Math 140B, Winter 2018
Exam 2 Solutions

(1) Since \( f \) is continuous on \([a, b]\), it is uniformly continuous. Let \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that for all \( x, y \in [a, b] \), \( |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{\alpha(b) - \alpha(a)} \). Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\) such that \( \Delta x_i < \delta \) for all \( i \). Then

\[
M_i - m_i = \sup_{x,y \in [x_{i-1}, x_i]} |f(x) - f(y)| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)}
\]

for each \( i \), so we have that

\[
U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha(x_i) \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta \alpha(x_i) = \varepsilon,
\]

so \( f \in \mathcal{R}(\alpha) \).

(2) See the solution for Problem (B) on HW 6. Use \( g(x) = x^2 \) and compute that

\[
\int_a^b g(x) \, dx = \frac{b^3 - a^3}{3}.
\]

(3)(a) By assumption, \( \{x \in [a, b] : f(x) \geq 1\} \) is finite. Let \( M = \max\{x_1, \ldots, x_n, 1\} \) where \( \{x_1, \ldots, x_n\} = \{x \in [a, b] : f(x) \geq 1\} \). Then \( f(x) \leq M \) for all \( x \), and \( f(x) \geq 0 \) for all \( x \) by assumption, so \( f \) is bounded.

(3)(b) Let \( \varepsilon > 0 \), and let \( \{x \in [a, b] : f(x) \geq \varepsilon\} = \{c_1, \ldots, c_k\} \). For \( 1 \leq i \leq k \), let \( I_i \) be an open interval containing \( c_i \) with length less than \( \varepsilon/k \). By shrinking the \( I_i \)'s if necessary, we may assume they are disjoint (for notational convenience). Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\) made up of \( a, b \), and all the endpoints of the \( I_i \)'s. Define \( A = \{i \in \{1, \ldots, n\} : \exists 1 \leq m \leq k \text{ such that } c_m \in [x_{i-1}, x_i]\} \), that is, \( A \) keeps track of the “bad” intervals in \( P \). Then we have that

\[
U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i
\]

\[
= \sum_{i \in A} (M_i - m_i) \Delta x_i + \sum_{i \not\in A} (M_i - m_i) \Delta x_i
\]

\[
\leq M \sum_{i \in A} \frac{\varepsilon}{k} + 2\varepsilon \sum_{i \not\in A} \Delta x_i
\]

\[
= M \varepsilon + 2\varepsilon (b - a) \tag{1}
\]

where (1) follows because \( |A| = k \), and for \( i \in A \), \( \Delta x_i < \varepsilon/k \), and \( M_i - m_i \leq M \) always holds (where \( M \) is the bound for \( f \) from (a)) because \( M_i \leq M \) and \( m_i \geq 0 \). If \( i \not\in A \), then \( f(x) \leq \varepsilon \), so \( M_i - m_i \leq 2\varepsilon \) by the triangle inequality. Since \( M \) and \( b - a \) do not depend on \( \varepsilon \), this shows that \( f \) is integrable.

(3)(c) Proof 1:
By the same reasoning as in (b), \( U(P, f) = \sum_{i \in A} M_i \Delta x_i + \sum_{i \in \mathcal{A}} \Delta x_i | P, f ) = \sum_{i \in \mathcal{A}} \Delta x_i = M \epsilon + \epsilon(b - a). \) Since this works for every \( \epsilon, \) this implies that \( \int_a^b f(x)dx \leq \epsilon. \) Since \( f(x) \geq 0 \) for all \( x, \) \( L(P, f) \geq 0 \) for all \( P, \) and so \( \int_a^b f(x)dx \geq 0. \) Putting these together implies that \( \int_a^b f(x)dx = 0. \)

**Proof 2:** This proof shows a more general type of argument that can be quite useful.

Observe that \( \{ x \in [a, b] : f(x) > 0 \} = \bigcup_{n \in \mathbb{N}} \{ x \in [a, b] : f(x) > \frac{1}{n} \} \) by the archimedean principle. By assumption, each \( \{ x \in [a, b] : f(x) > \frac{1}{n} \} \) is finite, so this means that \( \{ x \in [a, b] : f(x) > 0 \} \) is countable. Since \( f \geq 0 \) by assumption, this means that \( \{ x \in [a, b] : f(x) = 0 \} \) is uncountable. (To see this: suppose for contradiction that it is not dense. Then there exists some open set \( (c, d) \) in \( [a, b] \) such that \( (c, d) \cap \{ x \in [a, b] : f(x) = 0 \} = \emptyset, \) i.e. \( (c, d) \subseteq \{ x \in [a, b] : f(x) > 0 \}, \) so \( \{ x \in [a, b] : f(x) > 0 \} \) is countable.)

This means that every interval in \( [a, b] \) contains a point where \( f \) is zero, so \( L(P, f) = 0 \) for every partition \( P. \) Since we know \( f \) is integrable by (b), this implies that \( \int_a^b f(x)dx = \sup_P L(P, f) = 0. \)

\( (4) \) We need to show that for all \( \epsilon > 0, \) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) and all \( x \in [0, 1], |f_n(x) - f(x)| < \epsilon. \)

Let \( \epsilon > 0 \) and let \( \delta > 0 \) be such that for all \( x, y \in [0, 1], |x - y| < \delta \implies |f(x) - f(y)| < \epsilon. \) By definition of \( f_n, \) for any \( x \in [0, 1], |f_n(x) - f(x)| = |\frac{f_n(x) - f(x)}{\frac{n}{n+1}}| \leq \frac{1}{n+1}. \) Choose \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \delta. \) Then for all \( n \geq N \) and all \( x \in [0, 1], |\frac{n}{n+1} - x| < \delta, \) so \( |f_n(x) - f(x)| < \epsilon \) by choice of \( \delta. \)