Math 140B, Winter 2018

Homework 5 Solutions

(A)(1) Proof 1: Observe that $J_{f,x}$ is an increasing function of $r$: if $R > r$, then $(x-r,x+r) \cap [a,b] \subseteq (x-R,x+R) \cap [a,b]$, and the diameter of a subset cannot exceed the diameter of the larger set, so $J_{f,x}(r) \leq J_{f,x}(R)$. Define $G(r) = J_{f,x}(r)$ for $r > 0$, and $G(r) = 0$ if $r \in [-1,0]$. Then $G$ is monotone increasing on $[-1,1]$, so by Theorem 4.29, $G(0+)$ exists. Since $G(r) = J_{f,x}(r)$ for $r > 0$, this means $J_{f,x}(0+) = G(0+)$ exists, as desired. (Note that we extended $J_{f,x}$ to be defined for negative $r$ because Theorem 4.29 only allows $x$ in the interior of $(a,b)$.)

Proof 2: Define $a_n = J_{f,x}(1/n)$. As above, this sequence decreases as $n$ increases, and is bounded below by 0, i.e. $a_n$ is a bounded monotone sequence, and hence converges. Let $L = \lim_{n \to \infty} a_n$. We will now verify that $\lim_{r \to 0^+} J_{f,x}(r) = L$. Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $J_{f,x}(1/N) - L < \varepsilon$. (Note that $J_{f,x}(1/N) \geq L$ because the limit of a decreasing sequence is the infimum, so we omit the superfluous absolute value. Indeed, as proof 1 uses, $L = \inf_{r>0} J_{f,x}(r)$, not just the inf over the 1/n’s.) Let $0 < r < 1/N$. Then $J_{f,x}(r) \leq J_{f,x}(1/N)$, so $J_{f,x}(r) - L \leq J_{f,x}(1/N) - L < \varepsilon$. Moreover, there exists some $m$ such that $r > 1/m$, and we have $J_{f,x}(r) \geq J_{f,x}(1/m) \geq L$, so we have actually shown that if $0 < r < 1/N$, then $|J_{f,x}(r) - L| < \varepsilon$, as desired.

(2) First, assume that $f$ is continuous at $x$. Let $\varepsilon > 0$ and let $R > 0$ be such that $y \in (x-R,x+R) \implies |f(y) - f(x)| < \varepsilon/2$. Then $	ext{diam}((f((x-R,x+R) \cap [a,b]))) \leq \varepsilon$ by the triangle inequality: for any $y,z \in (x-R,x+R)$, $|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(y)| < \varepsilon/2 + \varepsilon/2$. Since $J_{f,x}(r)$ decreases as $r$ decreases, this implies that $J_{f,x}(r) \leq \varepsilon$ for all $r < R$, so $0 \leq J_f(x) \leq \varepsilon$. This works for every $\varepsilon > 0$, so $J_f(x) = 0$.

Now, suppose that $J_f(x) = 0$. Let $\varepsilon > 0$, and let $r > 0$ be such that $J_{f,x}(r) < \varepsilon$. Then by definition of $J_{f,x}(r)$, for all $y \in (x-r,x+r) \cap [a,b]$, $|f(x) - f(y)| < \varepsilon$, as desired.

(3) Fix $\varepsilon > 0$ and define $B_\varepsilon = \{x \in [a,b] | J_f(x) < \varepsilon\}$. We will show that $B_\varepsilon$ is open. Let $x \in B_\varepsilon$ and let $r > 0$ be such that $J_{f,x}(r) < \varepsilon$. Let $y \in (x - \frac{r}{2}, x + \frac{r}{2}) \cap [a,b]$. Then $(y - \frac{r}{2}, y + \frac{r}{2}) \subseteq (x-r,x+r) \cap [a,b]$ by the triangle inequality, and so $J_{f,y}(r/2) < \varepsilon$ too. This shows that the open neighbourhood $(x - \frac{r}{2}, x + \frac{r}{2}) \cap [a,b]$ is contained in $B_\varepsilon$, so it’s open.

(4) Let $D = \{x \in [a,b] | f$ is discontinuous at $x\}$. By part (2), $f$ is discontinuous at $x$ if and only if $J_f(x) = 0$, so $D = \{x \in [a,b] | J_f(x) > 0\}$. 

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There does not exist a function \( \delta > 0 \) such that for every \( x \in (0, 1) \), there exists some \( y \in (0, 1) \) with \( |x - y| < \delta \) and \( f(y) = 0 \).

Assume for contradiction that such an \( \delta \) exists. By the Archemedian property, there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < J_f(x) \). Then \( x \in \{ y \in [a, b] | J_f(y) \geq \frac{1}{n} \} \), so \( x \) is in the union as well, as desired.

Moreover, by part (3), each of the sets in the union is closed, and we are taking the union of countably many sets because \( \mathbb{N} \) is countable. (Note that when we say “is a countable union of”, it means that there’s countably many sets, not that the sets themselves are countable.)

\[ (5) \text{ Let } \{ t_n \}_{n \in \mathbb{N}} \text{ be an enumeration of } \mathbb{Q}. \text{ Define } f(x) = \begin{cases} \frac{1}{n} & \text{if } x = t_n, \\ 0 & \text{else} \end{cases}, \text{ as in problem (A) of HW 4. We will prove that } f \text{ is continuous at every irrational number and discontinuous at every rational number.} \]

First, let \( x \in \mathbb{Q} \). Then \( x = t_n \) for some \( n \in \mathbb{N} \). Let \( \delta > 0 \) and consider \( (x - \delta, x + \delta) \). By the density of the irrationals, there exists some irrational \( y \in (x - \delta, x + \delta) \), and it has \( f(y) = 0 \). Then for every \( \delta > 0 \), there exists some \( y \) with \( |x - y| < \delta \) such that \( |f(x) - f(y)| = f(t_n) = \frac{1}{n} \), meaning that this difference cannot be made less than \( \frac{1}{n} \), i.e. \( f \) is not continuous at \( x \).

Now, let \( x \notin \mathbb{Q} \). Let \( \varepsilon > 0 \) and let \( N = \max\{ n \in \mathbb{N}| \frac{1}{n} \geq \varepsilon \} \). This means that there are only finitely many points, \( t_1, \ldots, t_N \) at which \( f \) attains values larger than \( \varepsilon \). Let \( \delta = \min\{|t_1 - x|, \ldots, |t_N - x|\} \), which is positive because \( x \neq t_k \) for any \( k \). Then the interval \( (x - \delta, x + \delta) \) does not contain any of \( t_1, \ldots, t_N \), so if \( y \in (x - \delta, x + \delta) \), \( |f(y) - f(x)| \leq \frac{1}{N+1} < \varepsilon \). This is because the largest possible value would be if \( t_N+1 \in (x - \delta, x + \delta) \) and we choose \( y = t_{N+1} \). This works for every \( \varepsilon > 0 \), so \( f \) is continuous at \( x \), as desired.

\[ (6) \text{ Bonus! Claim: There does not exist a function } f \text{ such that } f \text{ is continuous on } \mathbb{Q} \text{ and discontinuous on } \mathbb{Q}^c. \]

Proof of claim. Assume for contradiction that such an \( f \) exists. By part (4), this means that \( \mathbb{Q}^c \) is a countable union of closed sets. Moreover, they must all be nowhere dense (i.e. have empty interior); otherwise they would contain a point of \( \mathbb{Q} \) by density of \( \mathbb{Q} \). Now, let \( \{ q_n \}_{n \in \mathbb{N}} \) be an enumeration of \( \mathbb{Q} \) and note that \( \{ q_n \} \) is a closed nowhere dense set for every \( n \). Then we have that \( \mathbb{R} = \mathbb{Q}^c \cup \bigcup_{n \in \mathbb{N}} \{ q_n \} \) is a countable union of nowhere dense sets, which contradicts Baire’s theorem that \( \mathbb{R} \) cannot be written as a countable union of closed, nowhere dense sets. (For \( \mathbb{R} \) (and intervals in \( \mathbb{R} \), this was Ch.2 #30 on HW1. This is all that’s needed for this problem, though a more general form holds, see Ch.3 #22.)

\[ \square \]
(B)(1) Fix $\varepsilon > 0$ and let $N$ be such that $A_2 = \{a_1, \ldots, a_N\}$. For $1 \leq k \leq N$, let $I_k$ be an open interval containing $a_k$ with length less than $\varepsilon/N$ and such that every pair of these intervals has a positive distance between them (this is not really required, but it makes it nice: if the intervals overlap then there’s actually more than $N$ “bad” intervals in the partition, but it still works. Since we can make it nice and disjoint if we want to, we may as well.). Let $C = [a, b] - \bigcup_{k=1}^{N} I_k$, which is compact because we’ve removed an open set from a compact set.

By definition of $C$ (it’s disjoint from $A_2$), for every $x \in C$ there exists $r_x > 0$ such that $J_{f,x}(r_x) < \varepsilon$. The neighbourhoods $N_{r_x/2}(x) = (x - r_x/2, x + r_x/2)$ are an open cover of $C$, so by compactness there exist $x_1, \ldots, x_n$ and $r_1, \ldots, r_n$ such that $C \subseteq \bigcup_{k=1}^{n} N_{r_k/2}(x_k)$ and $J_{f,x_k}(r_k) < \varepsilon$. Let $r = \frac{1}{2} \min\{r_1, \ldots, r_n\}$.

Claim: For all $x \in C$, $J_{f,x}(r) < \varepsilon$.

Proof of claim. Let $x \in C$. Since the collection of $N_{r_k/2}(x_i)$’s cover $C$, there exists $1 \leq i \leq n$ such that $x \in N_{r_i/2}(x_i)$. Suppose that $y, z \in (x - r, x + r)$. Then $|y - x_i| \leq |y - x| + |x - x_i| < r + \frac{r}{2} \leq r_i$ by definition of $r_i$ and similarly for $z$, so in fact $y, z \in N_{r_i}(x_i).$ This proves that $(x - r, x + r) \subseteq (x_i - r_i, x_i + r_i).$ Then we have that $J_{f,x_i}(r) \leq J_{f,x_i}(r_i) < \varepsilon$ from the definition of $J_{f,x}$. \square

Now, let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ containing the endpoints of every $I_k$, and such that every interval not contained in some $I_k$ has length less than $r$. Let $M = \sup_{x \in [a, b]} |f(x)|$. Define $A = \{i \in \{0, \ldots, n-1\} | x_i$ is not the left endpoint of any of the $I_k\}$. Then we have that

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$\leq \sum_{k=1}^{N} 2M \varepsilon N + \sum_{i \in A} (M_i - m_i) \Delta x_i$$

$$\leq 2M \varepsilon + \varepsilon(b - a),$$

where we’ve used that if $[x_i, x_{i+1}]$ is not one of the $I_k$’s, then the fact that $\Delta x_i < r$ by choice implies that $M_i - m_i < \varepsilon$ by the claim. We then further simplify that $\sum \Delta x_i \leq b - a$. Since $2M \varepsilon + \varepsilon(b - a) \to 0$ as $\varepsilon \to 0$, this proves that $f$ is integrable on $[a, b]$.

(2) We will verify that each of the $A_i$ from the previous part are finite, which will imply $f$ is integrable by $B(1)$. Assume for contradiction that there exists some $A_i$ that is infinite. Since it’s a bounded infinite set, by Bolzano-Weierstrass, it has an accumulation point, say $z \in [a, b]$. By assumption, $\lim_{y \to z} f(y) = L$ exists, so there exists $\delta > 0$ such that $0 < |y - z| < \delta \implies |f(y) - L| < \varepsilon/4$. Then, by the triangle inequality, for all $x, y \in (z - \delta, z + \delta) - \{z\}$, $|f(x) - f(y)| < \varepsilon/2$.

Since $z$ is a limit point of $A_i$, there exists $y \in A_i \cap (z - \delta, z + \delta)$ with $y \neq z$. Since $J_f(y) \geq \varepsilon$, there exists $r > 0$ such that $J_{f,y}(r) > \varepsilon/2$. By shrinking $r$ if necessary, we may assume further that $(y - r, y + r) \subseteq (z - \delta, z + \delta)$ and $z \notin (y - r, y + r)$. Since $J_{f,y}(r) > \varepsilon/2$, there exist
(C)#12. Let $M = \sup_{x \in [a,b]} |f(x)|$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{2M}$. Define $g(t) = \frac{t-x_i}{x_{i+1}-x_i} f(x_{i+1}) + \frac{x_{i+1}-t}{x_{i+1}-x_i} f(x_i)$, where $t \in [x_{i-1}, x_i]$. Note that $g$ is a piecewise linear function on each $[x_{i-1}, x_i]$ in $P$. Let $M_m = \sup_{x \in [x_{i-1}, x_i]} |f(x)|$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. We will show that $U(P, (f-g)^2, \alpha) < \varepsilon^2$, which implies that $\left( \int_a^b (f-g)^2 \, d\alpha \right)^{1/2} \leq \|f-g\|_2 < \varepsilon$, as desired. Notice that the maximum value of $(f-g)^2$ on $[x_{i-1}, x_i]$ will be at most $(M_m - m_i)^2$, where $M_m = \sup_{x \in [x_{i-1}, x_i]} |f(x)|$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. This is because the maximum value of $g$ is at most $M_m$ and the minimum value is at least $m_i$. Then we have that

\[
U(P, (f-g)^2, d\alpha) \leq \sum_{i=0}^{n-1} (M_m - m_i)^2 \Delta \alpha(x_i)
\leq 2M \sum_{i=0}^{n-1} (M_m - m_i) \Delta \alpha(x_i)
= 2MU(P, f, \alpha)
< \varepsilon^2,
\]
as desired.

#15. Use integration by parts with $F(x) = x, G(x) = \frac{1}{2} f(x)^2$. Then $\int_a^b F(x)G'(x) \, dx = F(b)G(b) - F(a)G(a) - \int_a^b \frac{1}{2} f(x)^2 \, dx$, i.e.

\[
\int_a^b x f(x) f'(x) \, dx = -\frac{1}{2} \int_a^b f(x)^2 \, dx = -\frac{1}{2},
\]
where we’re using that $G(b) = G(a) = 0$ by assumption.

To see the next inequality, apply the Hölder inequality (from problem 10) with $p = q = 2$ (i.e. the Schwarz inequality) to the functions $xf(x)$ and $f'(x)$ to obtain

\[
\left| \int_a^b x f(x) f'(x) \, dx \right| \leq \left( \int_a^b x^2 f(x)^2 \, dx \right)^{1/2} \left( \int_a^b (f'(x))^2 \, dx \right)^{1/2}.
\]

Squaring this and using the previous part implies that $\frac{1}{4} \leq \int_a^b x^2 f(x)^2 \, dx \int_a^b (f'(x))^2 \, dx$, as desired. Going this far is sufficient for this problem.
It is actually true that equality occurs in the Schwarz inequality if and only if the two functions are proportional (this is on the next HW), so equality would occur if and only if there exists some $c \in \mathbb{R} - \{0\}$ such that $f'(x) = cx f(x)$ for all $x$. Letting $y = f(x)$, this means solving $y' = cxy$, which can be done using separation of variables to obtain that $y = e^{c \frac{x^2}{2}} + D$ for some constant $D$. However, it is not possible to choose $D$ to satisfy both $f(a) = f(b) = 0$, unless $c = 0$, in which case $D = -1$ and $f$ is the constant zero function, which is impossible because $\int_a^b f^2 dx = 1$. 