A. (1) First, observe the following general fact. If \( f : (x - \varepsilon, x + \varepsilon) \to \mathbb{R} \) is differentiable at \( x \), i.e. if \( f'(x) \) exists, then for every sequence \( x_n \) converging to \( x \) and satisfying \( x_n \neq x \) for every \( n \in \mathbb{N} \), we will have \( f'(x) = \lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} \). This can be verified from the definition of the limit of a function and the definition of the limit of a sequence. In words, the idea is that if we know that \( f'(x) \) exists, then we can find the value by taking any such sequence \( x_n \). In particular, choosing any two different sequences must give the same value for \( f'(x) \).

For this problem, we will show that \( f'(0) \) does not exist, because by taking two different sequences, we will find different values for \( f'(0) \).

**Claim:** \( f \) is not differentiable at 0.

**Proof of claim.** Assume for contradiction that \( f'(0) \) exists. Consider the sequence \( x_n = \frac{1}{p_n} \). Then we can evaluate that

\[
\frac{f(x_n) - f(0)}{x_n - 0} = \frac{f_n(x_n) - f(0)}{x_n - 0} = p_n^{-1/p_n},
\]

which converges to 1 as \( n \to \infty \). Thus, we must have that \( f'(0) = 1 \).

Now, consider the sequence \( y_n = 3^{-n} \). Then

\[
\frac{f(y_n) - f(0)}{y_n - 0} = \frac{f_n(y_n) - f(0)}{y_n - 0} = 3^{-n/3},
\]

which converges to 0 as \( n \to \infty \), implying that \( f'(0) = 0 \). Since we cannot simultaneously satisfy \( f'(0) = 1 \) and \( f'(0) = 0 \), \( f'(0) \) cannot exist. Thus, \( f \) is not differentiable at 0.

**Remark:** an even easier sequence we could choose for the second case is \( y_n = -1/n \). Then \( f(y_n) = 0 \) for all \( n \), so we see that \( \frac{f(y_n) - f(0)}{y_n - 0} = 0 \) for all \( n \), again implying \( f'(0) = 0 \).

A. (2) We will first prove the hint. Fix \( \varepsilon > 0 \) and let \( \delta = \frac{\varepsilon}{2M} \). Let \( |x| < \delta \).

**Case 1:** \( x > 0 \). Fix \( n \in \mathbb{N} \). By the mean value theorem applied to \( g'_n \) on \([0, \delta]\), there exists \( y \in (0, \delta) \) such that \( \frac{g'_n(x) - g'_n(0)}{x - 0} = g'_n(y) \), so \( |g'_n(x)| \leq |g'_n(y)||x| \leq M\delta \leq \varepsilon/2 < \varepsilon \).

**Case 2:** \( x < 0 \). Use the mean value theorem applied to \( g'_n \) on \([-\delta, 0]\) to get the same conclusion.

**Case 3:** \( x = 0 \). Then \( |g'_n(0)| = |0| = 0 < \varepsilon \).

Now, we will show that \( g'(0) \) exists and equals 0. That is, we will show that for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < |x| < \delta \implies |\frac{g(x) - g(0)}{x - 0}| < \varepsilon \). Let \( \varepsilon > 0 \) and let \( \delta = \frac{\varepsilon}{2M} \) and let \( 0 < |x| < \delta \).

**Case 1:** \( x \neq p_n^{-m} \) for all \( n, m \in \mathbb{N} \). Then \( \left| \frac{g(x) - g(0)}{x - 0} \right| = 0 < \varepsilon \).
Case 2: \( x = p_n^{-m} \) for some \( n, m \in \mathbb{N} \). Then, since \( g_n(0) = g(0) = 0 \), we have that \( g(x) - g(0) \sim g_n(x) - g_n(0) = \frac{g_n(x) - g_n(0)}{x - 0} \). By the mean value theorem applied to \( g_n \) on \([0, x]\), there exists some \( y \in (0, x) \) such that \( g_n(x) - g_n(0) = g_n'(y) \). Then by the hint we proved above, since \(|y| < |x| < \delta\), we have \(|g_n'(y)| < \varepsilon\), as desired.

B. Rudin Ch.5. (7) Observe that, since \( f(x) = g(x) = 0 \),

\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} = \lim_{t \to x} \left( \frac{f(t)-f(x)}{g(t)-g(x)} \right).
\]

By assumption, \( f'(x) \) and \( g'(x) \) exist, i.e. the limits \( \lim_{t \to x} \frac{f(t)-f(x)}{g(t)-g(x)} \) and \( \lim_{t \to x} \frac{g(t)-g(x)}{t-x} \) exist. This means that we can bring the limit in:

\[
\lim_{t \to x} \left( \frac{f(t)-f(x)}{g(t)-g(x)} \right) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \lim_{t \to x} \frac{g(t)-g(x)}{t-x} = f'(x) \frac{g'(x)}{g'(x)}.
\]

(9) We will show that \( f'(0) \) exists and \( f'(0) = 3 \). Fix \( x > 0 \). By the mean value theorem applied to \( f \) on \([0, x]\), there exists \( c_x \in (0, x) \) such that \( f(x) - f(0) = f'(c_x) \). Moreover, as \( x \to 0 \) from above, \( c_x \to 0 \) from above, so \( \lim_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{c_x \to 0+} f'(c_x) = 3 \). A similar argument on \([x, 0]\) for \( x < 0 \) shows that the limit from the left is also 3, thus \( f'(0) \) exists and equals 3.

(15) We will show the hint first. Let \( h > 0 \). By Taylor’s theorem, there exist some \( \xi \in (x, x+2h) \) such that

\[
f(x + 2h) = f(x) + f'(x)(x + 2h - x) + \frac{1}{2} f''(\xi)(2h)^2.
\]

Rearranging this gives that \( f'(x) = \frac{f(x + 2h) - f(x)}{2h} - h f''(\xi) \). Then we have

\[
|f'(x)| \leq \left| \frac{f(x + 2h) - f(x)}{2h} \right| + h |f''(\xi)|
\leq \left| \frac{f(x + 2h) - f(x)}{2h} \right| + h M_2
\leq \frac{M_0}{h} + h M_2
\]
Thus, by taking supremum over all \( x \) on the left hand side, \( M_1 \leq \frac{M_0}{h} + hM_2 \) for all \( h > 0 \).
If \( M_0 = 0 \), taking \( h \to 0 \) shows that \( M_1 = 0 \) and so the desired inequality is satisfied as \( 0 \leq 0 \).
If \( M_2 = 0 \), taking \( h \to \infty \) gives \( M_1 = 0 \) as well, so again the desired inequality holds. Now, suppose \( M_0 \neq 0 \) and \( M_2 \neq 0 \). Let \( h = \sqrt{\frac{M_0}{M_2}} \). Then we have
\[
M_1 \leq \frac{M_0}{\sqrt{M_0/M_2}} + \sqrt{\frac{M_0}{M_2}}. M_2.
\]
Rearranging this and squaring yields the desired inequality \( M_1^2 \leq 4M_0M_2 \).

We will now show equality is achieved in the given example. It is clear that \( |f(x)| \leq 1 \), and since \( |f(0)| = 1 \), we have that \( M_0 = 1 \). It is easy to compute that \( f'(x) = 4x \) if \( x \in (-1,0) \), and \( f'(x) = \frac{4}{(x^2+1)^2} \) if \( x > 0 \). By a similar argument to 9, it follows that \( f'(0) = 0 \). Thus, we have that \( |f'(x)| \leq 4 \) by inspection, and as \( x \to -1 \), \( f'(x) \to -4 \), so \( M_1 = 4 \). Similarly compute that \( M_2 = 4 \). We see that equality is indeed achieved.

We now show that the result holds for vector-valued \( f : (a, \infty) \to \mathbb{R}^n \). Let \( M_0 = \sup_{x \in (a, \infty)} \| f(x) \| \), \( M_1 = \sup_{x \in (a, \infty)} \| f'(x) \| \), and \( M_2 = \sup_{x \in (a, \infty)} \| f''(x) \| \). Fix \( y \in \mathbb{R}^n \), and consider the function \( g_y : (a, \infty) \to \mathbb{R} \) given by \( g_y(x) = y \cdot f(x) \). Note that the result will hold for \( g_y \) by the first part of this problem: for all \( x \in (a, \infty) \),
\[
\sup_{z \in (a, \infty)} |g_y'(z)|^2 \leq 4 \sup_{z \in (a, \infty)} |g_y(z)| \sup_{z \in (a, \infty)} |g_y''(z)|.
\]
From the definition of \( g_y \), it is easy to verify that \( \sup_{z \in (a, \infty)} |g_y(z)| \leq \| y \| M_0 \), and \( \sup_{z \in (a, \infty)} |g_y''(z)| \leq \| y \| M_2 \).
Fix \( x \in (a, \infty) \). We will show that \( \| f'(x) \|^2 \leq 4M_0M_2 \). If \( f'(x) = 0 \), there is nothing to prove. Assume \( f'(x) \neq 0 \). By the above, we have that for all \( y \in \mathbb{R}^n \), \( \| y \cdot f'(x) \|^2 \leq 4\| y \|^2 M_0M_2 \).
Take \( y = \frac{f'(x)}{\| f'(x) \|} \). Then \( y \cdot f'(x) = \frac{f'(x) \cdot f'(x)}{\| f'(x) \|} = \| f'(x) \| \), and \( \| y \| = 1 \). Thus, for this specific \( y \), we obtain \( \| f'(x) \|^2 \leq 4M_0M_2 \). Taking the supremum over \( x \) yields the desired result.

(17) By Taylor’s theorem applied to \( \beta = 1, \alpha = 0 \), there exists some \( s \in (0,1) \) such that
\[
f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{3!} f^{(3)}(s).
\]
Similarly, with \( \beta = -1 \) and \( \alpha = 0 \), there exists some \( t \in (-1,0) \) such that
\[
f(-1) = f(0) - f'(0) + \frac{1}{2} f''(0) - \frac{1}{3!} f^{(3)}(t).
\]
Using the given values of \( f \) in the problem statement, this leaves us with two equations:

\[
1 = \frac{1}{2} f''(0) + \frac{1}{6} f^{(3)}(s), \quad \text{and} \quad 0 = \frac{1}{2} f''(0) - \frac{1}{6} f^{(3)}(t).
\]

Subtracting the second from the first and multiplying through by 6 yields that \( 6 = f^{(3)}(s) + f^{(3)}(t) \). It follows that either \( f^{(3)}(s) \geq 3 \) or \( f^{(3)}(t) \geq 3 \) (if both were less than 3, their sum would be less than 6), which is the desired statement.

For the given function \( f(x) = \frac{1}{2}(x^3 + x^2) \), compute that \( f^{(3)}(x) = 3x \), so we have that \( f^{(3)}(1) = 3 \), i.e. equality is achieved.

22. (a) Suppose for contradiction that there are two fixed points \( x \) and \( y \). Without loss of generality, we may assume that \( x < y \). By the mean value theorem, there exists some \( c \in (x, y) \) such that \( \frac{f(x) - f(y)}{x - y} = f'(c) \). But since \( f(x) = x \) and \( f(y) = y \), this implies that \( f'(c) = 1 \), contradicting our assumption that \( f'(t) \neq 1 \) for all \( t \). Thus, there is at most one fixed point.

(b) Suppose that \( t \) is a fixed point, i.e. that \( f(t) = t \). Then we would have that \( \frac{1}{1 + e^t} = 0 \), which is impossible. Thus, \( f \) has no fixed points. Compute that \( f'(t) = 1 - \frac{e^t}{(1 + e^t)^2} \). Since \( 0 < \frac{e^t}{(1 + e^t)^2} < 1 \), it follows that \( 0 < f'(t) < 1 \) for all \( t \).

(c) Let \( x_1 \in \mathbb{R} \) and for \( n \in \mathbb{N} \) define \( x_{n+1} = f(x_n) \). We will show this sequence converges, and that the limit is a fixed point of \( f \). If \( x_n = x_{n+1} = f(x_n) \) for any \( n \in \mathbb{N} \), then \( x_n \) is a fixed point of \( f \), and there is nothing to prove. Thus, assume that \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N} \). We may apply the mean value theorem to \( f \) on \([x_n, x_{n+1}] \) (or \([x_{n+1}, x_n] \) if \( x_n > x_{n+1} \)); there exists a real number \( c \) between \( x_n \) and \( x_{n+1} \) such that \( \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(c) \). We have shown that for all \( n \in \mathbb{N} \), \( |f(x_{n+1}) - f(x_n)| \leq A|x_{n+1} - x_n| \), i.e.

\[
|x_{n+2} - x_{n+1}| \leq A|x_{n+1} - x_n|.
\]

This lets us bound \( |x_{n+1} - x_n| \) in terms of \( |x_2 - x_1| \), as follows.

**Claim:** For all \( n \in \mathbb{N} \), \( |x_{n+2} - x_{n+1}| \leq A^n |x_2 - x_1| \).

This claim can be easily proven by induction. Now, suppose that \( m > n \) are natural numbers. Then we have that

\[
|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|
\]

\[
\leq |x_2 - x_1| (A^{m-2} + A^{m-3} + \cdots + A^{n-1})
\]

\[
= |x_2 - x_1| A^{n-1} (A^{m-n-3} + \cdots + 1)
\]
Now, notice that since $0 \leq A < 1$, \( \sum_{n=0}^{\infty} A^n = \frac{1}{1-A} < \infty \). Then using the fact that \( A^{m-n-3} + \cdots + 1 \leq \sum_{n=0}^{\infty} A^n \) in the above computations, we have that \( |x_m - x_n| \leq |x_2 - x_1| A^{n-1} \frac{1}{1-A} \). Since \( A^n \to 0 \) as \( n \to \infty \) (because \( 0 \leq A < 1 \)), it follows that \( |x_m - x_n| \to 0 \) as \( m > n \to \infty \), i.e. \( (x_n) \) is a Cauchy sequence, hence is convergent.

Let \( x = \lim_{n \to \infty} x_n \). We will show that \( f(x) = x \). Observe that \( f(x) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) \) by continuity of \( f \). Moreover, by definition of \( x_{n+1} \), \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x \), so \( f(x) = x \), as desired.