Homework due Friday, January 26, at 3:00 pm.

A.

(1) Let $P = \{p_1, p_2, \ldots \}$ denote the set of odd prime numbers listed in increasing order, i.e., $p_1 < p_2 < \cdots$. Define functions $f_1, f_2, \ldots$ and $f$ on $(-1, 1)$ as follows. For every $n \in \mathbb{N}$ define

$$f_n(x) = x^{1 + \frac{1}{p_n}},$$

and define

$$f(x) = \begin{cases} f_n(x) & x = \frac{1}{p_m^n} \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Is $f$ differentiable at 0?

(2) For all $n \in \mathbb{N}$ let $g_n : (-1, 1) \rightarrow \mathbb{R}$ be a twice differentiable function. Further, assume that

(a) $g_n'(0) = 0$ for all $n$, and

(b) there exists some $M \in \mathbb{R}$ so that for all $n \in \mathbb{N}$ and all $x \in (-1, 1)$ we have $|g_n''(x)| \leq M$.

Define

$$g(x) = \begin{cases} g_n(x) & x = \frac{1}{p_m^n} \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Prove that $g$ is differentiable at 0.

(Hint: Using (a) and (b) above prove that for every $\epsilon > 0$ there exists some $\delta > 0$ so that if $|x| < \delta$, then $|g_n'(x)| < \epsilon$ simultaneously for all $n$.)

B. Rudin, Chapter 5 (page 114), problems #7, 9, 15, 17, 22.