Math 140B, Winter 2018
Homework 4 Solutions

A. It is clear that given any partition $P$ of $[0, 1]$, $L(P, f) = 0$. This is because $f$ is only nonzero on a countable set of points, so the infimum on any interval will be zero. Thus, to show that $f$ is integrable, we need to show that for all $\varepsilon > 0$, there exists a partition $P$ of $[0, 1]$ such that $U(P, f) < \varepsilon$.

Let $\varepsilon > 0$ and define $N = \max\{n \in \mathbb{N} : \frac{1}{n} \geq \varepsilon\}$ (note that such $N$ exists by the Archimedian principle). The idea is that only $t_1, \ldots, t_N$ have a large enough contribution to the upper sum to be noteworthy: those that are less than $\varepsilon$ are very small already. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[0, 1]$ such that $x_{i+1} - x_i < \frac{\varepsilon}{N}$ for all $i$ and $t_1, \ldots, t_N \notin P$ (this is to ensure that each “bad” $t_i$ can lie in only one interval, so $|I| \leq N$). Define $I = \{i \in \{0, \ldots, n-1\} : \text{there exists } 1 \leq k \leq N \text{ such that } t_k \in [x_i, x_{i+1})\}$. Then

$$U(P, f) = \sum_{i=0}^{N-1} M_i \Delta x_i$$

$$= \sum_{i \in I} M_i \Delta x_i + \sum_{i \notin I} M_i \Delta x_i$$

$$< \sum_{i=1}^{N} \frac{1}{i} \varepsilon + \sum_{i \notin I} \varepsilon \Delta x_i$$

$$= \frac{\varepsilon}{N} \sum_{i=1}^{N} \frac{1}{i} + \varepsilon \sum_{i \notin I} \Delta x_i$$

$$\leq \frac{\varepsilon}{N} \cdot N + \varepsilon$$

$$= 2\varepsilon.$$  \hfill (1)

where (1) is because for $i \notin I$, $M_i \leq \varepsilon$ by definition of $I$ and $N$. Note also that $|I| < N$ is possible, if some interval $[x_i, x_{i+1})$ contains two different $t_k, t_\ell$ with $k, \ell \in \{1, \ldots, N\}$. However, we obtain an upper bound by supposing there are $N$ intervals, each with one of these $t_k$'s in them. (2) follows because $\sum_{i=1}^{N} \frac{1}{i} \leq N$ in the first sum and in the second sum, $\sum_{i \notin I} \Delta x_i < 1$ because it cannot exceed the length of the interval $[0, 1]$.

Remark: It doesn’t matter that we got $2\varepsilon$ instead of $\varepsilon$. As long as we get some bound that goes to zero as $\varepsilon \to 0$, it’s fine. If you prefer, go through and replace $\varepsilon$ with $\varepsilon/2$ where we defined $N$ and $P$, and you will get $\varepsilon$ at the end. Also, note that using the rougher estimate of $M_i = 1$ for $i \in I$ would be sufficient.

B.(a) Since $f$ is integrable on $[a, b]$, we know that for every $\varepsilon > 0$, there exists a partition $P$ such that $U(P, f) - L(P, f) = \sum(M_i - m_i)\Delta x_i < \varepsilon$. Notice that a function that is integrable on $[a, b]$ is also integrable on every subinterval $[c, d] \subseteq [a, b]$ (exercise).

Let $P_1 = \{x_0 < x_1 < \ldots < x_k\}$ be a partition of $[a, b]$ such that $U(P_1, f) - L(P_1, f) < b - a$. This implies that there is some $[x_i, x_{i+1}]$ on which $M_i - m_i < 1$: if this were not true, then the total sum would exceed $b - a$. Let $I_1 := [a_1, b_1]$ be one of these intervals in $P_1$. 

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Since $f$ is integrable on $I_1$, if $U(P_2, f) - L(P_2, f) < \frac{1}{2}(b_1 - a_1)$. As above, this implies that there is some subinterval $[x_j, x_{j+1}]$ of $P_2$ such that $M_j - m_j < \frac{1}{2}$. Let $I_2 := [a_2, b_2]$ be a closed subinterval of $[x_j, x_{j+1}]$ with $a_2 \neq x_k$ and $b_2 \neq x_{j+1}$. (As will become clear, we do not want to have one of the endpoints in every interval we're defining. This could lead to only one-sided continuity of the point we construct.)

Proceed inductively: given $I_n = [a_n, b_n]$, choose a partition $P_n$ of $I_n$ such that $U(P_n, f) - L(P_n, f) < \frac{1}{n}(b_n - a_n)$. Then some subinterval $[x_k, x_{k+1}]$ has $M_k - m_k < \frac{1}{n}$. Let $I_{n+1} := [a_{n+1}, b_{n+1}]$ be a subinterval of $[x_k, x_{k+1}]$ with $a_{n+1} \neq x_k$, $b_{n+1} \neq x_{k+1}$.

Then we have a sequence of nested compact intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, so $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. Let $x \in \bigcap_{n \in \mathbb{N}} I_n$.

**Claim:** $f$ is continuous at $x$.

**Proof of claim.** Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon$. Since $x \in I_{N+1}$, it is contained in some open subinterval of $I_N$, say $(x - \delta, x + \delta)$ for some $\delta > 0$. This is because $x \neq a_N$ and $x \neq b_N$, because these points are not in $I_{N+1}$ by construction, so $x$ is actually in the open interval $(a_N, b_N)$.

Let $M_N = \sup_{y \in I_n} f(y)$ and $m_N = \inf_{y \in I_n} f(y)$. By construction of $I_N$, $M_N - m_N < \frac{1}{N}$. This says that the maximum distance between any two points of $I_N$ is bounded above by $\frac{1}{N}$. In particular, for all $y \in (x - \delta, x + \delta)$, $|f(x) - f(y)| < \frac{1}{N} < \varepsilon$, which shows that $f$ is continuous at $x$. \qed

**B.(b)** We use the following characterization of density: let $A \subseteq [a, b]$. Then $A$ is dense in $[a, b]$ if and only if for every open interval $(c, d) \subseteq [a, b]$, $(c, d) \cap A \neq \emptyset$. (Exercise to verify this.)

Let $(c, d) \subseteq [a, b]$ be any open subinterval, and let $[c', d'] \subseteq (c, d)$. Since $f$ is integrable on $[c', d']$, one can use the same argument as in (a), substituting $d' - a'$ in place of $b - a$ in the first step, to show that there exists some point $x \in [c', d']$ at which $f$ is continuous. This shows that every open subinterval of $[a, b]$ contains a point at which $f$ is continuous, which implies the set of points at which $f$ is continuous is dense in $[a, b]$.

**C.(6)** Let $P$ be the Cantor set. Let $0 < \varepsilon < 1$. We claim to be able to cover $P$ with open intervals whose lengths add up to less than $\varepsilon$. To see this, recall that $P$ is the intersection of sets $E_n$ made up of $2^n$ closed intervals each of size $3^{-n}$. Since $(2/3)^n \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $(2/3)^N < \frac{\varepsilon}{2}$. By slightly extending each of these, say from $[a_k, b_k]$ to $(a_k - \frac{\varepsilon}{2^{N+1}}, b_k + \frac{\varepsilon}{2^{N+1}})$, we get a collection of open intervals covering $P$, and the total added length is equal to $\frac{2^{-N}}{2^{N+1}} < \frac{\varepsilon}{2}$. (Note also that there is still a positive distance between two intervals: the distance between $b_k$ and $a_{k+1}$ was $3^{-N}$, and is now at least $3^{-(N+1)}$. This matters for making sure the partition we define later really is made up of intervals, not single points.)

Thus, if $1 \leq k \leq 2^N$, $I_k := (a'_k, b'_k)$, where $a'_k = a_k - \frac{\varepsilon}{2^{k+1}}$ and $b'_k = b_k + \frac{\varepsilon}{2^{N+1}}$, these cover $P$ and have total length less than $\varepsilon$. Because $a'_1 < 0$ and $b'_2 > 1$, we shall change these to $a'_1 = 0$ and
\(b_{2N} = 1\) to fix the first and last intervals as \([0, b'_1)\) and \((a'_{2N}, 1]\), as we would like the intervals to stay in \([0, 1]\). Note that intervals of the form \([0, b)\) and \((a, 1]\) are open in \([0, 1]\) even though they are not open in \(\mathbb{R}\).

Let \(K = [0, 1] - \bigcup_{k=1}^{2^N} I_k\). Then \(K\) is closed because we are removing an open set from a closed set. Moreover, it is compact. Since \(f\) is continuous on \(K\) by assumption, \(f\) is uniformly continuous on \(K\): there exists \(\delta > 0\) such that for all \(x, y \in K\), \(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon\). Form a partition \(Q = \{x_0 < x_1 < \ldots < x_n\}\) of \([0, 1]\) such that (1) every \(a'_k\) and every \(b'_k\) is in \(Q\), (2) every \(x_i\) in \(Q\) that is not equal to an \(a'_k\) or \(b'_k\) is not contained in any \(I_k\) for \(1 \leq k \leq 2^N\), and satisfies \(x_{i+1} - x_i < \delta\) for all \(i\).

Let \(M = \sup_{x \in [0, 1]} |f(x)|\). Then

\[
U(P, f) - L(P, f) = \sum_{i=0}^{n-1} (M_i - m_i) \Delta x_i
\]

\[
\leq \sum_{k=1}^{2^N} 2M(b'_k - a'_k) + \sum_{i \text{ with } x_i \not\in a'_k} (M_i - m_i) \Delta x_i
\]

\[
= 2M \sum_{k=1}^{2^N} (b'_k - a'_k) + \sum_{i \text{ with } x_i \not\in a'_k} (M_i - m_i) \Delta x_i
\]

\[
\leq 2M \varepsilon + \varepsilon(b - a),
\]

where the last line uses the fact that if we are in an interval \([x_i, x_{i+1}]\) satisfying (2) in our definition of \(Q\), then the uniform continuity of \(f\) implies that \(M_i - m_i < \varepsilon\). On the other intervals, we use the rough estimate that \(M_k - m_k \leq M - (-M) = 2M\). Since \(2M \varepsilon + \varepsilon(b - a) \to 0\) as \(\varepsilon \to 0\), we conclude that \(f\) is integrable on \([0, 1]\).

(8) Let \(P\) be the partition of \([1, n]\) of the form \(\{1, 2, 3, \ldots, n\}\). Since \(f\) is decreasing on \([1, \infty),\) it is clear that \(L(P, f) = f(2) + f(3) + \cdots + f(n)\), and \(U(P, f) = f(1) + f(2) + \cdots + f(n-1)\). Thus,

\[
\sum_{k=2}^{n} f(k) \leq \int_{1}^{n} f(x)dx \leq \sum_{k=1}^{n-1} f(k).
\]

Suppose that the series \(\sum_{n=1}^{\infty} f(n)\) converges. Then the limit is an upper bound for the sequence \(a_n = \int_{1}^{n} f(x)dx\). Then \(a_n\) is a bounded monotonically increasing sequence (because \(f \geq 0\)), and hence converges. Suppose now that \(a_n\) converges to some \(a\). Then \(a + f(1)\) upper bound for the monotonically increasing sequence \(b_n = \sum_{k=1}^{n} f(k)\), so \(b_n\) converges as well. Thus, the integral converges if and only if the series converges.

We still need to check for generic \(b > 1\). Let \(b > 1\) and let \(n \in \mathbb{N}\) such that \(n \leq b < n + 1\). Then, because \(f\) is decreasing and non-negative,

\[
\int_{1}^{n} f(x)dx \leq \int_{1}^{b} f(x)dx \leq \int_{1}^{n+1} f(x)dx.
\]

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If the sequence $a_n$ converges, it follows that the limit $\lim_{b \to \infty} \int_a^b f(x) \, dx$ exists by the squeeze theorem. Since we have established $a_n$ converges if and only if the series converges, this yields the desired conclusion.

(10)(a) Notice that $e^x$ is a convex function (you can verify that it’s concave up everywhere to justify this. Recall that $f : \mathbb{R} \to \mathbb{R}$ is convex if for all $t \in [0, 1]$ and all $x, y \in \mathbb{R}$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$). Observe that

$$uv = \exp(\log(uv)) = \exp(\log(u) + \log(v)) = \exp\left(\frac{1}{p} \log(u) + \frac{1}{q} \log(v)\right) = \exp\left(\frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q)\right) \leq \frac{1}{p} \exp(\log(u^p)) + \frac{1}{q} \exp(\log(v^q)) \leq \frac{u^p}{p} + \frac{v^q}{q}$$

(10)(b) By part (a), for all $x$ we have that

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}.$$  

By integrating this, we obtain that

$$\int_a^b f(x)g(x) \, dx \leq \frac{1}{p} \int_a^b f(x)^p \, dx + \frac{1}{q} \int_a^b g(x)^q \, dx = \frac{1}{p} + \frac{1}{q} = 1$$

by the assumptions in the problem statement.

(10)(c) Claim: If $\int_a^b |f(x)|^p \, dx = 0$, then $\int_a^b f(x)g(x) \, dx = 0$.

This will be assigned as a problem on another HW, so a full solution will not be provided here. As a hint, first use the fact that $g$ is bounded to justify that proving $\int_a^b |f(x)| \, dx = 0$ is sufficient. Try to use convexity of $x^p$ to justify that

$$\left(\frac{L(P,|f|)}{b-a}\right)^p \leq \frac{L(P,|f|^p)}{b-a}$$

for every partition $P$ of $[a, b]$ to argue that $\int_a^b |f(x)| \, dx = 0$.

By the claim, the inequality holds as $0 \leq 0$ if $\int_a^b |f|^p \, dx = 0$ or $\int_a^b |g|^q \, dx = 0$, so assume that
both are strictly positive. Define
\[ F(x) = \frac{|f(x)|}{(\int_a^b |f(x)|^p d\alpha)^{1/p}} \quad \text{and} \quad G(x) = \frac{|g(x)|}{(\int_a^b |g(x)|^q d\alpha)^{1/q}}. \]

Applying part (b) to \( F \) and \( G \) implies that \( \int_a^b F(x)G(x) d\alpha \leq 1 \), and rearranging this implies that \( \int_a^b |f(x)||g(x)| d\alpha \leq \left( \int_a^b |f(x)|^p d\alpha \right)^{1/p} \left( \int_a^b |g(x)|^q d\alpha \right)^{1/q} \). The desired result then follows by using that \( \left| \int_a^b f g d\alpha \right| \leq \int_a^b |f||g| d\alpha \).

\((10)(d)\) Since these improper intervals are only defined with the assumption that the limits actually exist, we can use that the inequality holds for all \( b > 0 \), and then taking the limits preserves the inequality by the usual limit laws.

\((11)\) Note that the Schwarz inequality referred to in this problem is \((10)(d)\) with \( p = q = 2 \) (as is stated at the very end of problem \((10)\)). We have that
\[
\|f - h\|_2^2 = \int_a^b |f - h|^2 d\alpha \quad \text{by definition}
\]
\[
= \int_a^b |f - g + g - h|^2 d\alpha
\]
\[
\leq \int_a^b (|f - g| + |g - h|)^2 d\alpha \quad \text{by the usual triangle inequality}
\]
\[
= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g||g - h| d\alpha + \int_a^b |g - h|^2 d\alpha
\]
\[
\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \quad \text{by the Schwarz inequality on the middle term}
\]
\[
= (\|f - g\|_2 + \|g - h\|_2)^2
\]
so the result follows by taking square roots of both sides.