : m \in \mathbb{N}\} \) have a limit point in \( X \)?

1. **Bonus problems**

(13) Let \( p \) be a prime number. For any integer \( m \in \mathbb{Z} \) define
\[ \nu_p(m) = \begin{cases} \text{power of } p \text{ in the prime factorization of } m & \text{if } m \neq 0 \\ \infty & \text{if } m = 0 \end{cases}. \]

Define the following norm on \( \mathbb{Q} \).
\[ \left|\frac{m}{n}\right|_p = \begin{cases} p^{\nu_p(n) - \nu_p(m)} & \text{if } \frac{m}{n} \neq 0 \\ 0 & \text{if } \frac{m}{n} = 0 \end{cases}. \]

(a) Show that this definition is well defined, i.e. if \( \frac{m}{n} = \frac{m'}{n'} \), then \( \left|\frac{m}{n}\right|_p = \left|\frac{m'}{n'}\right|_p \).

(b) For any two \( r, s \in \mathbb{Q} \) define \( d_p(r, s) = |r - s|_p \). Show that \((\mathbb{Q}, d_p)\) is a metric space.

(c) Show that \( \mathbb{Z} \) is a bounded subset of \( \mathbb{Q} \) with respect to this metric.

(d) Does \( \{p^n : n \in \mathbb{N}\} \) have a limit point in \( \mathbb{Q} \)?

(14) Let \( A \) and \( B \) be two sets. Assume that there are 1-1 maps \( f : A \to B \) and \( g : B \to A \). Prove that there exists a one to one correspondence between \( A \) and \( B \).

Hint: Let \( D = \bigcup_{n=0}^{\infty} (g \circ f)^n(A - g(B)) \).
\[ h(a) = \begin{cases} f(a) & a \in D \\ g^{-1}(a) & a \in A \setminus D \end{cases}. \]

Show that \( h \) is well-defined, 1-1, and onto.