Homework due Friday, March 16, at 3:00 pm.

A. Let $\{f_n\} \subset C([a,b])$ be a sequence. Prove that there exists some (nonempty) interval $(c,d) \subset [a,b]$ so that $\{f_n\}$ is uniformly bounded on $(c,d)$.

(Hint: Rudin, chapter 2, problem 30 is useful.)

B.

1. Let $\{f_n\} \subset C([a,b])$ be a sequence of equicontinuous functions. Assume $\{f_n(a)\}$ is bounded. Prove or disprove: $\{f_n\}$ is uniformly bounded.

2. Let $K$ be a compact metric space. Let $\{f_n\} \subset C(K)$ be a sequence of equicontinuous functions. Assume $\{f_n(x)\}$ is bounded for some $x \in K$. Prove or disprove: $\{f_n\}$ is uniformly bounded.

C. Rudin, Chapter 7 (page 165), problems 16, 18, 20, 22.

(Bonus Problem)

1. Let $\{f_n\} \subset C([a,b])$ and assume $f_n \to f$ pointwise on $[a,b]$. Prove that $f$ is continuous on a dense subset of $[a,b]$.

2. Let $f : [a,b] \to \mathbb{R}$ be a function that is differentiable everywhere on $[a,b]$. Prove that $f'$ is continuous on a dense subset of $[a,b]$.

(Hint: First note that it suffice to prove that $f$ is continuous at some point $x \in (a,b)$. Assume for a contradiction that $f$ is discontinuous everywhere on $(a,b)$. For every $n \in \mathbb{N}$, consider $A_n = \{x \in (a,b) : J_f(x) \geq 1/n\}$. Then for some $n_0$ there exist some nonempty open interval $I$ so that $I \subset A_{n_0}$. For each $m \in \mathbb{N}$ define $A_m = \{x \in I : |f_n(x) - f(x)| < \frac{1}{m n_0} \text{ for all } n > m\}$. Then there exists some $m_0$ so that $A_{m_0}$ has nonempty interior. Draw a contradiction from this.)

(Bonus Problem) Let $\alpha \in [0,1]$ be an irrational number. Define $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ by $R_\alpha(z) = e^{2\pi i \alpha} z$.

Let $f : \mathbb{S}^1 \to \mathbb{C}$ be a continuous function. Let $z_0 \in \mathbb{S}^1$, prove that

$$\frac{1}{n} \sum_{j=0}^{n-1} f(R_\alpha^j(z_0)) \to \int_0^1 f(e^{2\pi i \theta}) \, d\theta.$$

(Hint: First prove this for $f(z) = z^n$, for $n \in \mathbb{Z}$. Then use Stone-Weierstrass theorem.)

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*(Bonus Problem) Let $c \in (0, 1)$. Prove that the set of functions $f : [0, 1] \to \mathbb{R}$ which are continuous on $[0, 1]$ and differentiable at $c$ is a countable union of nowhere dense subsets of $C([0, 1])$.

(Hint: Let $n, N \in \mathbb{N}$, define $A_N(n) = \{f \in C(0, 1) : \frac{f(c+\frac{1}{n})-f(c)}{\frac{1}{n}} \leq N\}$. Fix $N$ and define $B_N = \bigcap_n A_N(n)$. Then, consider $\bigcup_N B_N$.)