Math 109, Fall 2017

Discussion about common areas of confusion on Exam 2

**Problem 1:** Several common mistakes were:

- Supposing that $B \neq C$, and using this to say “since $x \in B$, $x \notin C$”. This does not follow: $B \neq C$ does not mean that there are no elements in $B \cap C$. It means that there is some element that is in only one of the sets, i.e. $\exists x \in B - C \text{ or } x \in C - B$. You must to explicitly say you are working with such an element $x$ if you want to conclude that $x \notin C$. (E.g. write “Since $B \neq C$, there exists some $x \in B - C$ or $x \in C - B$. Case 1: Let $x \in B - C$.”)

- When proving $B = C$, you should take an element $x \in B$ and use the fact that $A \cap B = A \cap C$ and $A \cup B = A \cup C$ to conclude that $x \in C$, and then similarly show $x \in C \implies x \in B$. It is not a good idea to begin by assuming that $x \in A \cap B$: this is can be worked out, but you have to be sure that you also consider the possibility that $x \in B - A$! (Your proof must consider all possible $x \in B$, not just those that are also in $A$.)

- A frequent argument was of the form: suppose $B = C$ and want to show that $A \cup B = A \cup C$. Let $x \in B$. Then $x \in A \cup B$, and $x \in A \cup C$ (because $B = C$), so $A \cup B = A \cup C$. The problem is this: to conclude that $A \cup B = A \cup C$, you need to show that $y \in A \cup B \iff y \in A \cup C$. We only proved it for $x \in B$. What if $x \in A - B$? We have not proven it for all possible $y$’s yet, so cannot conclude the set equality yet. You must prove this second case first.

- In a proof by contradiction, of the form: suppose $B \neq C$. Then there exists $x \in B - C$ or $x \in C - B$. Assume $x \in B - C$. Case 1: $x \in A$. Then $x \in A \cap B$, but $x \notin A \cap C$, contradicting the assumption that $A \cap B = A \cap C$. Therefore $B = C$. The problem is this: we have reached a contradiction in this case, which only tells us that it is not possible that $x \in A$. We have proven that there are no elements of $A$ in $B - C$, but we have not yet concluded that $x \in B - C$ cannot exist. It just can’t be in $A$! (You only conclude $B \neq C$ once you show that you also reach a contradiction when $x \notin A$.) (This is a more minor error, but worth noting.)

**Problem 2:**

- Confusion about the definition of injective. If $f : A \to B$, to prove that $f$ is injective, you need to prove that for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. Equivalently, you can prove the contrapositive $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$.

  - Proving that $a_1 = a_2 \implies f(a_1) = f(a_2)$ does not say much: this is true for any function $f$: that every element of the domain must be mapped to a unique element of the codomain is part of the definition of a function.

  - The negation of $f$ being injective is that there exists some $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. It is not the same as $f(a_1) = f(a_2) \implies a_1 \neq a_2$. In fact, this implication is never true for any function: $f(a) = f(a)$ must always be true, but $a = a$! More generally, the negation of $p \implies q$ is not an implication, it is $p \land \neg(q)$.

- Writing $g(f(a)) = I_A$. This is not possible. $g(f(a))$ is an element of the set $A$. $I_A$ is a function $I_A : A \to A$ defined by $I_A(x) = x$ for all $x \in A$. You should write $g(f(a)) = I_A(a)$.

- $g \circ f = I_A$ does not mean $f$ and $g$ are inverses of each other. See Problem 17 in Problems II.
To prove that \( g : B \rightarrow A \) is surjective, one needs to prove that \( \forall a \in A, \exists b \in B, g(b) = a \). A common confusion was saying the definition is \( \exists b \in B, \forall a \in A, g(b) = a \). This statement cannot be true if \( A \) contains more than one element: by definition of \( g : B \rightarrow A \), every element of \( B \) is mapped under \( g \) to a unique element of \( A \): \( g(b) = a \). If \( A \) has two distinct elements \( a_1, a_2 \), we cannot have \( g(b) = a_1 \) and \( g(b) = a_2 \), as then \( b \) is not mapped to a unique element of \( A \).

**Problem 3:**

- Misuse of “=” and “ \( \iff \) ”. This goes back to the previous point. These symbols are not interchangeable—use “=” if two expressions/values are actually equal to each other, and use “ \( \iff \) ” if you are asserting that two statements are equivalent (imply each other). It does not make sense, for example, to say
  \[
  \left( x + \frac{1}{x} \right) \left( x^k + \frac{1}{x^k} \right) \iff x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}}.
  \]
  These expressions are related by equality, not implication!

**Problem 4:**

- It seems like many people knew the sort of calculation that they were supposed to do without really understanding the definition of a limit or the fundamental logic of the proof. Recall that we say \( \lim_{n \to \infty} a_n = \ell \) if
  \[
  \forall \epsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+ ((n \geq N) \implies (|a_n - \ell| < \epsilon)).
  \]
  In other words, for arbitrary \( \epsilon > 0 \), there is some positive integer \( N \) (usually depending on \( \epsilon \)) such that for any integer \( n \geq N \), we have that \( a_n \) is within \( \epsilon \) of the limit \( \ell \). Your proof needs to somewhere contain this underlying logic. For this problem, a basic proof outline would be something like:
Suppose $\epsilon \in \mathbb{R}^+$ and let $N$ be an integer larger than $\frac{11+6\epsilon}{9\epsilon}$. Then if $n \in \mathbb{Z}^+$ is such that $n \geq N$, by transitivity we have that $n > \frac{11+6\epsilon}{9\epsilon}$, which implies that $\frac{|4n + 1|}{3n - 2} - \frac{4}{3} < \epsilon$. Thus, by the definition of a limit, $\lim_{n \to \infty} \frac{4n + 1}{3n - 2} = \frac{4}{3}$.

The rest of the proof is just demonstrating the last implication (in bold above). This is the longest and most involved part, but without the appropriate language surrounding it, it is not a real proof, just a computation. In this class, we want to see that you know how to structure a proof and that you understand the definitions, not just that you can do algebraic manipulations.

- Most of you used backwards reasoning to prove that $n > \frac{11+6\epsilon}{9\epsilon}$ implies $\frac{|4n + 1|}{3n - 2} - \frac{4}{3} < \epsilon$ for $\epsilon > 0$ and $n \in \mathbb{Z}^+$. This is of course completely fine, but you should indicate that you are using backwards reasoning, both with words and $\iff$ symbols. This shows you understand that what you are starting with is actually what you hope to achieve, and not something that is generally true or something that you are assuming to be true.