Math 140B, Winter 2018
Homework 8 Solutions

(A)(1) Observe that for every $x \in [0,1/2]$, $|(Tf)(x) - (TG)(x)| \leq \int_0^x |f(t) - g(t)|dt \leq \int_0^x \|f - g\|_{\infty} dt = x\|f - g\|_{\infty} \leq \frac{1}{2} \|f - g\|_{\infty}$. Since this works for every $x$, it follows that $\|Tf - TG\|_{\infty} \leq \frac{1}{2} \|f - g\|_{\infty}$.

Now, let $m > n$. Then we have that

$$\|T^{n+1}f - T^nf\|_{\infty} \leq \|T^nf - T^{n-1}f\|_{\infty} + \cdots + \|T^{n+1}f - T^n f\|_{\infty}$$

$$\leq \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n}\right) \|Tf - f\|_{\infty}$$

$$\leq \frac{1}{2^n} \left(\sum_{k=1}^{\infty} \frac{1}{2^k}\right) \|Tf - f\|_{\infty}.$$

Let $\varepsilon > 0$ and choose $N$ sufficiently large so that $\frac{1}{2^n} < \left(\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}\right) \|Tf - f\|_{\infty}$ Then, by above, if $m > n > N$, $\|T^mf - T^n f\|_{\infty} < \varepsilon$. (Remark: Alternatively, notice that since the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, $\frac{1}{2^1} + \cdots + \frac{1}{2^n} < \varepsilon$ for $n,m$ sufficiently large by the Cauchy criterion for convergence of a series.)

(3) We will show that $T^n f$ converges uniformly to 0. By problem C on HW 6, we know that $(T^{n+1}f)(x) = \frac{1}{n+1} \int_0^x (x-t)^n f(t)dt$. Since $x, t \in [0,1/2]$, $|x-t|^n \leq 2^{-n}$. Since $f$ is continuous on a compact set, it is bounded (by $\|f\|_{\infty}$ by definition of $\|\cdot\|_{\infty}$). Then we see that

$$|(T^{n+1}f)(x)| \leq \frac{1}{n+1} \int_0^x |x-t|^n |f(t)| dt \leq \frac{\|f\|_{\infty}}{2^n n!} \int_0^x dt \leq \frac{\|f\|_{\infty}}{2^n n!}.$$

So, given $\varepsilon > 0$, choose $N$ such that $\frac{\|f\|_{\infty}}{2^n n!} < \varepsilon$. Then for all $n \geq N$ and all $x \in [0,1/2]$, $|(T^{n+1}f)(x)| < \varepsilon$, as desired.

(4) (Bonus) The only difference with the computation in (3) is that $|x-t|^n \leq 2^n$ here. Thus, by the same method as in (3), we get $|(T^{n+1}f)(x)| \leq 2^n \frac{\|f\|_{\infty}}{n!}$. Since this quantity also converges to zero as $n \to \infty$, we get the uniform convergence to zero again.

(B)(1) Observe that $S^1$ is a compact subset of $C$. Thus, $f$ is a continuous function on a compact
set, and so it is bounded and also uniformly continuous. To prove the uniform boundedness of \( F_n \), we need to show that there exists \( M \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \) and all \( z \in S^1 \), \( |F_n(z)| \leq M \).

Let \( n \in \mathbb{N} \) and \( z \in S^1 \). Then we have that
\[
|F_n(z)| \leq \frac{1}{n} \sum_{k=1}^{n} |f(R_n^k(z))| \leq \frac{1}{n} \sum_{k=1}^{n} \|f\|_{\infty} = \|f\|_{\infty},
\]
where we're using that \( |f(R_n^k(z))| \leq \|f\|_{\infty} \) holds because \( R_n^k(z) \in S^1 \) and \( \|f\|_{\infty} \) is the max of \( f \) on \( S^1 \) by definition. Thus, \( (F_n) \) is uniformly bounded by \( \|f\|_{\infty} \).

Now, we will show equicontinuity. Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that for all \( w, z \in S^1 \), \( |w - z| < \delta \Rightarrow |f(w) - f(z)| < \varepsilon \). Observe also that \( |R_n(w) - R_n(z)| = \left| e^{2\pi i \alpha} (w - z) \right| = |w - z| \) because \( |e^{2\pi i \alpha}| = 1 \) (it is an element of \( S^1 \) by definition of \( R_n \)). Induction then implies that \( |R_n^k(w) - R_n^k(z)| = |w - z| \) for all \( k \in \mathbb{N} \). Now, suppose that \( |w - z| < \delta \) and let \( n \in \mathbb{N} \). Then we have that
\[
|F_n(w) - F_n(z)| \leq \frac{1}{n} \sum_{k=1}^{n} \left| f(R_n^k(w)) - f(R_n^k(z)) \right| \leq \frac{1}{n} \sum_{k=1}^{n} \varepsilon = \varepsilon.
\]

(2) (Bonus) Let \( F_n \) be a subsequence of \( (F_n) \) that converges uniformly to \( F \). We will show that \( F(R_n(z)) = F(z) \) for all \( z \in S^1 \) by showing that \( |F(R_n(z)) - F(z)| < \varepsilon \) for every \( \varepsilon > 0 \). Observe that for any \( n \), we have
\[
|F(R_n(z)) - F(z)| \leq |F(R_n(z)) - F_n(z)| + |F_n(z) - F(z)|.
\]
Let \( \varepsilon > 0 \). By the uniform convergence of \( F_n \to F \), for \( n \) sufficiently large, the first and third terms are less than \( \varepsilon/3 \). We need to bound the middle term. To do this, observe that
\[
F_n(R_n(z)) = \frac{1}{n} \sum_{k=1}^{n} f(R_n^k(z)) = \frac{1}{n} \sum_{i=2}^{n+1} f(R_n^{i-1}(z)).
\]
Thus, \( F_n(R_n(z)) - F_n(z) = \frac{1}{n} \sum_{i=2}^{n+1} (f(R_n^{i-1}(z)) - f(R_n(z))) \).

Then we have that \( |F_n(R_n(z)) - F_n(z)| \leq \frac{2\|f\|_{\infty}}{n} \). So, choose \( n \) large enough to also make \( \frac{2\|f\|_{\infty}}{n} < \varepsilon/3 \) to get the desired result.

(C) We will follow the hint. Since \( f \) is continuous on \([-1, 1]\) it is uniformly continuous there. Let \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that for all \( x, y \in [-1, 1] \), \( |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \). Observe that since \( f \) is continuous and \( f(1) = f(-1) = 0 \), this implies that if \( |x - 1| < \delta \) or \( |x + 1| < \delta \), then \( |f(x)| < \varepsilon \). So, if we consider any \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \), we have three possibilities: \( x, y \in [-1, 1] \), \( x \in [-1, 1] \) and \( y \not\in [-1, 1] \), or \( x, y \not\in [-1, 1] \). In the first case, we already have \( |f(x) - f(y)| < \varepsilon \). In the third case, \( f(x) = f(y) = 0 \), so \( |f(x) - f(y)| = 0 < \varepsilon \). Consider the second case: \( x \in [-1, 1] \) and \( y \not\in [-1, 1] \). Then \( |x - y| < \delta \) requires that either \( |x - 1| < \delta \) or \( |x + 1| < \delta \) to see this: suppose \( y > 1 \). Then we must have \( x \leq 1 < y \), so \( 1 - x < y - x = \delta \). Similarly if \( y < 1 \). Then \( |f(x)| = |f(x) - f(y)| < \varepsilon \) by the above discussion too. Thus, we have proven \( f \) is uniformly continuous on all of \( \mathbb{R} \).

Now, let \( \varepsilon > 0 \) and let \( \delta > 0 \) be such that for all \( x, y \in \mathbb{R} \), \( |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \).

By assumption (c), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \int_{|t|>\delta} f_n(t)dt < \varepsilon \). Suppose
that \(n \geq N\). By definition, 
\[
|f \ast \varphi_n(x) - f(x)| = \left| \int_{-1}^{1} f(x-t) \varphi_n(t) dt - f(x) \right|
\]
(using that \(\varphi_n = 0\) outside \([-1,1]\) to restrict the bounds of the integral). By assumption (b), \(\int_{-1}^{1} \varphi_n(t) dt = 1\), so \(f(x) = \int_{-1}^{1} f(x) \varphi_n(t) dt\). Then we have that
\[
|f \ast \varphi_n(x) - f(x)| = \left| \int_{-1}^{1} f(x-t) \varphi_n(t) dt - \int_{-1}^{1} f(x) \varphi_n(t) dt \right|
\]
\[
\leq \int_{-1}^{1} |f(x-t) - f(x)| \varphi_n(t) dt
\]
\[
= \int_{|t| \leq \delta} |f(x-t) - f(x)| \varphi_n(t) dt + \int_{|t| > \delta} |f(x-t) - f(x)| \varphi_n(t) dt
\]
\[
\leq \varepsilon \int_{|t| \leq \delta} \varphi_n(t) dt + 2\|f\|_{\infty} \int_{|t| > \delta} \varphi_n(t) dt
\]
\[
\leq \varepsilon (1 + 2\|f\|_{\infty})
\]
where the last line is using that, because \(\varphi_n(t) \geq 0\) for all \(n\) and \(t\), \(\int_{|t| \leq \delta} \varphi_n(t) dt \leq \int_{-1}^{1} \varphi_n(t) dt = 1\) by (b). Since this bound goes to zero and doesn’t depend on \(x\), the convergence is uniform.

**Remark:** \(\int_{|t| > \delta} \varphi_n(t) dt\) means \(\int_{-\delta}^{\delta} \varphi_n(t) dt + \int_{\delta}^{1} \varphi_n(t) dt\).

(C)\#7. Observe that \(f_n\) converges pointwise to 0. We will show that the convergence is uniform.

Fix \(n \in \mathbb{N}\) and compute that \(f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2}\) is zero only at \(x = \pm \frac{1}{\sqrt{n}}\). Thus, the only places a local max or min can occur is at these points. Moreover, \(\lim_{x \to \infty} f_n(x) = 0\) and similarly \(\lim_{x \to -\infty} f_n(x) = 0\). Since \(f'_n < 0\) for \(|x| > \frac{1}{\sqrt{n}}\), we see that \(|f_n(x)| \) decreasing (but bounded) the further \(x\) is from \(\left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right]\) - so the maximum of \(|f_n|\) must occur in \(\left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right]\). Computing
\[
|f_n(\pm 1/\sqrt{n})| = \frac{1}{2n^{3/2}}
\]
shows that this is the maximum of \(|f_n|\) on \(\mathbb{R}\). Thus, given \(\varepsilon > 0\), choose \(N\) sufficiently large so that \(\frac{1}{2N^{3/2}} < \varepsilon\). Then for all \(x \in \mathbb{R}\) and all \(n \geq N\), \(|f_n(x) - 0| < \varepsilon\), as desired.

Notice that \(f'_n(x) \to 0\) if \(x \neq 0\) and \(f'_n(0) \to 1\). Since the limit \(f \equiv 0\), \(f'(x) = 0\) for all \(x\). Thus, the equality stated is satisfied for all \(x \neq 0\) but not at \(x = 0\).

**Remark:** This problem shows that even if \(f_n \to f\) uniformly and \(f_n\) and \(f\) are differentiable, the derivatives do not converge uniformly (or even pointwise).

#11. To show uniform convergence, we will show \(\sum f_n g_n\) is uniformly Cauchy (i.e. the Cauchy criterion for uniform convergence is satisfied). As per the hint, we will try to adapt the proof of

Theorem 3.42. Let \(S_N(x) = \sum_{n=1}^{N} f_n(x)\). By assumption (a), there exists some \(M \in \mathbb{R}\) such that \(|S_N(x)| \leq M\) for all \(N\) and all \(x\).
Let \( m, n \in \mathbb{N} \) and suppose without loss of generality that \( m > n \). Using Theorem 3.41 (which basically comes down to the observation that \( S_N(x) - S_{N-1}(x) = f_N(x) \)), we have that

\[
\sum_{k=n}^{m+1} f_k(x)g_k(x) \leq \sum_{k=n}^{m} |S_k(x)(g_k(x) - g_{k+1}(x))| + |S_{m+1}(x)g_{m+1}(x) - S_{n-1}g_n(x)|
\]

\[
\leq M \sum_{k=n}^{m} (g_k(x) - g_{k+1}(x)) + M (g_n(x) - g_{m+1}(x))
\]

\[
= 2M (g_n(x) - g_{m+1}(x)) \tag{1}
\]

where (1) follows because \( g_k(x) \geq g_{k+1}(x) \) for all \( x \) and all \( k \), so we can drop the absolute values: \( g_n(x) - g_{m+1}(x) \geq 0 \) and \( g_k(x) - g_{k+1}(x) \geq 0 \). (2) is just noticing that the sum is telescoping and equals \( g_n(x) - g_{m+1}(x) \). Let \( \varepsilon > 0 \). Since \( g_n \to 0 \) uniformly, there exists \( N \in \mathbb{N} \) such that for all \( n, m \geq N \) and all \( x \), \( |g_n(x)| \leq \varepsilon/4M \). Then if \( m > n \geq N \), we have that

\[
\left| \sum_{k=n}^{m+1} f_k(x)g_k(x) \right| < \varepsilon,
\]

as desired.

**Remark:** We could also make \( g_n(x) - g_{m+1}(x) \) small using the Cauchy criterion for uniform convergence instead of just making each term small.

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\#15. We will show that \( f \) is constant. We’ll prove the contrapositive: if \( f \) is not constant, then \((f_n)\) is not equicontinuous. Assume that \( f \) is not constant. Then there exist \( x, y \in \mathbb{R} \) such that \( f(x) \neq f(y) \). Then \( |f(x) - f(y)| > 0 \). To show \((f_n)\) is not equicontinuous, we need to show there exists some \( \varepsilon > 0 \) such that for all \( \delta > 0 \), there exists \( n \in \mathbb{N} \) and \( a, b \in \mathbb{R} \) such that \( |a - b| < \delta \) but \( |f_n(a) - f_n(b)| \geq \varepsilon \).

Let \( \varepsilon = |f(x) - f(y)| \). Let \( \delta > 0 \) and choose \( n \) sufficiently large so that \( \frac{|x - y|}{n} < \delta \). Then using \( a = x/n \) and \( b = y/n \), we have that \( |a - b| < \delta \). But, by definition of \( f_n \), \( |f_n(x/n) - f_n(y/n)| = |f(x) - f(y)| = \varepsilon \). Thus, \((f_n)\) is not equicontinuous.