A. (we will relate this to integrals later) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For every $x \in [a, b]$ define the function $J_{f,x} : (0, \infty) \to \mathbb{R}$ by
\[ J_{f,x}(r) = \text{diam}\left(f((x-r, x+r) \cap [a, b])\right). \]

(1) Prove that $\lim_{r \to 0^+} J_{f,x}(r)$ exists for every $x \in [a, b]$. Denote this limit by $J_f(x)$.

(2) Prove that $f$ is continuous at $x$ if and only if $J_f(x) = 0$.

(3) Show that for every $\varepsilon > 0$ the set 
\[ \{x \in [a, b] : J_f(x) \geq \varepsilon\} \]
is a closed set.

(4) Show that the set of discontinuity $f$ is a union of (at most) countably many closed sets.

(5) Construct (with justification) a function on $\mathbb{R}$ which is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{Q}^c$.

(6) (Bonus problem) Does there exist any function on $\mathbb{R}$ which is discontinuous on $\mathbb{Q}^c$ and continuous on $\mathbb{Q}$?

B. Let the notation be as in problem A.

(1) Assume $f$ is a function with the following property. For every $\varepsilon > 0$ the set
\[ A_\varepsilon = \{x \in [a, b] : J_f(x) \geq \varepsilon\} \]
is a finite set. Prove that $f \in \mathcal{R}$ on $[a, b]$.

(Hint: Remove small neighborhoods of points in $A_\varepsilon$. The remaining set, $B$ say, is compact. Show that there exists some $r > 0$ so that $J_{f,x}(r) < \varepsilon$, $\forall x \in B$; construct a partition $P$ using these.)

(2) Assume $\lim_{y \to x} f(y)$ exists for all $x \in [a, b]$. Prove that $f \in \mathcal{R}$ on $[a, b]$.

(Hint: Prove that the condition in B(1) holds under this assumption.)

C. Rudin, Chapter 6 (page 147), problems # 12, 15, 16, 17.