1. Proof by induction

(1) Recall that the Fibonacci sequence is defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all integer $n > 1$. Prove that

$$\sum_{i=0}^{n} a_i^2 = a_na_{n+1}$$

for all integer $n \geq 0$.

**Answer:** We prove this by induction on $n$.

**Base case:** $n = 0$. We have

$$\sum_{i=0}^{0} a_i^2 = 0 = a_0a_1.$$

**Inductive step:** Assume the conclusion holds for an integer $k \geq 0$. That is:

$$\sum_{i=0}^{k} a_i^2 = a_ka_{k+1}.$$

Need to prove that

$$\sum_{i=0}^{k+1} a_i^2 = a_{k+1}a_{k+2}.$$

**Proof of the inductive step:** We have

$$\sum_{i=0}^{k+1} a_i^2 = \sum_{i=0}^{k} a_i^2 + a_{k+1}^2$$

definition of $\sum$

$$= a_ka_{k+1} + a_{k+1}^2$$

by inductive hypothesis

$$= a_{k+1}(a_k + a_{k+1})$$

Recursive formula

as we claimed.

Hence by induction we have $\sum_{i=0}^{n} a_i^2 = a_na_{n+1}$ for all integer $n \geq 0$. □

(2) Let $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for every positive integer $n$. Prove that

(a) For any positive integer $n$, we have that $a_n < a_{n+1}$.

(b) For any positive integer $n$, we have that $a_n < \frac{1+\sqrt{5}}{2}$.

**Answer:** We begin with the proof of part (a).

We will prove this using induction on $n$.

**Base case:** $n = 1$. Recall from homework problems that for any two real numbers $a$ and $b$ we have

$$|a| \leq |b| \Leftrightarrow a^2 \leq b^2$$

and

$$|a| = |b| \Leftrightarrow a^2 = b^2$$

as we claimed.
From this, we have
\[ a_1 = 1 < \sqrt{2} = a_2 \iff 1 < 2. \]

Hence, the claim holds for \( n = 1 \).

Inductive step: Assume \( a_k < a_{k+1} \) for a positive integer \( k \). We need to prove \( a_{k+1} < a_{k+2} \).

Proof of the inductive step: We have
\[
\begin{align*}
a_k < a_{k+1} \Rightarrow 1 + a_k < 1 + a_{k+1} & \quad \text{properties of inequalities} \\
\Rightarrow \sqrt{1 + a_k} < \sqrt{1 + a_{k+1}} & \quad \text{by (1.1)} \\
\Rightarrow a_{k+1} < a_{k+2} & \quad a_{k+1} = \sqrt{1 + a_k} \text{ and } a_{k+2} = \sqrt{1 + a_{k+1}}
\end{align*}
\]
as we claimed.

Hence, by induction we have \( a_n < a_{n+1} \).

We now turn to part (b). Again we use induction on \( n \).

Base case: \( n = 1 \). Using (1.1) and properties of inequalities, we have
\[
a_1 = 1 < \frac{1 + \sqrt{5}}{2} \iff 1 < \sqrt{5} \iff 1 < 5.
\]

Hence the claim holds for \( n = 1 \).

Inductive step: Assume \( a_k < \frac{1 + \sqrt{5}}{2} \) for some positive integer \( k \). We need to show that \( a_{k+1} < \frac{1 + \sqrt{5}}{2} \).

Proof of the inductive step: We have
\[
\begin{align*}
a_k < \frac{1 + \sqrt{5}}{2} \Rightarrow 1 + a_k < 1 + \frac{1 + \sqrt{5}}{2} & \quad \text{properties of inequalities} \\
\Rightarrow \sqrt{1 + a_k} < \sqrt{\frac{3 + \sqrt{5}}{2}} & \quad \text{by (1.1)} \\
\Rightarrow a_{k+1} < \sqrt{\frac{3 + \sqrt{5}}{2}} & \quad a_{k+1} = \sqrt{1 + a_k}.
\end{align*}
\]

Moreover, \( \frac{1 + \sqrt{5}}{2} \) is a positive number and we have
\[
\left( \frac{1 + \sqrt{5}}{2} \right)^2 = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2}.
\]

Hence \( \frac{1 + \sqrt{5}}{2} \) is a positive number and we have
\[
\left( \frac{1 + \sqrt{5}}{2} \right)^2 = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2}.
\]

Therefore, the claim in the inductive step follows from (1.2).

Thus, by induction we have \( a_n < \frac{1 + \sqrt{5}}{2} \). \( \square \)

(3) Prove that any integer \( m \geq 12 \) can be written as \( 5x + 4y \) for some non-negative integers \( x \) and \( y \).

**Answer:** We can prove this by strong induction on \( m \). See the solution to HW 5 part A.

(4) Let \( n \) be a positive integer. Prove that there exist a unique integer \( k \) and a unique integer \( 0 \leq r < 4 \) so that \( n = 4k + r \).
Answer: Let us first prove the uniqueness. Suppose there are integers $k_1, k_2$ and $0 \leq r_1, r_2 < 4$ so that

$$n = 4k_1 + r_1 = 4k_2 + r_2.$$  

We need to prove $k_1 = k_2$ and $r_1 = r_2$.

We get from (1.3) that $4(k_1 - k_2) = r_2 - r_1$. Hence $4| r_2 - r_1$. So by a proposition from class we have

$$4 \mid r_2 - r_1.$$  

Note, however, that $0 \leq r_1 < 4$ and $0 \leq r_2 < 4$, therefore, $-4 < r_2 - r_1 < 4$. That is $|r_2 - r_1| < 4$. This and (1.4) imply that $r_2 - r_1 = 0$, that is $r_1 = r_2$.

Let us denote the common value $r_1 = r_2$ by $r$. Therefore, (1.3) gives

$$4k_1 + r = 4k_2 + r$$  

which implies $4(k_1 - k_2) = 0$. Since $4 \neq 0$, we get that $k_1 - k_2 = 0$, that is, $k_1 = k_2$.

All together, we have proved the uniqueness.

Let us now prove the existence. We prove this by strong induction on $n$.

Base case: We verify this for $n = 1, 2, 3, 4$. Indeed we have

$$1 = 4 \times 0 + 1, \quad 2 = 4 \times 0 + 2, \quad 3 = 4 \times 0 + 3, \quad \text{and} \quad 4 = 4 \times 1 + 0.$$  

Hence the claim holds for $n = 1, 2, 3, 4$.

Inductive step: Now let $m > 4$ be an integer and let us assume that the conclusion holds for all integers $1 \leq \ell < m$. That is: for all integers $1 \leq \ell < m$ there are integers $k'$ and $0 \leq r' < 4$ so that $\ell = 4k' + r'$. We need to prove the conclusion also holds for $m$.

Note that

$$m = (m - 4) + 4.$$  

Since $m > 4$ we have $1 \leq m - 4 < m$. Therefore, inductive hypothesis applies to $\ell = m - 4$ and we get: there are integers $k'$ and $0 \leq r' < 4$ so that $m - 4 = 4k' + r'$.

Hence,

$$m = m - 4 + 4 = 4k' + r' + 4 = 4(k' + 1) + r'.$$  

That is $m$ satisfies the claim with $k = k' + 1$ and $r = r'$. So the proof is complete by strong induction. □

2. LANGUAGE OF SET THEORY AND FUNCTIONS

(1) List all the subsets of $\{\{\}, 1, 3, \{3, 4\}\}$.

Answer: This set has four elements, so your answer should have 16 elements.

(2) Let $A, B$ and $C$ be three sets, further, assume that $A \cup B \subseteq A \cup C$ and $A \cap B \subseteq A \cap C$. Prove that $B \subseteq C$.

Answer: See solution to HW 6, Problems II, #4.

(3) Let $A$ and $B$ be two nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. 
(4) Let \( A, B, C \) and \( D \) be four sets. Prove that
(a) \((A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\).
(b) \((A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)\). Provide an example to show that this inclusion may be proper, i.e. \((A \times B) \cup (C \times D)\) need not be equal to \((A \cup C) \times (B \cup D)\).

**Answer:** This was discussed in class. Please see Proposition 7.7.4 in the book.

(5) Let \( f : A \to B \) and \( g : B \to C \) be two functions.
(a) If \( f \) and \( g \) are injective, then \( g \circ f \) is injective.
(b) If \( f \) and \( g \) are surjective, then \( g \circ f \) is surjective.
(c) If \( f \) and \( g \) are bijective, then \( g \circ f \) is bijective.

**Answer:**
(a) Suppose \( x_1, x_2 \in A \) are so that \( g \circ f(x_1) = g \circ f(x_2) \). That is
\[
g(f(x_1)) = g(f(x_2)).
\]
Then since \( g \) is injective we have \( f(x_1) = f(x_2) \). Now, since \( f \) is injective we get from this that \( x_1 = x_2 \), as we claimed.
(b) Let \( z \in C \), we need so fine some \( x \in A \) so that \( g \circ f(x) = z \). That is \( g(f(x)) = z \). First note that since \( g \) is surjective, there exists some \( y_0 \in B \) so that \( g(y_0) = z \). Now since \( f \) is surjective, there exists some \( x_0 \in A \) so that \( f(x_0) = y_0 \). We thus get that
\[
g(f(x_0)) = g(y_0) = z.
\]
Hence the claim holds with \( x = x_0 \).
(c) Combine the proofs of (a) and (b) to get a proof of (c).

(6) Determine, with justification, whether the following functions are injective, surjective, or bijective.
(a) Let \( Y \subseteq X \), and define \( f : \mathcal{P}(X) \to \mathcal{P}(Y) \) by \( f(A) = A \cap Y \).
(b) Let \( Y \subseteq X \), and define \( f : \mathcal{P}(X) \to \mathcal{P}(Y) \) by \( f(A) = A \cup Y \).
(c) Let \( Y \subseteq X \) and assume that \( X \) and \( Y \) are non-empty sets. Define \( f : \mathcal{P}(X) \to \{A \in \mathcal{P}(X) | Y \subseteq A \} \) by \( f(A) = A \cup Y \).
(d) \( f : \mathbb{Z} \times \mathbb{Z} \to \{\ell \in \mathbb{Z} | \ell = 3k \text{ for some integer } k\} \) by \( f(m, n) = 3m + 6n \).
(e) \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) by \( f(m, n) = 2m + 11n \).
(f) \( f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by \( f(m, n) = 2^m3^n \).

**Answer:**
(a) The function is surjective but not injective. To see that \( f \) is surjective, consider any \( A \in \mathcal{P}(Y) \) and observe that since \( Y \subseteq X \), we also have \( A \in \mathcal{P}(X) \). Also, since \( A \subseteq Y \), \( A \cap Y = A \). Hence \( f(A) = A \), so \( f \) is surjective.
To see that \( f \) is not injective, observe that \( X \neq Y \), but \( f(X) = X \cap Y = Y \cap Y = f(Y) \).
(b) The function is bijective. See HW 6, Problems II, #8, part (iv).
(c) The function is surjective but not injective. To see that \( f \) is surjective, let \( B \in \{A \in \mathcal{P}(X) | Y \subseteq A \} \). Since \( \{A \in \mathcal{P}(X) | Y \subseteq A \} \subseteq \mathcal{P}(X) \), then we also have \( B \in \mathcal{P}(X) \). Then since \( Y \subseteq B \), we have \( f(B) = B \cup Y = B \), so \( f \) is surjective.
To see that \( f \) is not injective, observe that \( f(\emptyset) = Y = Y \cup Y = f(Y) \), but by assumption \( Y \neq \emptyset \), so \( f \) is not injective.

(d) The function is surjective but not injective. To see that \( f \) is surjective, let \( \ell = 3k \) for some integer \( k \). Then since \( -k \) and \( k \) are integers, we have \((-k, k) \in \mathbb{Z} \times \mathbb{Z} \) and \( f(-k, k) = -3k + 6k = 3k = \ell \). Hence, \( f \) is surjective.

To see that \( f \) is not injective, observe that \( f(2, 0) = 6 = f(0, 1) \).

(e) The function is surjective but not injective. To see that \( f \) is surjective, let \( k \) be an integer. Then since \( 6k \) and \(-k \) are integers, we have \((6k, -k) \in \mathbb{Z} \times \mathbb{Z} \) and \( f(6k, -k) = 2 \times 6k - 11k = (12 - 11)k = k \). Hence, \( f \) is surjective.

To see that \( f \) is not injective, observe that \( f(11, 0) = 22 = f(0, 2) \).

(f) The function is injective but not surjective. To see that \( f \) is injective, suppose \( f(m_1, n_1) = f(m_2, n_2) \) for some \( m_1, n_1, m_2, n_2 \in \mathbb{N} \). We will argue by contradiction that \( m_1 = m_2 \). Suppose that \( m_1 \neq m_2 \). Then either \( m_1 > m_2 \) or \( m_2 > m_1 \), and we can suppose without loss of generality that it is the former. Then

\[
(2.1) \quad f(m_1, n_1) = f(m_2, n_2) \iff 2^{m_1}3^{n_1} = 2^{m_2}3^{n_2} \iff 2^{m_1-m_2} = 3^{n_2-n_1}
\]

Since \( m_1 > m_2 \), \( m_1 - m_2 \) is a positive integer. Thus \( 2^{m_1-m_2} \) is even (one can easily prove this using the recursive definition of \( 2^n \)). On the other hand, \( 3^{n_2-n_1} \) is either not an integer (for \( n_2 < n_1 \)) or it is odd (again, this can be proved using the recursive definition and induction). This is a contradiction. Thus it must be the case that \( m_1 = m_2 \), and if we substitute this into the last line of (2.1) above, we see that this implies \( n_1 = n_2 \) as well. Thus \( f \) is injective.

To see that \( f \) is not surjective, show that there are no natural numbers \( m, n \) such that \( 2^m3^n = 5 \). To do this, you can prove that if \( m \geq 3 \) or \( n \geq 2 \), then \( 2^m3^n > 5 \). The rest of the cases (where \( m \leq 2 \) and \( n \leq 1 \)) can be computed explicitly, and are not equal to 5.

(7) (a) Give an example, with justification, of a function \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) which is 1-1 but not onto.

(b) Give an example, with justification, of a function \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) which is onto but not 1-1.

(c) List all the functions from \( \{1, 2, 3\} \) to \( \{1, 2, 3\} \). Is the following statement true: Let \( f : \{1, 2, 3\} \to \{1, 2, 3\} \). Then \( f \) is 1-1 if and only if it is onto.

**Answer:**

(a) There are many possible examples. One example is \( f(n) = n + 1 \). It is 1-1 because \( n_1 + 1 = n_2 + 1 \implies n_1 = n_2 \) by a property of inequalities. On the other hand, it is not onto since \( f(n) = n + 1 = 1 \) implies \( n = 0 \), so there is no positive integer \( n \) such that \( f(n) = 1 \).

The last part of the previous problem also serves as an example.

(b) There are many possible examples. One example is the function defined by \( f(1) = 1 \) and \( f(n) = n - 1 \) for \( n \geq 2 \). It is surjective because for any positive integer \( n \), \( f(n+1) = n + 1 - 1 = n \). However, it is not injective because \( f(1) = 1 = f(2) \).

(c) For each element \( x \) in your domain you have three choices for the output \( f(x) \) in your codomain. Hence, there should be \( 3^3 \) items in
3. QUANTIFIERS

(1) Use quantifiers to write the following statements.
(a) the sequence \( \{a_n\} \) converges to the real number \( \ell \).
(b) the sequence \( \{a_n\} \) does not converges to the real number \( \ell \).
(c) the sequence \( \{a_n\} \) does not converge to any real number.

Answer: (a) \( \forall \epsilon \in \mathbb{R}^+, \exists M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+ \left( n \geq M \Rightarrow (|a_n - \ell| < \epsilon) \right) \).
(b) \( \exists \epsilon \in \mathbb{R}^+, \forall M \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \left( (n \geq M) \land (|a_n - \ell| \geq \epsilon) \right) \).
(c) \( \forall \ell \in \mathbb{R}, \exists \epsilon \in \mathbb{R}^+, \forall M \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \left( (n \geq M) \land (|a_n - \ell| \geq \epsilon) \right) \).

(2) Use the definition of the limit of a sequence to show the following.
(a) \( \lim_{n \to \infty} \frac{n}{3n^3 - 1} = 0 \).
(b) \( \lim_{n \to \infty} \frac{2n + 1}{3n + 5} = \frac{2}{3} \).

Answer: (a) We need to show the following
\( \forall \epsilon \in \mathbb{R}^+, \exists M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+ \left( (n \geq M) \Rightarrow \left( \left| \frac{n}{3n^3 - 1} \right| < \epsilon \right) \right) \).

Let \( \epsilon > 0 \) be a positive real number. We use backward argument to find \( M \) which satisfies the above.

\[
\left| \frac{n}{3n^3 - 1} \right| < \epsilon \iff \frac{n}{3n^3 - 1} < \epsilon \\
\quad \iff n < \epsilon(3n^3 - 1) \\
\quad \iff n < 3\epsilon n^3 - \epsilon \\
\quad \iff \epsilon < 3\epsilon n^3 - n \\
\quad \iff \epsilon < n(3\epsilon n^2 - 1) \\
\quad \iff \epsilon < 3\epsilon n^2 - 1 \\
\quad \text{since } 0 < \epsilon \text{ and } 1 \leq n \\
\quad \iff \frac{\epsilon + 1}{3\epsilon} < n \\
\quad \iff \sqrt{\frac{\epsilon + 1}{3\epsilon}} < n \\
\quad \text{since } \frac{\epsilon + 1}{3\epsilon} > 0
\]

Hence if we set \( M \) to be the smallest integer bigger than \( \sqrt{\frac{\epsilon + 1}{3\epsilon}} \), we get the claim.

(b) We need to show the following
\( \forall \epsilon \in \mathbb{R}^+, \exists M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+ \left( (n \geq M) \Rightarrow \left( \left| \frac{2n}{3n + 5} - \frac{2}{3} \right| < \epsilon \right) \right) \).
Let $\epsilon > 0$ be a positive real number. We use backward argument to find $M$ which satisfies the above.

$$\left| \frac{2n}{3n+5} - \frac{2}{3} \right| < \epsilon \iff \left| -\frac{10}{9n+15} \right| < \epsilon$$

$$\iff \frac{10}{9n+15} < \epsilon \quad \text{since} \quad \left| -\frac{10}{9n+15} \right| = \frac{10}{9n+15}$$

$$\iff 10 < \epsilon(9n + 15) \quad \text{since} \quad 9n + 15 > 0$$

$$\iff 10 - 15\epsilon < 9\epsilon < n$$

$$\iff \frac{10 - 15\epsilon}{9\epsilon} < n \quad \text{since} \quad \epsilon > 0$$

Hence if we set $M$ to be the smallest integer bigger than $\frac{10 - 15\epsilon}{9\epsilon}$, we get the claim.

(3) Prove or disprove by a counter example the following statement.

(a) $\forall a \in (0, 1), \exists b \in (0, 1), b < a$.

(b) $\exists a \in (0, 1), \forall b \in (0, 1), b > a$.

where $(0, 1)$ is, as usual, the open interval $(0, 1)$.

**Answer:** (a) This is true. Indeed note that for any real number $a$ we have

$$a > 0 \iff \frac{a}{2} > 0 \quad \text{since} \quad 2 > 0$$

$$\iff a > \frac{a}{2} > 0.$$  

Hence, if $0 < a < 1$, then $0 < \frac{a}{2} < a < 1$ which implies $b = \frac{a}{2}$ satisfies the claim.

(b) This is false. The negation of this statement is

$$\forall a \in (0, 1), \exists b \in (0, 1), b \leq a,$$

which is true by the previous part of the problem.

(4) Prove that

$$\lim_{n \to \infty} \frac{(-1)^n}{2n+1} \neq \frac{1}{2}.$$  

**Answer:** As per part (b) of the first problem in this section, we need to prove the statement

$$\exists \epsilon \in \mathbb{R}^+, \forall M \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \left( (n \geq M) \land \left( \left( \frac{(-1)^n}{2n+1} - \frac{1}{2} \right) \geq \epsilon \right) \right).$$

Consider $\epsilon = 1/2$. For any $M \in \mathbb{Z}^+$, let $n$ be the first odd number $\geq M$ (note that such a number exists, since either $M$ or $M + 1$ is odd). Then since $n$ is odd, $(-1)^n < 0$, which (by properties of inequalities) means

$$(-1)^n < 0 \Rightarrow \frac{(-1)^n}{2n+1} < 0 \Rightarrow \frac{(-1)^n}{2n+1} - \frac{1}{2} < -\frac{1}{2} < 0.$$  

Since $\frac{(-1)^n}{2n+1} - \frac{1}{2} < 0$, we have that

$$\left| \frac{(-1)^n}{2n+1} - \frac{1}{2} \right| = - \left( \frac{(-1)^n}{2n+1} - \frac{1}{2} \right).$$
Thus by multiplying the last step of (3.1) by -1 (using a property of inequalities) and substituting for (3.2) above, we have

$$\left| \frac{(-1)^n n - 1}{2n + 1} - \frac{1}{2} \right| > \frac{1}{2}.$$ 

Thus we have demonstrated the claim with $\epsilon = 1/2$. 