Math 140B, Winter 2018  
Homework 7 Solutions

(A) We need to prove that \( \int_a^b f(x)dx = \int_{-b}^{-a} f(-x)dx \). Note that the change of variables theorem from Rudin cannot be applied, because \( \varphi(x) = -x \) is not an increasing function. Instead, we will mimic the proof.

Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\). Define \( Q = \{y_0, \ldots, y_n\} \) where \( y_n = -x_{n-k} \) (so \( Q = -P \)), which is a partition of \([-b, -a]\). Observe that doing this actually gives a bijection between partitions of \([a, b]\) and of \([-b, -a]\). Define \( g(x) = f(-x) \), \( M_i = \sup_{x \in [x_i-1, x_i]} f(x) \) and \( N_i = \sup_{y \in [y_{i-1}, y_i]} g(y) \).

**Claim:** \( N_i = M_{n-i+1} \) and \( \Delta y_i = y_i - y_{i-1} = \Delta x_{n-i+1} = x_{n-i+1} - x_{n-i} \).

**Proof of claim.**

\[
N_i = \sup\{g(y) | y \in [y_{i-1}, y_i]\} \\
= \sup\{g(y) - y \in [x_{n-i}, x_{n-i+1}]\} \quad \text{by definition of the } y_k's \\
= \sup\{f(-y) - y \in [x_{n-i}, x_{n-i+1}]\} \quad \text{by definition of } g \\
= \sup\{f(x)|x \in [x_{n-i}, x_{n-i+1}]\} \\
= M_{n-i}
\]

**Remark:** The above is the same as the book’s observation that \( f \) takes the same values on \([x_{n-i}, x_{n-i+1}]\) as \( g \) does on \([y_{i-1}, y_i]\).

\( \Delta y_i = \Delta x_{n-i+1} \) follows from the definition of the \( y_k \)'s. \(\square\)

By the claim, we then have that \( U(Q, g) = \sum_{i=1}^n N_i \Delta y_i = \sum_{i=1}^n M_{n-i+1} \Delta x_{n-i+1} = \sum_{i=1}^n M_i \Delta x_i = U(P, f) \) (we are just adding from the \( n \)-th term to the first instead). By a similar argument, \( L(Q, g) = L(P, f) \). Since this works for all partitions \( P \) of \([a, b]\), we have that

\[
L(P, f) = L(Q, g) \leq \int_{-b}^{-a} g(x)dx \leq \int_{-b}^{-a} f(x)dx \leq U(Q, g) = U(P, f)
\]

for every partition \( P \) of \([a, b]\) (and hence also for every \( Q \) of \([-b, -a]\)). Taking the \( \sup \) over all \( P \) implies that \( \int_a^b f(x)dx \leq \int_{-a}^{-b} g(x)dx \), and taking the \( \inf \) over all \( P \) implies that \( \int_{-b}^{-a} g(x)dx \leq \int_{-b}^{-a} f(x)dx \). Putting these together implies that

\[
\int_a^b f(x)dx = \int_{-b}^{-a} g(x)dx = \int_{-b}^{-a} g(x)dx,
\]

so \( g \) is integrable on \([-b, -a]\) with integral equal to the integral of \( f \) over \([a, b]\). By the definition of \( g \), this is the desired conclusion.
(B) Let \( x \in [a, b] \) and \( n \geq 1 \). By Taylor’s theorem, there exists some \( y \in [a, b] \) such that
\[
f(x) = p_n(x) + f^{(n+1)}(y) (x-a)^{n+1}/(n+1)!
\]
Then we have that
\[
|f(x) - p_n(x)| = \left| \frac{f^{(n+1)}(y)}{(n+1)!} (x-a)^{n+1} \right| \leq \frac{2^{n+1}(x-a)^{n+1}}{(n+1)!} \leq \frac{2^{n+1}(b-a)^{n+1}}{(n+1)!}.
\]
Let \( \varepsilon > 0 \). Since \( \frac{2^{n+1}(b-a)^{n+1}}{(n+1)!} \to 0 \) as \( n \to \infty \), there exists \( N \) such that \( n \geq N \implies \left| \frac{2^{n+1}(b-a)^{n+1}}{(n+1)!} \right| < \varepsilon \). Then, for any \( n \geq N \) and every \( x \in [a, b] \), \( |f(x) - p_n(x)| < \varepsilon \), so \( p_n(x) \to f(x) \) uniformly, as desired.

(C) Ch.7#1. Let \( (f_n) \) be a sequence of bounded functions which converge uniformly to \( f \) on some domain \( E \) (the domain is not important in this problem). Let \( M_n = \sup |f_n| \) for each \( n \).

We’ll first show that \( f \) is also bounded. Since \( f_n \to f \) uniformly, there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) and for all \( x \in E \), \( |f(x) - f_n(x)| < 1 \). Then, in particular, \( |f(x) - f_N(x)| < 1 \), so \( |f(x)| < 1 + |f_N(x)| \leq 1 + M_N \) for all \( x \), that is, \( 1 + M_N \) is a bound for \( |f| \).

Now, let \( n \geq N \) and \( x \in E \). Then we have \( |f_n(x) - f(x)| < 1 \), so \( |f_n(x)| < 2 + M_N \). This shows that \( 2 + M_n \) is a bound for \( |f_n| \) for every \( n \geq N \). Define \( M = \max \{ M_1, \ldots, M_{N-1}, 2 + M_N \} \), which clearly is a uniform bound for \( (f_n) \), i.e. \( |f_n(x)| \leq M \) for all \( n \in \mathbb{N} \) and all \( x \in E \).

#2. Let \( f \) and \( g \) be the functions to which \( f_n \) and \( g_n \) are converging uniformly, respectively. Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) and all \( x \in E \), \( |f_n(x) - f(x)| < \varepsilon/2 \) and \( |g_n(x) - g(x)| < \varepsilon/2 \). Then we have that
\[
|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon,
\]
as desired.

Now, suppose that \( (f_n) \) and \( (g_n) \) are sequences of bounded functions. By the (proof of the) previous problem, this means there exists \( M \) such that for all \( n \in \mathbb{N} \) and all \( x \in E \), \( |f_n(x)| \leq M \), \( |g_n(x)| \leq M \). Let \( \varepsilon > 0 \) and choose \( N \) such that both \( |f_n(x) - f(x)| < \varepsilon/2M \) and \( |g_n(x) - g(x)| < \varepsilon/2M \). Then for every \( x \in E \), we have
\[
|f(x)g_n(x) - f_n(x)g_n(x)| \leq |f(x)(g_n(x) - f_n(x))g_n(x)| + |f_n(x)(g_n(x) - g(x))| + |f_n(x)||g_n(x) - f_n(x)|
\]
\[
= |f(x)||g_n(x) - f_n(x)||g_n(x)| + |f_n(x)||g_n(x) - g(x)| + |f_n(x)||f_n(x) - f_n(x)|
\]
\[
\leq M|g(x) - g_n(x)| + M|f(x) - f_n(x)| < \varepsilon,
\]
so \( f_n g_n \to fg \) uniformly.
#3. Define $f_n(x) = g_n(x) = x - \frac{1}{n}$ from $\mathbb{R} \to \mathbb{R}$. By the Archimedean principle, $f_n \to x$ uniformly. However, $(f_n(x))^2 = x^2 - \frac{2}{n} + \frac{1}{n^2}$ does not converge uniformly to $x^2$ (but does converge pointwise). To see this, observe that for any $n \in \mathbb{N}$, $|(f_n(n))^2 - n^2| \geq \frac{2n}{n^2} = 2$, so the convergence is not uniform (we have verified the negation of the definition of uniform convergence, using $\varepsilon = 2$.)

Remark: I like the above example because it shows that the statement is not even true for squaring a uniformly convergent sequence. An easier example is below.

Alternatively, define $f_n : \mathbb{R} \to \mathbb{R}$, $g_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = x$ for all $n$ and $g_n(x) = \frac{1}{n}$ for all $x$ and $n$. Then $f_n \to x$ uniformly, $g_n \to 0$ uniformly, and $f_n g_n \to 0$ pointwise, but $|f_n(n) - 0| = 1$ for every $n$, so the convergence is not uniform.