(1) Since $f$ is continuous on $[a, b]$, it is uniformly continuous. Let $\varepsilon > 0$ and let $\delta > 0$ be such that for all $x, y \in [a, b]$, $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{\alpha(b) - \alpha(a)}$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$ for all $i$. Then $M_i - m_i = \sup_{x, y \in [x_{i-1}, x_i]} |f(x) - f(y)| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)}$ for each $i$, so we have that $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha(x_i) \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta \alpha(x_i) = \varepsilon$, so $f \in \mathcal{R}(\alpha)$.

(2) See the solution for Problem (B) on HW 6. Use $g(x) = x^2$ and compute that $\int_{a}^{b} g(x)dx = \frac{b^3 - a^3}{3}$.

(3)(a) By assumption, $\{x \in [a, b] : f(x) \geq 1\}$ is finite. Let $M = \max\{f(x_1), \ldots, f(x_n), 1\}$ where $\{x_1, \ldots, x_n\} = \{x \in [a, b] : f(x) \geq 1\}$. Then $f(x) \leq M$ for all $x$, and $f(x) \geq 0$ for all $x$ by assumption, so $f$ is bounded.

(3)(b) Let $\varepsilon > 0$, and let $\{x \in [a, b] : f(x) \geq \varepsilon\} = \{c_1, \ldots, c_k\}$. For $1 \leq i \leq k$, let $I_i$ be an open interval containing $c_i$ with length less than $\varepsilon/k$. By shrinking the $I_i$'s if necessary, we may assume they are disjoint (for notational convenience). Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ made up of $a, b$, and all the endpoints of the $I_i$'s. Define $A = \{i \in \{1, \ldots, n\} : \exists! m \leq k \text{ such that } c_m \in [x_{i-1}, x_i]\}$; that is, $A$ keeps track of the “bad” intervals in $P$. Then we have that

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i \in A} (M_i - m_i) \Delta x_i + \sum_{i \not\in A} (M_i - m_i) \Delta x_i$$

$$\leq M \sum_{i \in A} \frac{\varepsilon}{k} + 2\varepsilon \sum_{i \not\in A} \Delta x_i$$

$$= M \frac{\varepsilon}{k} + 2\varepsilon (b - a)$$

(1)

where (1) follows because $|A| = k$, and for $i \in A$, $\Delta x_i < \varepsilon/k$, and $M_i - m_i \leq M$ always holds (where $M$ is the bound for $f$ from (a)) because $M_i \leq M$ and $m_i \geq 0$. If $i \not\in A$, then $f(x) \leq \varepsilon$, so $M_i - m_i \leq 2\varepsilon$ by the triangle inequality. Since $M$ and $b - a$ do not depend on $\varepsilon$, this shows that $f$ is integrable.

(3)(c) Proof 1:
By the same reasoning as in (b), \( U(P, f) = \sum_{i \in A} M_i \Delta x_i + \sum_{i \notin A} M_i \Delta x_i \leq M \sum_{i \in A} \frac{\epsilon}{n} + \epsilon \sum_{i \notin A} \Delta x_i = M \epsilon + \epsilon (b - a) \). Since this works for every \( \epsilon \), this implies that \( \int_a^b f(x) \, dx \leq 0 \). Since \( f(x) \geq 0 \) for all \( x \), \( L(P, f) \geq 0 \) for all \( P \), and so \( \int_a^b f(x) \, dx \geq 0 \). Putting these together implies that \( \int_a^b f(x) \, dx = 0 \).

**Proof 2:** This proof shows a more general type of argument that can be quite useful.

Observe that \( \{ x \in [a, b] : f(x) > 0 \} = \bigcup_{n \in \mathbb{N}} \{ x \in [a, b] : f(x) > \frac{1}{n} \} \) by the archimedian principle. By assumption, each \( \{ x \in [a, b] : f(x) > \frac{1}{n} \} \) is finite, so this means that \( \{ x \in [a, b] : f(x) > 0 \} \) is countable. Since \( f \geq 0 \) by assumption, this means that \( \{ x \in [a, b] : f(x) = 0 \} = \{ x \in [a, b] : f(x) > 0 \}^c \) must be dense in \([a, b]\). (To see this: suppose for contradiction that it is not dense. Then there exists some open set \((c, d)\) in \([a, b]\) such that \((c, d) \cap \{ x \in [a, b] : f(x) = 0 \} = \emptyset\), i.e. \((c, d) \subseteq \{ x \in [a, b] : f(x) > 0 \}\), so \( \{ x \in [a, b] : f(x) > 0 \} \) is uncountable.)

This means that every interval in \([a, b]\) contains a point where \( f \) is zero, so \( L(P, f) = 0 \) for every partition \( P \). Since we know \( f \) is integrable by (b), this implies that \( \int_a^b f(x) \, dx = \sup_P L(P, f) = 0 \).

(4) We need to show that for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) and all \( x \in [0, 1] \), \( |f_n(x) - f(x)| < \epsilon \).

First, notice that since \( f \) is continuous on \([0, 1]\), a compact set, it is uniformly continuous. Let \( \epsilon > 0 \) and let \( \delta > 0 \) be such that for all \( x, y \in [0, 1] \), \( |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \).

By definition of \( f_n \), for any \( x \in [0, 1] \), \( |f_n(x) - f(x)| = |f\left(\frac{nx}{n+1}\right) - f(x)| \). Observe that \( \left| \frac{nx}{n+1} - x \right| = \left| \frac{x}{n+1} \right| \leq \frac{1}{n+1} \). Choose \( N \in \mathbb{N} \) such that \( \frac{1}{N+1} < \delta \). Then for all \( n \geq N \) and all \( x \in [0, 1] \), \( \frac{nx}{n+1} - x \) < \( \delta \), so \( |f_n(x) - f(x)| < \epsilon \) by choice of \( \delta \).