Math 140A Homework 9 (Solutions)

Problem 1 (Rudin, Chapter 3 #7). Prove that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

Solution. Observe that
\[
a_n - 2\sqrt{\frac{a_n}{n}} + \frac{1}{n^2} = \left(\sqrt{\frac{a_n}{n}} - \frac{1}{n}\right)^2 \geq 0,
\]
so it follows that $\frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \left( a_n + \frac{1}{n^2} \right)$.

Since the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ both converge, the comparison test (Theorem 3.25(a) in Rudin) implies that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ also converges.

Problem 2 (Rudin, Chapter 3 #8). If $\sum_{n=1}^{\infty} a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution. Let $\varepsilon > 0$ be given. Let $A = \sum_{n=1}^{\infty} a_n$, and, for each $n \in \mathbb{N}$, let $A_n = \sum_{k=1}^{n} a_k$, so that $A_n \to A$ as $n \to \infty$. Since the sequence $\{A_n\}$ is convergent, it is also bounded (Theorem 3.2(c) in Rudin), so there exists a $C > 0$ such that $|A_n| \leq C$ for all $n \in \mathbb{N}$. Since $\{b_n\}$ is monotone and bounded, there exists $b \in \mathbb{R}$ such that $b_n \to b$ as $n \to \infty$ (Theorem 3.14 in Rudin). Consequently, $\{b_n\}$ is Cauchy (Theorem 3.11 in Rudin), so there exists an $N_1 \in \mathbb{N}$ such that $|b_m - b_n| < \varepsilon/(3C)$ whenever $m, n \geq N_1$. Since $A_n \to A$ and $b_n \to b$ as $n \to \infty$, Theorem 3.3(c) in Rudin implies that $A_n b_n \to Ab$ and $A_{n-1} b_n \to Ab$ as $n \to \infty$. Thus, there exist $N_2, N_3 \in \mathbb{N}$ such that $|A_n b_n - Ab| < \varepsilon/3$ and $|A_{n-1} b_m - Ab| < \varepsilon/3$ whenever $n \geq N_2$ and $m \geq N_3$. Moreover, the following summation by parts formula holds (cf. Theorem 3.41 in Rudin):
\[
\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}),
\]
and since $\{b_n\}$ is monotone, each difference $b_k - b_{k+1}$ has the same sign, so
\[
\sum_{k=m}^{n-1} |b_k - b_{k+1}| = \left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) \right| = |b_m - b_n|
\]
whenever \( m \leq n \). Now, let \( N = \max\{N_1, N_2, N_3\} \). If \( n \geq m \geq N \), then applying the triangle inequality to (2.1) and using (2.2) yields
\[
\left| \sum_{k=m}^{n} a_k b_k \right| \leq |A_n b_n - A_{m-1} b_m| + \left| \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}) \right|
\]
\[
\leq |A_n b_n - Ab| + |Ab - A_{m-1} b_m| + \sum_{k=m}^{n-1} |A_k| |b_k - b_{k+1}|
\]
\[
\leq |A_n b_n - Ab| + |A_{m-1} b_m - Ab| + C \sum_{k=m}^{n-1} |b_k - b_{k+1}|
\]
\[
\leq |A_n b_n - Ab| + |A_{m-1} b_m - Ab| + C |b_m - b_n|
\]
\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + C \frac{\varepsilon}{3C} = \varepsilon.
\]

Now Theorem 3.22 in Rudin implies that \( \sum_{n=1}^{\infty} a_n b_n \) converges.

**Problem 3** (Rudin, Chapter 3 #11). Suppose \( a_n > 0 \), \( s_n = a_1 + \cdots + a_n \), and \( \sum_{n=1}^{\infty} a_n \) diverges.

(a) Prove that \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) diverges.

(b) Prove that
\[
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}
\]
and deduce that \( \sum_{n=1}^{\infty} \frac{a_n}{s_n} \) diverges.

(c) Prove that
\[
\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}
\]
and deduce that \( \sum_{n=1}^{\infty} \frac{a_n}{s_n^2} \) converges.

(d) What can be said about
\[
\sum_{n=1}^{\infty} \frac{a_n}{1+na_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}.
\]

**Solution.** (a) There are two cases to consider:

(1) Suppose \( \{a_n\} \) is a bounded sequence, so that there exists an \( M > 0 \) such that \( a_n \leq M \) for all \( n \in \mathbb{N} \). Then we have
\[
\frac{a_n}{1+a_n} \geq \frac{a_n}{1+M}
\]

for all \( n \in \mathbb{N} \), and the series \( \sum_{n=1}^{\infty} \frac{a_n}{1+M} \) diverges since it is a nonzero multiple of the divergent series \( \sum_{n=1}^{\infty} a_n \). By the comparison test (Theorem 3.25(b) in Rudin), it follows that \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) also diverges.

(2) Suppose \( \{a_n\} \) is unbounded, so that for every \( M > 0 \) there exists an \( n \in \mathbb{N} \) such that \( a_n > M \). Also, note that

\[
1 - \frac{a_n}{1+a_n} = \frac{1}{1+a_n} \leq \frac{1}{a_n}.
\] (3.1)

Pick \( n_1 \in \mathbb{N} \) such that \( a_{n_1} > 1 \) and, having chosen \( n_1, \ldots, n_k \), choose \( n_{k+1} \in \mathbb{N} \) such that \( a_{n_{k+1}} > \max\{k+1, a_{n_k}\} \). Then \( \{a_{n_k}\} \) is a non-decreasing subsequence of \( \{a_n\} \) satisfying \( a_{n_k} \geq k \) for all \( k \in \mathbb{N} \). Let \( \varepsilon > 0 \), and (using the Archimedean property) choose \( K \in \mathbb{N} \) such that \( 1/K < \varepsilon \). Then by (3.1), for \( k \geq K \) we have

\[
\left| \frac{a_{n_k}}{1+a_{n_k}} - 1 \right| = 1 - \frac{a_{n_k}}{1+a_{n_k}} \leq \frac{1}{a_{n_k}} \leq \frac{1}{a_{n_K}} \leq \frac{1}{K} < \varepsilon.
\]

It follows that

\[
\lim_{k \to \infty} \frac{a_{n_k}}{1+a_{n_k}} = 1,
\]

so in particular, the sequence \( \left\{ \frac{a_n}{1+a_n} \right\} \) does not converge to 0. By Theorem 3.23 in Rudin, we conclude that \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) diverges.

(b) Let \( N \in \mathbb{N} \) be given. Since the sequence \( \{s_n\} \) is non-decreasing, we have

\[
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}. \tag{3.2}
\]

Since the series \( \sum_{n=1}^{\infty} a_n \) diverges, the sequence \( \{s_n\} \) is unbounded (Theorem 3.24 in Rudin). Thus, there exists a \( k \in \mathbb{N} \) such that \( s_{N+k} \geq 2s_N \). From (3.2) it follows that

\[
\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq 1 - \frac{s_N}{s_{N+k}} \geq 1 - \frac{s_N}{2s_N} = \frac{1}{2}.
\]

Thus, the series \( \sum_{n=1}^{\infty} \frac{a_n}{s_n} \) does not satisfy the Cauchy criterion, so it diverges (Theorem 3.22 in Rudin).

(c) For \( n > 1 \), we have

\[
\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n^2} \leq \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}. \tag{3.3}
\]
Note that
\[
\sum_{k=2}^{n} \left( \frac{1}{s_{k-1}} - \frac{1}{s_k} \right) = \frac{1}{s_1} - \frac{1}{s_n} = \frac{1}{a_1} - \frac{1}{s_n}.
\]
Since \( s_n \to \infty \) as \( n \to \infty \) (Theorem 3.24 in Rudin), it follows that
\[
\sum_{n=2}^{\infty} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{1}{a_1} - \lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{a_1}.
\]
By the comparison test (Theorem 3.25(a) in Rudin) and (3.3), we conclude that \( \sum_{n=1}^{\infty} \frac{a_n}{s_n} \) converges.

(d) The sum \( \sum_{n=1}^{\infty} \frac{a_n}{1 + na_n} \) always converges by the comparison test (Theorem 3.25(a) in Rudin) since
\[
\frac{a_n}{1 + n^2a_n} \leq \frac{1}{n^2}
\]
and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (Theorem 3.28 in Rudin). However, the sum \( \sum_{n=1}^{\infty} \frac{a_n}{1 + n^2a_n} \) could converge or diverge, depending on the sequence \( \{a_n\} \). On the one hand, if \( a_n = 1 \) for all \( n \in \mathbb{N} \), then \( \sum_{n=1}^{\infty} \frac{a_n}{1 + na_n} = \sum_{n=1}^{\infty} \frac{1}{1 + n} = \sum_{n=2}^{\infty} \frac{1}{n} \) diverges by Theorem 3.28 in Rudin. On the other hand, if
\[
a_n = \begin{cases} 
1/\sqrt{n} & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\
1/n^2 & \text{otherwise.}
\end{cases}
\]
for all \( n \in \mathbb{N} \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges by Theorem 3.28 in Rudin since
\[
\sum_{n=1}^{\infty} a_n \geq \sum_{k=1}^{\infty} a_{k^2} = \sum_{k=1}^{\infty} \frac{1}{k}.
\]
However,
\[
\sum_{n=1}^{\infty} \frac{a_n}{1 + na_n} \leq \sum_{k=1}^{\infty} \frac{a_{k^2}}{1 + k^2a_{k^2}} + \sum_{n=1}^{\infty} \frac{1/n^2}{1 + n(1/n^2)}
\]
\[
\leq \sum_{n=1}^{\infty} \frac{1/k}{1 + k^2(1/k)} + \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,
\]
so \( \sum_{n=1}^{\infty} \frac{a_n}{1 + na_n} \) converges.
**Problem 4** (Rudin, Chapter 3 #21). Prove the following analogue of Theorem 3.10(b): If \( \{E_n\} \) is a sequence of closed nonempty and bounded sets in a complete metric space \( X \), if \( E_n \supseteq E_{n+1} \), and if
\[
\lim_{n \to \infty} \operatorname{diam} E_n = 0,
\]
then \( \bigcap_{n=1}^{\infty} E_n \) consists of exactly one point.

**Solution.** Pick \( x_n \in E_n \) for every \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \) be given. Choose \( N \in \mathbb{N} \) such that \( \operatorname{diam} E_n < \varepsilon \) whenever \( n \geq N \). If \( m, n \geq N \), then \( x_m \in E_m \subseteq E_N \) and \( x_n \in E_n \subseteq E_N \), so
\[
d(x_n, x_m) \leq \operatorname{diam} E_N < \varepsilon.
\]
Thus, \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists an \( x \in X \) such that \( x_n \to x \) as \( n \to \infty \).

Suppose \( x \notin E_n \) for some \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \) be given, and choose \( N \in \mathbb{N} \) such that \( d(x_k, x) < \varepsilon \) whenever \( k \geq N \). In particular, \( d(x_{\max\{n,N\}}, x) < \varepsilon \), and \( x_{\max\{n,N\}} \in E_n \) since \( E_{\max\{n,N\}} \subseteq E_n \). Moreover, since \( x \notin E_n \), it follows that \( x \neq x_{\max\{n,N\}} \). Thus, since \( \varepsilon \) was arbitrary, it follows that \( x \) is a limit point of \( E_n \). But \( E_n \) is closed, so \( x \in E_n \) — a contradiction. It follows that \( x \in \bigcap_{n=1}^{\infty} E_n \).

Lastly, suppose \( y \in \bigcap_{n=1}^{\infty} E_n \), and let \( \varepsilon > 0 \). Choose \( n \in \mathbb{N} \) such that \( \operatorname{diam} E_n < \varepsilon \). Then \( x, y \in E_n \), so
\[
d(x, y) \leq \operatorname{diam} E_n < \varepsilon.
\]
It follows that \( d(x, y) = 0 \), so \( x = y \). We conclude that \( \bigcap_{n=1}^{\infty} E_n = \{x\} \).

**Problem 5** (Rudin, Chapter 3 #22). Suppose \( X \) is a nonempty complete metric space, and \( \{G_n\} \) is a sequence of dense open subsets of \( X \). Prove Baire’s theorem, namely, that \( \bigcap_{n=1}^{\infty} G_n \) is not empty. (In fact, it is dense in \( X \).) Hint: Find a shrinking sequence of neighborhoods \( E_n \) such that \( E_n \subseteq G_n \), and apply Exercise 21.

**Solution.** We will prove the stronger statement that \( \bigcap_{n=1}^{\infty} G_n \) is dense in \( X \). To see this, let \( U \) be a nonempty open subset of \( X \). It suffices to show that \( U \cap \bigcap_{n=1}^{\infty} G_n \) is nonempty. We will show this by first constructing a sequence \( \{E_n\} \) of nonempty open subsets of \( X \) satisfying the following three properties.
(1) \( E_{n+1} \subseteq E_n \) for each \( n \in \mathbb{N} \);
(2) \( E_n \subseteq U \cap \bigcap_{k=1}^{n} G_k \) for each \( n \in \mathbb{N} \);
(3) \( \text{diam } E_n \leq 1/n \) for each \( n \in \mathbb{N} \).

We will construct such a sequence inductively.

Since \( G_1 \) is dense in \( X \), it follows that \( U \cap G_1 \) is nonempty. Pick any point \( x_1 \in U \cap G_1 \). Since \( U \cap G_1 \) is open, we can choose an \( \varepsilon > 0 \) such that \( N_{\frac{1}{2} \min\{1, \varepsilon\}}(x_1) \subseteq U \cap G_1 \). Let

\[
E_1 = N_{\frac{1}{2} \min\{1, \varepsilon\}}(x_1).
\]

Note that \( E_1 \subseteq N_{\varepsilon}(x_1) \subseteq U \cap G_1 \). Moreover, if \( x, y \in E_1 \), then

\[
d(x, y) \leq d(x, x_1) + d(y, x_1) \leq \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = 1,
\]

so \( \text{diam } E_1 \leq 1 \).

Next, suppose we have chosen nonempty open sets \( E_1, \ldots, E_n \) such that \( E_{k+1} \subseteq E_k \) for all \( k \in \{1, \ldots, n-1\} \), \( E_k \subseteq U \cap \bigcap_{j=1}^{k} G_j \) for all \( k \in \{1, \ldots, n\} \), and \( \text{diam } E_k \leq 1/k \) for all \( k \in \{1, \ldots, n\} \). Since \( G_{n+1} \) is dense in \( X \), it follows that \( E_n \cap G_{n+1} \) is nonempty. Pick \( x_{n+1} \in E_n \cap G_{n+1} \). Since \( E_n \cap G_{n+1} \) is open, there exists an \( \varepsilon_{n+1} > 0 \) such that \( N_{\varepsilon_{n+1}}(x_{n+1}) \subseteq E_n \cap G_{n+1} \). Let

\[
E_{n+1} = N_{\frac{1}{2} \min\{1, \varepsilon_{n+1}\}}(x_{n+1}).
\]

Observe that

\[
E_{n+1} \subseteq N_{\varepsilon_{n+1}}(x_{n+1}) \subseteq E_n \cap G_{n+1}
\]

\[
\subseteq \left( U \cap \bigcap_{k=1}^{n} G_k \right) \cap G_{n+1} = U \cap \bigcap_{k=1}^{n+1} G_k
\]

and \( \overline{E_{n+1}} \subseteq E_n \). Moreover, if \( x, y \in E_{n+1} \), then

\[
d(x, y) \leq d(x, x_{n+1}) + d(y, x_{n+1}) \leq \frac{1}{2(n+1)} + \frac{1}{2(n+1)} = \frac{1}{n+1},
\]

so \( \text{diam } E_{n+1} \leq 1/(n+1) \).

Thus, we have inductively constructed a sequence of open sets \( \{E_n\} \) satisfying properties (1), (2), and (3) above. By (1), we have

\[
\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \overline{E_n}.
\]
Moreover, by (3) and the Archimedean property, it follows that \( \text{diam} \; E_n \to 0 \) as \( n \to \infty \). By Theorem 3.10(a) in Rudin, \( \text{diam} \; E_n \to 0 \) as \( n \to \infty \) as well. By Problem 4 and (5.1), it follows that \( \bigcap_{n=1}^{\infty} E_n \) is nonempty. But by (2) we have

\[
\bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{n=1}^{\infty} \left( U \cap \bigcap_{k=1}^{n} G_k \right) = U \cap \bigcap_{n=1}^{\infty} G_n,
\]

so we conclude that \( U \cap \bigcap_{n=1}^{\infty} G_n \) is nonempty. Consequently, \( \bigcap_{n=1}^{\infty} G_n \) is dense in \( X \).

**Remark.** Problem 5 is one form of Baire’s theorem (also known as the Baire category theorem\(^1\)), which is a very important result in topology and analysis. For example, in functional analysis, it is used to prove the open mapping theorem, the inverse mapping theorem, the closed graph theorem, and the Banach-Steinhaus theorem (also known as the uniform boundedness principle). Another application of Baire’s theorem is the result that says that every complete metric space with no isolated points is necessarily uncountable (in particular, \( \mathbb{R} \) is uncountable).

**Problem 6** (Rudin, Chapter 4 #1). Suppose \( f \) is a real function defined on \( \mathbb{R} \) which satisfies

\[
\lim_{h \to 0} \left[ f(x + h) - f(x - h) \right] = 0
\]

for every \( x \in \mathbb{R} \). Does this imply that \( f \) is continuous?

**Solution.** No. One possible counterexample is

\[
f(x) = \begin{cases} 
1, & \text{if } x = 0, \\
0, & \text{if } x \neq 0.
\end{cases}
\]

**Problem 7** (Rudin, Chapter 4 #3). Let \( f \) be a continuous real function on a metric space \( X \). Let \( Z(f) \) (the zero set of \( f \)) be the set of all \( p \in X \) at which \( f(p) = 0 \). Prove that \( Z(f) \) is closed.

**Solution.** Note that \( \{0\} \) is a closed subset of \( \mathbb{R} \) (recall that singletons are always closed in metric spaces), and since \( f \) is continuous, \( Z(f) = f^{-1}(\{0\}) \) is closed as well (cf. the Corollary to Theorem 4.8 in Rudin).

\(^1\)No relation to categories in category theory.