Math 109, Fall 2017
Homework 3 Solutions

A.(a) Let $x$ and $y$ be given real numbers. We will prove this by cases.

Case (i): Both $x$ and $y$ are non-negative. Then $x + y \geq 0$ as well, so $|x + y| = x + y$, $|x| = x$, and $|y| = y$. Therefore $|x + y| \leq |x| + |y|$ in this case.

Case (ii): Exactly one of $x, y$ is non-negative, the other is negative. Since $x$ and $y$ are interchangable in this part (what we do to $x$ is the same as what we do to $y$), we may assume without loss of generality that $x \geq 0$ and $y < 0$. Then $|x| = x$ and $|y| = -y$. There are two sub-cases to consider: if $x + y \geq 0$, then $|x + y| = x + y$. Since $y \leq -y$ (since $y$ is negative), we therefore have $|x + y| = x + y \leq x - y = |x| + |y|$. If instead $x + y < 0$, then $|x + y| = -x - y$. Since $-x \leq x$ (since $x \geq 0$), $|x + y| = -x - y \leq x - y = |x| + |y|$.

Case (iii): Both $x$ and $y$ are negative. Then $x + y < 0$ as well, so $|x + y| = -x - y$, $|x| = -x$, and $|y| = -y$. Then, as in case (i), the quantities are actually equal. Hence, in all cases $|x + y| \leq |x| + |y|$.

(b) Let $x$ and $y$ be given real numbers. Observe that $x = (x - y) + y$, so $|x| = |(x - y) + y| \leq |x - y| + |y|$ by part (a). So we have $|x| - |y| \leq |x - y|$. Similarly, $y = (y - x) + x$, so $|y| \leq |y - x| + |x|$. Since $|y - x| = |x - y|$, we have shown that $-|y| \leq |x - y|$. Thus, $|x - y| + |y| \leq |x| + |y|$ as well, so putting the two together implies that $||x| - |y|| \leq |x - y|$.

(c) Let $x$ and $y$ be given real numbers. We will prove this by cases.

Case (i) - Both $x$ and $y$ are non-negative. Then $xy \geq 0$ and so $|xy| = xy$, while $|x||y| = (x)(y) = xy$.

Case (ii) - Both $x$ and $y$ are negative. Then $xy > 0$ and so $|xy| = xy$, while $|x||y| = (-x)(-y) = xy$.

Case (iii) - Exactly one of $x, y$ is non-negative and the other is negative. As in part (a), without loss of generality we may assume that $x \geq 0$ and $y < 0$. Then $xy \leq 0$ and so $|xy| = -xy$, while $|x||y| = (x)(-y) = -xy$. Hence, in all cases $|xy| = |x||y|$.

(d) Let $x$ be a given real number. We will again use cases.

Case (i) - $x > 0$. Then $|x| = x$ and so $|x|^2 = x^2$.

Case (ii) - $x < 0$. Then $|x| = -x$ and so $|x|^2 = (-x)^2 = x^2$.

Hence, in both cases $|x|^2 = x^2$. The fact that $|x|^2 = x^2$ follows from the fact that $x^2 \geq 0$.

(e) Let $x$ be a given real number. By part (d), we have that $|x|^2 = x^2$. Also, $|x| \geq 0$ by the definition of $|.|$. Hence, by the definition of square root given in the problem, $\sqrt{x^2} = |x|$. (Take $z = |x|$ and $y = x^2$ in the definition. Since $|z| \geq 0$ and $|y|^2 = z^2 = y = x^2$, we have proven that $|x|$ satisfies the definition for being the square root of $x^2$.)

(f) Let $x$ and $y$ be given real numbers. First, suppose that $|x| \leq |y|$. Then by part (d) and the fact that $|x|, |y| \geq 0$, $x^2 = |x|^2 \leq |x||y| \leq |y|^2 = y^2$ by properties of inequalities.

For the forward implication, we will prove the contrapositive. Assume that $|x| > |y|$. We
have
\[ x^2 = |x|^2 > |x||y| \] by a property of inequalities since $|x| > 0$
\[ \geq |y||y| \] by a property of inequalities since $|y| \geq 0$
\[ = |y|^2 = y^2 \]

B. Assume that $d|n$ and $d|(n+1)$. Then by problem D(ii) on HW 2, $d|[(n+1) - 1]$, so $d|1$. Since $d$ is assumed to be positive, the only possibility is that $d = 1$.

To do it without using D(ii), simply apply the definition of divides: since $d|n$ and $d|n + 1$, there exist integers $k, \ell$ such that $n = dk$ and $n + 1 = d\ell$. Then $dk + 1 = d\ell$, so $1 = d(\ell - k)$.

Since $\ell$ and $k$ are integers, so is $\ell - k$, so we’ve proven $d|1$, and so $d = 1$ as above.

C. Problems I

12. We will use induction on $n$. For the base case $n = 1$, note that $4^1 + 5 = 9 = 3 \times 3$, so $3|4^1 + 5$.

For the induction step, assume there is some $k \geq 1$ such that $3|4^k + 5$. We will prove $3|4^{k+1} + 5$. The main idea is to write $4^{k+1} + 5$ as “$4^k + 5$ with something done to it”. Observe that $4^{k+1} + 5 = 4 \cdot 4^k + 5 = 4(4^k + 5) - 15$. By the induction hypothesis, $3|4^k + 5$. Since $3|15$ as well, $3|(4(4^k + 5) - 15) = 4^{k+1} + 5$ by D(ii) on HW 2, as desired.

Therefore, by induction, $3|4^n + 5$ for all positive integers $n$.

**Alternative way of doing the induction step:** By the induction hypothesis, we know that $4^k + 5 = 3\ell$ for some integer $\ell$. Therefore $4^k = 3\ell - 5$, so $4^{k+1} = 4(3\ell - 5) = 12\ell - 20$. We want to look at $4^{k+1} + 5$, so add 5 to this, and we have that $4^{k+1} + 5 = 12\ell - 15 = 3(4\ell - 5)$. Since $4\ell$ and $5$ are integers, so is $4\ell - 5$, so we’ve proven $3|4^{k+1} + 5$, as desired.

13. We will use induction on $n$. The base case is $n = 4$, since we are told to prove it for all $n \geq 4$. Observe $4! = 24$ and $2^4 = 16$. Since $4! > 2^4$, the base case holds.

For the induction step, assume there is some $k \geq 4$ such that $k! > 2^k$. We want to show that $(k + 1)! > 2^{k+1}$. By the induction hypothesis, $k! > 2^k$. Since $k + 1 > 0$, we can multiply by $k + 1$ to obtain $(k + 1)! = (k + 1)! > (k + 1)2^k$. Since $k \geq 4$, $k + 1 > 2$, so $(k + 1)2^k > 2 \cdot 2^k = 2^{k+1}$. Therefore $(k + 1)! > 2^{k+1}$, as desired.

It follows by induction that $n! > 2^n$ for all integers $n \geq 4$.

14. Let $x$ be any (fixed!) real number such that $x > -1$. (We will prove that it’s true for all $n$ when this one fixed $x$ is used. Since we don’t assume anything about $x$ beyond $x > -1$, the same argument proves it’s true for every $x > -1$.) We will prove by induction on $n$ that $(1 + x)^n \geq 1 + nx$ for all integers $n \geq 0$. For the base case of $n = 0$, observe $(1 + x)^0 = 1 = 1 + 0x = 1$.

Assume that $k \geq 0$ is such that $(1 + x)^k \geq 1 + kx$. Then
\[
(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + x)(1 + kx)
\]
by the induction hypothesis and because $x > -1$
\[= 1 + kx + x + kx^2\]
\[\geq 1 + (k + 1)x\]
because $kx^2 \geq 0$ since $k \geq 0$ and $x^2 \geq 0$

Therefore, by induction, $(1 + x)^n \geq 1 + nx$ for all $n \geq 0$ and all $x > -1$.

**Remark:** The second line requires $x > -1$ because we multiplied the inequality $(1 + x)^k \geq 1 + kx$ by $1 + x$. If $x < -1$, this would have been multiplying by a negative number, so the inequality would flip direction.

2
16. We will use induction on $n$. For the base case of $n = 1$, note that

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{1+1}.$$ 

For the inductive step, let $k$ be a positive integer and assume that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$ 

Then

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \text{ by the inductive hypothesis}$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

as desired.

Hence, it follows that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all positive integers $n$ by induction.