This examination contains 4 problems. This is a closed book exam.

Show all your computations and justify/explain your answers.
Electronic devices are NOT allowed.

(1) (10 pts) Prove the following theorem: Let \( f : [a, b] \to \mathbb{R} \) be a continuous function and let \( \alpha : [a, b] \to \mathbb{R} \) be increasing. Then \( f \in R(\alpha) \) on \([a, b]\).

(2) (10 pts) Let \( f : [a, b] \to \mathbb{R} \) be continuous. Prove that there exists some \( c \in [a, b] \) such that
\[
\int_a^b x^2 f(x) \, dx = \left( \frac{b^3 - a^3}{3} \right) f(c).
\]

(3) Let \( f : [a, b] \to \mathbb{R} \) be a function such that \( f(x) \geq 0 \) for all \( x \in [a, b] \). Moreover, assume that for every \( \varepsilon > 0 \) the set \( \{ x \in [a, b] : f(x) \geq \varepsilon \} \) is a finite set.
(a) (5 pts) Prove that \( f \) is bounded on \([a, b]\).
(b) (10 pts) Prove that \( f \in R \) on \([a, b]\).
(c) (5 pts) Compute \( \int_a^b f(x) \, dx \).

(4) (7 pts) Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. For every \( n \in \mathbb{N} \) define
\[
f_n : [0, 1] \to \mathbb{R} \quad \text{by} \quad f_n(x) = f\left( \frac{nx}{n+1} \right).
\]
Prove that \( \{f_n\} \) uniformly converges to \( f \) on \([0, 1]\).