Finitary Analysis in Homogeneous Spaces

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Abstract

In this paper we give an overview of recent developments pertaining to the quantitative aspects of dynamics of group actions on homogeneous spaces.

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1. Introduction

Dynamical systems have become a major player in several unexpected areas in modern mathematics. Homogeneous spaces and the moduli spaces of compact Riemann surfaces serve as two hubs where techniques from dynamical systems and analysis duel in a nearly magical fashion with the structure provided by the rich geometric, algebraic, and arithmetic properties of the underlying space.

Investigations in these directions have resulted in several breakthrough results with striking applications in other areas of mathematics. However, most of these celebrated achievements share the lacuna that they are not quantitative. It is much anticipated and a challenging task to develop finitary arguments in these contexts; this article aims at providing an overview of some of the quantitative results in this setting.

Let us begin by recalling the general frame work of homogeneous dynamics. Let $G \subset \mathrm{SL}_d(\mathbb{R})$ be a connected linear Lie group, and let $\Gamma \subset G$ be a lattice (a discrete subgroup with finite covolume). Let $W \subset G$ be a closed connected subgroup of G. The following problem has proven to be of fundamental importance:

Describe the behavior of the orbit Wx for every *point $x \in G/\Gamma$.*

Note that we demand information about the orbit of every point in the space not merely a typical point, which is a more common theme in ergodic theory. Note also that in the above generality, one cannot expect a meaningful answer to this problem. E.g., if $G = \mathrm{SL}_2(\mathbb{R})$ and W is the group of diagonal matrices in G, then individual orbits can have very complicated behavior and in particular the closure of orbits can be a fractal set, see e.g. [64].

If W is generated by unipotent elements*, however, Raghunathan had conjectured that for every $x \in G/\Gamma$ there exists a connected subgroup $W \subset L \subset G$ so that Lx is *periodic* and the closure of Wx equals Lx — an orbit Lx is periodic if the stabilizer of x in L is a lattice in L, see §2.

Raghunaths's conjecture in its full generality was proved by Ratner [88–90]. Prior to Ratner's seminal work, some important special cases of this conjecture were established by Margulis [75], and Dani and Margulis [25, 26].

As was alluded to above, these fundamental results are not quantitative, e.g., they do not provide any rate at which the orbit fills up its closure. Indeed Ratner's work relies on the pointwise ergodic theorem which is hard to effectivize. The work of Dani and Margulis uses minimal sets, which though formally ineffective, can be effectivized with some effort. However, this a rather challenging task; moreover, the rates one obtains are often poor, see §6 for further discussion.

We note that good effective bounds for equidistribution of unipotent orbits can have far reaching consequences. Indeed the Riemann hypothesis is equivalent to giving an error term of the form $O_\varepsilon(y^{\frac{3}{4}+\varepsilon})$ for equidistribution of periodic horocycles of period $1/y$ on

* An $d \times d$ matrix is called unipotent if all its complex eigenvalues are 1. A connected subgroup of $\mathrm{SL}_d(\mathbb{R})$ is called unipotent if all its elements are unipotent.

the modular surface [91, 100]. Motivated by related but less dramatic applications, one is interested in obtaining rates which have *polynomial nature*. In the generality that will be discussed in §6, however, such bounds seem beyond the reach of the current technology. That said, there have been some exciting developments in this direction which will be discussed in the sequel.

We bring this introduction to a close by mentioning that there have also been groundbreaking works in a similar vein to the rigidity phenomena which lie at the heart of this paper, but in different contexts: In fact the papers [30, 68] concern higher rank diagonalizable flows; the papers [5–7, 12] concern the classification of stationary measures; the papers [40, 41] concern the action of $\mathrm{SL}_2(\mathbb{R})$ on moduli spaces and apply also the method developed for stationary measures; and [4, 66, 79, 80] concern the case where Γ has infinite covolume. These works, with the exception of [12], are all qualitative and *any* effective account of these would be very intriguing.

2. Complexity of Periodic Orbits

Let $L \subset G$ be a closed subgroup. A point $x \in X = G/\Gamma$ is called *L-periodic* if

$$\mathrm{Stab}_L(x) = \{g \in L : gx = x\}$$

is a lattice in L. A periodic L-orbit (or simply a periodic orbit if L is clear from the context) is an orbit Lx where x is an L-periodic point. Note that a periodic L-orbit is always closed in X, see [87].

The rigidity results we will discuss here assert that the closure of an orbit Wx is a periodic orbit Lx of an intermediate subgroup $W \subset L \subset G$. It is therefore natural to expect that quantitative statements in this context will in general depend on delicate properties of the point x and the acting group W. Indeed already for an irrational rotation of a circle, Diophantine properties of the angle of rotation dictate the rate of equidistribution. In the more general context at the heart of our discussions here, periodic orbits of intermediate subgroups will play the role of rational numbers. Consequently, it is crucial to fix a measure of complexity for the periodic orbits which are obstructions to the density of an orbit in X.

Fix some open bounded neighborhood Ω of the identity in G. For a periodic orbit $Lx \subset X$, define

$$\mathrm{vol}(Lx) = \frac{m_L(Lx)}{m_L(\Omega)}, \tag{2.1}$$

where m_L is an arbitrary Haar measure on L and $m_L(Lx)$ is the covolume of $\mathrm{Stab}_L(x)$ in L with respect to m_L. This notion of volume will serve as our measure of the complexity of the periodic orbit,

We refer the reader to [31, §2.3] for basic properties of the above definition. Here we only mention that even though this notion depends on the choice of Ω, two different choices of Ω give rise to comparable definitions of vol, in the sense that their ratio is bounded above and below. Therefore, we ignore the dependence on Ω in the notation.

Given a periodic orbit Lx, we let μ_{Lx} denote the probability L-invariant measure on Lx. The G-invariant probability measure on X will be denoted by m_X.

The general theme of a finitary statement will be a dichotomy as follows: Unless there is an explicit obstruction with *low complexity*, the orbit Wx *fills up* X with an explicit rate — as we will see the quality of this rate varies in different situations.

3. Effective Equidistribution of Nilflows

Perhaps the first natural place to seek quantitative density theorems is the case of nilflows. Let X be a nilmanifold. That is, $X = G/\Gamma$ where G is a closed connected subgroup of the group of strictly upper triangular $d \times d$ matrices and $\Gamma \subset G$ is a lattice.

Rigidity results in this setting have been known for quite some time thanks to works of Weyl, Kronecker, L. Green, and Parry [2,85], and more recently Leibman [67].

Quantitative results, *with a polynomial error rate*, have also been established in this context and beyond the abelian case, see [46,53]. The complete solution was given by B. Green and T. Tao [53]. The following is a special case of the main result in [53].

Theorem 3.1 ([53]). *Let $X = G/\Gamma$ be a nilmanifold as above. There exists some $A \geq 1$ depending on $\dim G$ so that the following holds. Let $x \in X$, let $\{u(t) : t \in \mathbb{R}\}$ be a one parameter subgroup of G, let $0 < \eta < 1/2$, and let $T > 0$. Then at least one of the following holds for the partial trajectory $\{u(t)x : t \in [0, T]\}$.*

(1) *For every $f \in C^\infty(X)$ we have*
$$\left|\frac{1}{T}\int_0^T f(u(t)x)\,\mathrm{d}t - \int_X f\,\mathrm{d}m_X\right| \ll_{X,f} \eta$$
where the dependence on f is given using a certain Lipschitz norm.

(2) *For every $0 \leq t_0 \leq T$ there exists some $g \in G$ and some $H \subsetneq G$ so that $H\Gamma/\Gamma$ is periodic with $\mathrm{vol}(gH\Gamma/\Gamma) \ll_X \eta^{-A}$ and for all $t \in [0, T]$ with $|t - t_0| \leq \eta^A T$, we have*
$$\mathrm{dist}_X(u(t)x, gH\Gamma/\Gamma) \ll_X \eta$$
where dist_X is a metric on X induced from a right invariant Riemannian metric on G.

We refer the reader to [33] for this formulation and the deduction of it from the main result in [53]. Let us however highlight here the aforementioned dichotomy: either the orbit $\{u(t)x : t \in [0, T]\}$ is effectively equidistributed, part (1) in Theorem 3.1, or there is an explicit obstruction of low complexity which prevents this, part (2) in Theorem 3.1.

4. Horospherical Groups

Let G be a connected semisimple Lie group. A subgroup $W \subset G$ is called horospherical if there exists an ($\mathbb{R}$-diagonalizable) element $a \in G$ so that

$$W = W^+(a) := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to -\infty\}.$$

It is well known that $W \subset G$ is horospherical if and only if it is the unipotent radical of a proper parabolic subgroup of G. In particular, a horospherical subgroup is always unipotent*, but not vice versa; indeed, if W is horospherical, then $G/N_G(W)$ is compact where $N_G(W)$ denotes the normalizer of W in G.

The study of the action of a horospherical subgroup of G on G/Γ has a long history, and rigidity theorems á la Ratner in this case were established by Hedlund, Furstenberg, Veech, and Dani [20,21,24,48,59,97] prior to Ratner's theorems. Indeed, thanks to the fact that the behavior of individual orbits of a horospherical subgroup can be related to the decay of matrix coefficients, effective equidistribution, with a polynomial error rate, can also be established. The first works in this direction we are aware of are [16,64,91] as well as the more recent [45,93,95] and this has now been established in much greater generality [62,63,78,82]. Closely related is the case of translates of orbits of subgroups of G which are fixed by an involution [3,29,39].

We refer the reader to [78, Thm. 3.1] for the case of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ and to [62, Thm. 1.11] for the general case. Let us only mention here that in this case, obstructions to effective equidistribution of Wx, where $W = W^+(a)$, can be described using the rate of excursion of $\{a^{-n}x : n \in \mathbb{N}\}$ to infinity. Consequently, quantitative non-divergence of unipotent flows [22,23,27,65,74] play a crucial role in the analysis, see also the discussion in §6.2. In particular, when $X = G/\Gamma$ is compact, Wx is equidistributed in (X, m_X) with a polynomial rate for *every* point $x \in X$.

Another class of examples where one may attempt to bring properties of horospherical subgroups to bear are provided by semidirect product constructions. Let $G = H \ltimes V$ where H is a non-compact semisimple Lie group and V is an irreducible representation of H. One then investigates the action of a horospherical subgroup $W \subset H$ on G/Γ. This case is significantly more complicated that the case of horospherical subgroups, and only partial progress has been made in this direction. Indeed Strömbergsson [96] used analytic methods to settle the case of $G = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ with the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$, $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, and W the group of unipotent upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$; his method has also been used to tackle some other cases.

We end this section by mentioning that ideas developed in the homogeneous setting have also found applications in the study of horospherical foliation (strong unstable foliation) in the space of translation surfaces, see e.g. [42,70].

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* The fact that a horospherical subgroup is unipotent follows readily from the definition

5. Periodic Orbits of Semisimple Groups

Until roughly fifteen years ago, the source of quantitative treatments in this context could essentially be traced back to the settings discussed in §3 and §4. However, the situation has recently improved. In the remaining parts of this article, we discuss some of these advances.

One of the earliest works in this *new wave* was the landmark paper of Einsiedler, Margulis, and Venkatesh [32] concerning the periodic orbits of semisimple groups. Let $\boldsymbol{G}$ be a connected, semisimple algebraic $\mathbb{Q}$-group, and let G be the connected component of the identity in the Lie group $\boldsymbol{G}(\mathbb{R})$. Let $\Gamma \subset G$ be a congruence subgroup of $\boldsymbol{G}(\mathbb{Q})$, and put $X = G/\Gamma$. Let $H \subset G$ be a semisimple subgroup without any compact factors which has a finite centralizer in G.

The following is the main equidistribution theorem proved in [32].

Theorem 5.1 ([32]). *There exists some $\delta = \delta(G, H)$ so that the following holds. Let Hx be a periodic orbit. For every $V > 1$ there exists a subgroup $H \subset S \subset G$ so that Sx is periodic,* $\mathrm{vol}(Sx) \le V$, *and*

$$\left| \int_X f \,\mathrm{d}\mu_{Hx} - \int_X f \,\mathrm{d}\mu_{Sx} \right| \ll_{G,\Gamma,H} \mathcal{S}(f) V^{-\delta} \quad \text{for all } f \in C_c^\infty(X),$$

where $\mathcal{S}(f)$ denotes a certain Sobolev norm.

Theorem 5.1 is an effective version (of a special case) of a theorem by Mozes and Shah [84]. The polynomial nature of the error term, i.e., a (negative) power of V, in Theorem 5.1 is quite remarkable — effectivizations of dynamical arguments often yield much worse rates, see §6. The source of this polynomial rate is the uniform spectral gap for congruence quotients, which is used as a crucial input in [32].

As it was alluded to already, the fact that one deals with periodic orbits of semisimple groups in arithmetic quotients is an indispensable features of the ideas developed in [32] viz. the uniform spectral gap for congruence quotients. However, some of the other assumptions made in Theorem 5.1 may be relaxed. Indeed in a subsequent work, Einsiedler, Margulis, Mohammadi, and Venkatesh [31] proved an adelic statement which lifts two of the restrictions imposed in Theorem 5.1: the fact that H is assumed fixed (the estimates in Theorem 5.1 depend on H) and splitting assumption on H at the archimedean place (H has no compact factors).

Let $\boldsymbol{G}$ be a connected, semisimple, algebraic $\mathbb{Q}$-group* and set $X = \boldsymbol{G}(\mathbb{A})/\boldsymbol{G}(\mathbb{Q})$ where $\mathbb{A}$ denotes the ring of adeles. Then X admits an action of the locally compact group $\boldsymbol{G}(\mathbb{A})$ preserving the probability measure m_X. Let $\mathbf{H}$ be a semisimple, simply connected, algebraic $\mathbb{Q}$-group, and let $g \in \boldsymbol{G}(\mathbb{A})$. Fix also an algebraic homomorphism $\iota : \mathbf{H} \to \mathbf{G}$ defined over $\mathbb{Q}$ with finite central kernel. E.g., let $\boldsymbol{G} = \mathrm{SL}_d$ and $\boldsymbol{H} = \mathrm{Spin}(Q)$ for an integral quadratic form Q in d variables.

* The paper [31] allows for any number field, F, but unless X is compact, δ in Theorem 5.2 will depend on $\dim \boldsymbol{G}$ and $[F : \mathbb{Q}]$.

To this algebraic data and any $g \in \boldsymbol{G}(\mathbb{A})$, one associates a homogeneous set

$$Y := g\iota(\boldsymbol{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})) \subset X$$

and a homogeneous probability measure μ.

The following is a special case of the main theorem in [31].

Theorem 5.2 ([31]). *Assume further that $\boldsymbol{G}$ is simply connected. There exists some $\delta > 0$, depending only on $\dim \boldsymbol{G}$, so that the following holds. Let Y be a homogeneous set and assume that $\iota(\boldsymbol{H}) \subset \boldsymbol{G}$ is maximal. Then*

$$\left| \int_X f \,\mathrm{d}\mu - \int_X f \,\mathrm{d}m_X \right| \ll_{\boldsymbol{G}} \mathcal{S}(f) \operatorname{vol}(Y)^{-\delta} \quad \text{for all } f \in C_c^\infty(X),$$

where $\mathcal{S}(f)$ is a certain adelic Sobolev norm.

The flexibilities that Theorem 5.2 provides have interesting number theoretic applications. Indeed the following generalization of Duke's theorem is proved in [31].

Let $\mathcal{Q}_d = \mathrm{PO}_d(\mathbb{R}) \backslash \mathrm{PGL}_d(\mathbb{R}) / \mathrm{PGL}_d(\mathbb{Z})$ be the space of positive definite quadratic forms on $\mathbb{R}^d$ up to the equivalence relation defined by scaling and equivalence over $\mathbb{Z}$. Equip $\mathcal{Q}_d$ with the push-forward of the normalized Haar measure on $\mathrm{PGL}_d(\mathbb{R})/\mathrm{PGL}_d(\mathbb{Z})$.

Let Q be a positive definite integral quadratic form on $\mathbb{Z}^d$, and let $\mathrm{genus}(Q)$ (resp. spin genus(Q)) be its genus (resp. spin genus).

Theorem 5.3 ([31]). *Suppose $\{Q_n\}$ varies through any sequence of pairwise inequivalent, integral, positive definite quadratic forms. Then the genus (and also the spin genus) of Q_n, considered as a subset of $\mathcal{Q}_d$, equidistributes as $n \to \infty$ (with speed determined by a power of $|\mathrm{genus}(Q_n)|$).*

It is worth mentioning that when $d = 3, 4$, this theorem *even in its qualitative form* is new. When $d > 5$, the qualitative version of this theorem follows from an equidistribution theorem proved in [52], see also [43] for related analysis in the presence of a splitting condition at the archimedean place

Another application of Theorem 5.2 is an independent proof of property (τ) except for groups of type A_1. In particular, the paper [31] provides an alternative proof of the main result of Clozel in [19], albeit with weaker exponents, see [31, §4].

In addition to the ingredients involved in [32], the proof of Theorem 5.2 relies on Prasad's volume formula [86] and the work of Borel and Prasad [9]. These fundamental inputs are responsible for liberties supplied by Theorem 5.2.

The main problem which remains open in this direction is to prove an analogue of Theorem 5.2 which allows $\iota(\boldsymbol{H})$ to have an infinite centralizer; such a theorem would have quite interesting number theoretic applications, see [36]. Some progress has been made in this direction recently, the reader is invited to consult [1, 34, 35] for instance.

6. Effective Unipotent Dynamics

In view of Theorem 3.1, let us assume that G has non-compact semisimple subgroups, e.g., G is a non-compact semisimple linear Lie group. In light of the results discussed in §4, the analysis of the quantitative behavior of unipotent orbits in G/Γ is reduced to orbits of groups which are not horospherical. Not surprisingly, however, this task has proven quite challenging. In this section, we will discuss some recent progress made in this direction. The general theme of results in this section revolves around exploiting and effectivizing the *polynomial like behavior* of unipotent orbits.

6.1. Effective versions of the Oppenheim conjecture

The Oppenheim conjecture, proved by Margulis [75], states that if Q is a nondegenerate, indefinite quadratic form which is not a rational multiple of a form with integer coefficients, then for every $\varepsilon > 0$, there exists some $v \in \mathbb{Z}^3 \setminus \{0\}$ so that $|Q(v)| < \varepsilon$. Generalizations were also proved by Dani and Margulis prior to Ratner's theorems.

Later, Eskin, Margulis, and Mozes [37,38] proved quantitative (equidistribution) versions of the Oppenheim conjecture which relies on Ratner's equidistribution theorem [88–90], Linearization techniques of Dani and Margulis [28], and a system of inequalities for a certain Margulis function — an ingenious idea introduced in [37] which has become an indispensable tool in homogeneous dynamics and beyond, see §7.2. Similar results for inhomogeneous forms have also been established [73,76].

Effective results in this context have also been actively pursued. Indeed, the analytic approach (using the Hardy-Littlewood circle method) which had been employed prior to Margulis' work is by its nature effective. However, this approach is generally only applicable if either the number of variables is large or the form has special features, see, e.g., [60]. More recently, Buterus, Götze, Hille, and Margulis [18] have proved effective version of the Oppenheim conjecture (as well as the equidistribution versions [37]), with polynomial error rates, provided that the number of variable is at least 5. Their proof combines analytic techniques with ideas from geometry of numbers in the form of inequalities which are reminiscent of [37]. Analytic methods were also used in [92] and [11] to obtain polynomial estimates for a.e. form in certain families of forms in dimensions 3 and 4. The case of general forms in 3 and 4 variables, however, seem to be out of the reach of analytic methods.

Lindenstrauss and Margulis [69] proved an effective version of the Oppenheim conjecture for ternary form with poly-log error rates.

Theorem 6.1 ([69]). *There exist absolute constants $A \geq 1$ and $\kappa > 0$ so that the following holds.*

Let Q be an indefinite, ternary quadratic form with $\det Q = 1$ and let $\varepsilon > 0$. There exists $T_0(\varepsilon) > 0$ so that for any $T \geq T_0(\varepsilon)\|Q\|^A$ at least one of the following holds.

(1) *For every $\xi \in [-(\log T)^\kappa, (\log T)^\kappa]$ there is a primitive integer vector $v \in \mathbb{Z}^3$ with $0 < \|v\| < T^A$ satisfying*

$$|Q(v) - \xi| \ll (\log T)^{-\kappa}.$$

(2) *There is an integral quadratic form Q' with $|\det Q'| < T^{\varepsilon}$ so that*

$$\|Q - \lambda Q'\| \ll \|Q\| T^{-1}$$

where $\lambda = |\det Q'|^{-1/3}$.

The implied multiplicative constants are absolute and $\| \;\|$ denotes a norm on $\mathrm{Mat}_3(\mathbb{R})$.

Note in particular that if Q is a reduced, indefinite, ternary quadratic form which is not proportional to an integral form but has algebraic coefficients, then part (1) in Theorem 6.1 holds true for Q, see [69, Cor. 1.12].

The aforementioned dichotomy is again present in Theorem 6.1: unless there is an explicit obstruction (part (2) in Theorem 6.1), one obtains an effective density result.

The proof in [69] is rather involved and is based on effectivizing Margulis' original proof of the Oppenheim conjecture as well as the subsequent works by Dani and Margulis. This approach, is based on the study of the action of $\mathrm{SO}(Q)$, the isometry group of Q, on $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$, and relies on the notion of minimal sets from topological dynamics. Minimal sets are not suitable for quantitative arguments. Indeed, the paper [69] replaces this qualitative notion with a Diophantine condition in terms of the rate of escape to infinity under a certain one-parameter $\mathbb{R}$-diagonalizable subgroup. This novel ingredient plays a crucial role in obtaining an effective account — similar elements in more general contexts will be discussed in §6.2. It is worth mentioning that relying only on this input, one gets a rate that is $\ll \log(\log T)$. The stronger bound obtained in [69] is made possible thanks to a combinatorial lemma [69, §9] which is of independent interest.

6.2. Linearization of unipotent orbits

As it was mentioned before, Margulis and Dani developed a topological approach to settle certain special cases of Raghunathan's conjecture which relied on the notion of minimal sets. One of the first steps in effectivizing this topological argument would therefore be to replace minimal sets with an explicit Diophantine condition. This was established by Lindenstrauss, Margulis, Mohammadi, and Shah in [72] which may be thought of as an effective version of the *linearization technique* of Dani and Margulis [28].

The linearization technique has its roots in the techniques developed by Margulis [74] in his proof of the nondivergence of unipotent orbits. These nondivergence results are effective. Indeed, they were sharpened by Dani in [22,23] and have been given a very explicit and effective form by Kleinbock and Margulis in [65]. However, the author is not aware of an effective treatment of the main results in [28] prior to [72].

Let us recall the setting in [72]. Let $\boldsymbol{G}$ be a connected $\mathbb{Q}$-group, and put $G = \boldsymbol{G}(\mathbb{R})$. We assume that $\Gamma \subset G$ is an arithmetic lattice. More specifically, fix an embedding $\iota : \boldsymbol{G} \to \mathrm{SL}_N$ defined over $\mathbb{Q}$ so that $\iota(\Gamma) \subset \mathrm{SL}_N(\mathbb{Z})$. Using ι, we identify $\boldsymbol{G}$ with $\iota(\boldsymbol{G}) \subset \mathrm{SL}_N$, and hence will always assume that $G \subset \mathrm{SL}_N(\mathbb{R})$.

Define the following family

$$\mathcal{H} = \{\boldsymbol{H} \subset \boldsymbol{G} : \boldsymbol{H} \text{ is a connected } \mathbb{Q}\text{-subgroup and } \mathrm{R}(\boldsymbol{H}) = \mathrm{R}_u(\boldsymbol{H})\}$$

where $\mathrm{R}(\boldsymbol{H})$ (resp. $\mathrm{R}_u(\boldsymbol{H})$) denotes the solvable (resp. unipotent) radical of $\boldsymbol{H}$. Alternatively, $\boldsymbol{H} \in \mathcal{H}$ if and only if $\boldsymbol{H}$ is a connected $\mathbb{Q}$-subgroup which is generated by unipotent subgroups over the algebraic closure of $\mathbb{Q}$. By a theorem of Borel and Harish-Chandra, $\boldsymbol{H}(\mathbb{R}) \cap \Gamma$ is a lattice in $\boldsymbol{H}(\mathbb{R})$ for every $\boldsymbol{H} \in \mathcal{H}$. We always assume that $\boldsymbol{G} \in \mathcal{H}$.

Let $U \subset G$ be a (connected) unipotent subgroup of G, and put $X = G/\Gamma$. For every $\boldsymbol{H} \in \mathcal{H}$, put $H = \boldsymbol{H}(\mathbb{R})$. Define

$$N_G(U, H) := \{g \in G : Ug \subset gH\}.$$

Note that $N_G(U, H)$ is an $\mathbb{R}$-subvariety of G. Moreover, if $H \lhd G$ and $U \subset H$, then $N_G(U, H) = G$.

Put

$$\mathcal{S}(U) = \Big(\bigcup_{\substack{\boldsymbol{H} \in \mathcal{H} \\ \boldsymbol{H} \neq \boldsymbol{G}}} N_G(U, H)\Big)/\Gamma \quad \text{and} \quad \mathcal{G}(U) = X \setminus \mathcal{S}(U).$$

Points in $\mathcal{S}(U)$ are called *singular* with respect to U, and points in $\mathcal{G}(U)$ are called *generic* with respect to U — these are, a priori, different from the measure theoretically generic points in the sense of Furstenberg for the action of U on X equipped with m_X (see e.g. [50, p. 98] for a definition); however, any measure theoretically generic point is generic in this explicit sense as well. The aforementioned remarkable theorem of Ratner [90] states that for every $x \in \mathcal{G}(U)$, we have $\overline{Ux} = X$.

Dani and Margulis [28] proved that U orbits of points in $\mathcal{G}(U)$ *avoid* $\mathcal{S}(U)$. The paper [72] makes this principle quantitative with polynomial rates.

We need some more notation to state this quantitative result. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and put $\mathfrak{g}(\mathbb{Z}) := \mathfrak{g} \cap \mathfrak{sl}_N(\mathbb{Z})$. Let $\| \; \|$ denote the max norm on $\mathfrak{sl}_N(\mathbb{R})$ with respect to the standard basis. This induces a family of norms on $\wedge \mathfrak{sl}_N(\mathbb{R})$, which we continue to denote by $\| \; \|$.

Let $\boldsymbol{H} \in \mathcal{H}$ be a nontrivial proper subgroup of $\boldsymbol{G}$, and put

$$\rho_H := \wedge^{\dim \boldsymbol{H}} \mathrm{Ad} \quad \text{and} \quad V_H := \wedge^{\dim \boldsymbol{H}} \mathfrak{g}.$$

The representation ρ_H is defined over $\mathbb{Q}$.

Let $\mathbf{v}_{\boldsymbol{H}}$ be a primitive integral vector in $\wedge^{\dim \boldsymbol{H}} \mathrm{Lie}(\boldsymbol{G})$ corresponding to the Lie algebra of $\boldsymbol{H}$, i.e., we fix a $\mathbb{Z}$-basis for $\mathrm{Lie}(H) \cap \mathfrak{sl}_N(\mathbb{Z})$, and let $\mathbf{v}_{\boldsymbol{H}}$ be the corresponding wedge product. The vector $\mathbf{v}_{\boldsymbol{H}}$ embeds diagonally in $\wedge^{\dim \boldsymbol{H}} \mathfrak{g}$; we denote this diagonally embedded vector by v_H. Define

$$\eta_H(g) := \rho_H(g) v_H \quad \text{for every } g \in G.$$

In order to simplify the exposition, let us assume that U is a one-parameter unipotent subgroup of G. Fix some $z \in \mathfrak{g}$ with $\|z\| = 1$ so that $U = \{u(t) = \exp(tz) : t \in \mathbb{R}\}$. With this notation, for an element $\boldsymbol{H} \in \mathcal{H}$, we have

$$N_G(U, H) = \{g \in G : z \wedge \eta_H(g) = 0\}.$$

As it was observed before, $N_G(U, H)$ is a variety; therefore, it could change drastically under small perturbations of U. However, *effective* notions must be stable under small perturbations. One of the innovations of [72] is the introduction of the following effective notion of generic points:

Definition 6.2. Let $\varepsilon : \mathbb{R}^+ \to (0,1)$ be a monotone decreasing function, and let $t \in \mathbb{R}^+$. A point $g\Gamma$ is called (ε, t)-*Diophantine* for the action of $U = \{\exp(tz) : t \in \mathbb{R}\}$ if for all $\boldsymbol{H} \in \mathcal{H}$ with $\{e\} \neq \boldsymbol{H} \neq \boldsymbol{G}$

$$\|z \wedge \eta_H(g)\| \geq \varepsilon(\|\eta_H(g)\|) \qquad \text{if} \qquad \|\eta_H(g)\| < e^t. \tag{6.1}$$

A point is ε-*Diophantine* if it is (ε, t)-Diophantine for all $t > 0$.

Note that this is a condition on the *pair* $(U, g\Gamma)$. Unless $U \subset \boldsymbol{H}(\mathbb{R})$ for some (proper) $\boldsymbol{H} \lhd \boldsymbol{G}$, the set $\mathcal{G}(U)$ is nonempty, moreover any $x \in \mathcal{G}(U)$ is ε-Diophantine for some ε as above. In most interesting examples, the singular set $\mathcal{S}(U)$ is a dense subset of X. Therefore, $\mathcal{G}(U)$ is usually a G_δ-set without any interior points. For any $t \in \mathbb{R}^+$, on the other hand, the set of (ε, t)-Diophantine points in Definition 6.2 is a nice closed set with interior points (indeed, it is the closure of its interior points).

As was discussed in §2, and we have seen in prior sections, finitary statements require a measure of complexity for obstructions. In [72], the following measure of arithmetic complexity for subgroups in $\mathcal{H}$ is used. Define

$$\mathrm{ht}(\boldsymbol{H}) := \|\mathbf{v}_{\boldsymbol{H}}\|. \tag{6.2}$$

That is, the height of a $\mathbb{Q}$-group $\boldsymbol{H}$ is given by the height of the corresponding point in the Grassmanian of $\mathrm{Lie}(\boldsymbol{G})$, see [8, §1.5]. It is worth mentioning that for subgroups $\boldsymbol{H} \in \mathcal{H}$, $\mathrm{ht}(\boldsymbol{H})$ is closely related to the volume of the periodic orbit $H\Gamma/\Gamma$ as it was defined in §2, see [32, §17], [31, App. B], and [81, §6.2].

The space X is not necessarily compact; to deal with this issue, we fix an exhaustion of X by compact subsets as follows. For every $\eta > 0$, define

$$X_\eta = \{g\Gamma \in X : \min_{0 \neq v \in \mathfrak{g}(\mathbb{Z})} \|\mathrm{Ad}(g)v\| \geq \eta\}.$$

By (a generalization of) Mahler's compactness criterion, X_η is compact for every $\eta > 0$, see e.g. [72, Lemma 2.8]. Moreover, $\bigcup_{\eta > 0} X_\eta = G/\Gamma$.

For every $g \in \mathrm{SL}_N(\mathbb{R})$, in particular for every $g \in G$, we let

$$|g| = \max\{\|g\|, \|g^{-1}\|\}.$$

where $\|\ \|$ denotes the max norm on $\mathrm{SL}_N(\mathbb{R})$ with respect to the standard basis.

The following is the main result in [72] in the case of real groups.

Theorem 6.3 ([72]). *There are constants $A, D > 1$ depending only on N, and $E > 1$ depending on N, G and Γ, so that the following holds. Let $g \in G$, $t > 0$, $k \geq 1$, and $0 < \eta < 1/2$. Assume $\varepsilon : \mathbb{R}^+ \to (0,1)$ satisfies for every $s > 0$ that*

$$\varepsilon(s) \leq \eta^A s^{-A}/E.$$

Then at least one of the following three possibilities holds.

(1)

$$\left|\left\{\xi \in [-1,1] : \begin{array}{c} u(e^k\xi)g\Gamma \notin X_\eta \textit{ or} \\ u(e^k\xi)g\Gamma \textit{ is not } (\varepsilon, t)\textit{-Diophantine} \end{array}\right\}\right| < E\eta^{1/D}$$

(2) *There exist a nontrivial proper subgroup* $\boldsymbol{H} \in \mathcal{H}$ *with*

$$\mathrm{ht}(\boldsymbol{H}) \leq E(|g|^A + e^{At})\eta^{-A}$$

so that the following hold for all $\xi \in [-1,1]$*:*

$$\|\eta_H(u(e^k\xi)g)\| \leq E(|g|^A + e^{At})\,\eta^{-A}$$
$$\|z \wedge \eta_H(u(e^k\xi)g)\| \leq Ee^{-k/D}(|g|^A + e^{At})\,\eta^{-A}$$

where $U = \{\exp(tz) : t \in \mathbb{R}\}$.

(3) *There exist a nontrivial proper normal subgroup* $\boldsymbol{H} \lhd \boldsymbol{G}$ *with*

$$\mathrm{ht}(\boldsymbol{H}) \leq Ee^{At}\eta^{-A}$$

so that

$$\|z \wedge v_H\| \leq \varepsilon(\mathrm{ht}(\boldsymbol{H})^{1/A}\eta/E)^{1/A}.$$

Indeed the paper [72] proves versions of this theorem for friendly measures [72, Thm. 1.7] as well as S-arithmetic versions of this theorem [72, §3]. In particular, in view of [72, Thm. 3.2], and by using the restriction of scalars from number fields to $\mathbb{Q}$, the results in [72] are applicable also in the case of groups defined over a general number field.

The arguments in [72] rely on polynomial behavior of unipotent orbits as did the arguments in [28]. However, in addition to being *polynomially* effective, the results also differ from [28] in the following sense. They provide a compact subset of $\mathcal{G}(U)$ which is *independent* of the base point and to which a unipotent orbit returns unless there is an algebraic obstruction, see [72, Thm. 1.1 and Thm. 1.5]. Regarding nondivergence properties of unipotent orbits, such uniformity is well known and is due to Dani (see [23, 27]), but in this context it was not known prior to [72].

These features have been made possible using two main new ingredients. First is the use of an effective notion of a generic point, Definition 6.2. The second ingredient is the use of a certain subgroup in $\mathcal{H}$ which controls the *speed* of unipotent orbits in the representation space V_H, see [72, §4.7]. In addition, the arguments in [72] rely on effective versions of Nullstellensatz, [77, Thm. IV], as well as some local non-vanishing theorems related to Lojasiewicz inequality, [15, 54, 55].

6.3. Effective density of unipotent orbits

The paper [72] is the first in a series of papers which provide a general effective orbit closure theorem for unipotent orbits on arithmetic quotients. The second paper, which is in preparation, crucially relies on the results of [72].

The rate we obtain (for density of a unipotent orbit) are an iteration of logarithms in the size of the flow parameter where the number of iterations depends on $\dim G$.

7. Arithmetic Combinatorics and Polynomial Bounds

The discussion in §6 allude to the fact that effectivizing the *existing* arguments from unipotent dynamics often does not yield a polynomial rate. Indeed beyond the notable settings we discussed in §3–5, polynomial rates of density or equidistribution in this context are rather rare. In this section we discuss some recent progress made in this direction.

7.1. Random walks by toral automorphisms

Let $\Gamma \subset \mathrm{SL}_d(\mathbb{Z})$ be a Zariski dense subgroup which acts strongly irreducibly on $\mathbb{R}^d$ (that is, no nontrivial subspace of $\mathbb{R}^d$ is invariant under a finite index subgroup of Γ). Let ν be a finitely supported probability measure on Γ whose support generates Γ.

Furstenberg [47] showed that

$$\lambda_1(\nu) = \lim_{n\to\infty} \frac{1}{n} \log \|g_1 \cdots g_n\| \qquad \nu^{\mathbb{N}}\text{-a.s.}$$

is positive.

In a landmark paper [12], Bourgain, Furman, Lindenstrauss, and Mozes proved an equidistribution theorem for random walks on $\mathbb{T}^d$ corresponding to ν, with polynomial rates.

Theorem 7.1 ([12]). *For every $0 < \lambda < \lambda_1(\nu)$ there exists a constant $C = C(\nu, \lambda)$ so that if for a point $x \in \mathbb{T}^d$ the measure $\mu_n = \nu^{(n)} * \delta_x$ satisfies that for some $a \in \mathbb{Z}^d \setminus \{0\}$*

$$|\hat{\mu}_n(a)| > t > 0 \quad \textit{with} \quad n > C \log(2\|a\|/t),$$

then x admits a rational approximation p/q where $p \in \mathbb{Z}^d$ and $q \in \mathbb{N}$ satisfying

$$\|x - \tfrac{p}{q}\| < e^{-\lambda n} \quad \textit{and} \quad |q| < (2\|a\|/t)^C.$$

Indeed the main results in [12] allow for a more general class of subgroups Γ and measures ν. Let us also mention that the results in [12] have been further generalized in subsequent works, see, e.g., [57,58].

The argument in [12] is quite involved and relies on several ingredients. Here we only highlight one the main steps in the proof, which concerns bootstrapping the information about one large Fourier coefficient to a (large scale) structure for the set of large Fourier coefficients. Suppose $|\hat{\mu}_n(a)| > t$ for some large n and some nonzero a. Then using quantitative theory of random matrix products, one can show that for a suitable choice of $n_1 \le n$ the measure μ_{n_1} has Fourier coefficients which are $> t/2$ on a subset with a (small) positive dimension, [12, Prop. 6.2]. The next task is to deduce from this a possibly smaller scale $n_2 < n_1$, so that μ_{n_2} has large (polynomial in t) Fourier coefficients on a set whose large scale dimension is d. This is carried out in two steps, the first, and arguably more difficult, step is to bootstrap the dimension to $d - \varepsilon$ for a small ε (depending on ν), [12, Prop. 6.3]. The paper [12] uses ideas from additive combinatorics viz. discretized ring conjecture [10] to establish this improvement. After this is obtain, one can use more or less classical estimates from Fourier analysis to improve the dimension from $d - \varepsilon$ to d, [12, Prop. 6.5 and 6.11].

The three stages in the above outline: initial dimension, bootstrapping the dimension, and from high dimension to positive density are reminiscent of the three stages present in the work of Bourgain and Gamburd on random walks on compact groups [13,14] — these three stages will be revisited in the next section.

7.2. Quotients of $\mathbf{SL_2}(\mathbb{C})$ and $\mathbf{SL_2}(\mathbb{R}) \times \mathbf{SL_2}(\mathbb{R})$

We now turn to the question of density (or more ambitiously equidistribution) results in quotients of *semisimple* groups, with polynomial rates. For reasons we already discussed, this has proven quite a challenging task.

Lindenstrauss and Mohammadi [71] have very recently obtained first results in the literature which provide a polynomial rate for density of general orbits in a homogeneous space of a semisimple group, beyond the settings we discussed in §4 and §5.

Let us fix some notation. Let

$$G = \mathrm{SL}_2(\mathbb{C}) \quad \text{or} \quad G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}),$$

and let $\Gamma \subset G$ be a lattice. Put $X = G/\Gamma$.

Let dist be the right invariant metric on G which is defined using the killing form. This metric induces a metric dist_X on X. The injectivity radius of a point $x \in X$ may be defined using this metric. For every $\eta > 0$, let

$$X_\eta = \{x \in X : \text{injectivity radius of } x \text{ is} \geq \eta\};$$

this is closely related to the definition in §6.2, see e.g. [71, §3] and references there.

Let $H \subset G$ be one of the following:

$$\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C}) \quad \text{or} \quad \{(g,g) : g \in \mathrm{SL}_2(\mathbb{R})\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Let $P \subset H$ be the group of upper triangular matrices in H.

As before, let $\| \ \|$ denote the maximum norm on $\mathrm{Mat}_2(\mathbb{C})$ or $\mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R})$ with respect to the standard basis. For every $R > 0$ and every subgroup $L \subset G$, let

$$B_R^L = \{g \in L : \|g - I\| \leq R\}.$$

The following is one of the main results in [71]:

Theorem 7.2 ([71]). *Assume that Γ is an arithmetic lattice. For every $0 < \delta < 1/2$, every $x_0 \in X$, and large enough T (depending explicitly on δ and the injectivity radius of x_0) at least one of the following holds.*

(1) *For every $x \in X_{T^{-\kappa\delta}}$, we have*

$$\mathrm{dist}_X\left(x, B_{T^A}^P.x_0\right) \leq CT^{-\kappa\delta}.$$

(2) *There exists $x' \in X$ such that Hx' is periodic with* $\mathrm{vol}(Hx') \leq T^\delta$, *and*

$$\mathrm{dist}_X(x', x_0) \leq CT^{-1}.$$

Where A, κ, and C are positive constants depending on X.

The proof of Theorem 7.2 has a similar flavor to [49] by Gamburd, Jakobson, and Sarnak as well as to the work of Bourgain and Gamburd [13,14] and the aforementioned work of Bourgain, Furman, Lindenstrauss, and Mozes [12], see Theorem 7.1.

In particular, the three stages of the proof which were discussed in §7.1 are present here as well: in the first step, a Diophantine condition (in the form of a closing lemma) is used to show that unless part (2) in Theorem 7.2 holds, one can produce positive dimension at a certain scale (*initial dimension*). The arithmeticity of Γ is used in this step.

The second step, is the bootstrap phase in the following form: by passing to a larger scale and translating $B^P_{T\delta}.x_0$ with a random element of controlled size, one can obtain a set with *large dimension*. This step is carried out using a Margulis function argument. As it was mentioned before, Margulis functions were introduced in the context of homogeneous dynamics in [37] by Eskin, Margulis, and Mozes, and have become an indispensable tool in homogeneous dynamics and beyond.

The third step is to deduce effective density from large dimension. Two main ingredients are present in this step: first is a projection theorem which is based on the works of Wolff and Schlag [94, 99] and is an adaptation of [61]. This is used to move the additional dimension supplied by the bootstrap phase to the direction of a horospherical subgroup of G. The second ingredient is an argument due to Venkatesh [98] and is based on the following quantitative decay of correlations for the ambient space X: There exists $\kappa_X > 0$ so that

$$\left|\int \varphi(gx)\psi(x)\,\mathrm{d}m_X - \int \varphi\,\mathrm{d}m_X \int \psi\,\mathrm{d}m_X\right| \ll_G \mathcal{S}(\varphi)\mathcal{S}(\psi)e^{-\kappa_X \operatorname{dist}(e,g)} \tag{7.1}$$

for all $\varphi, \psi \in C^\infty(X)$, where $\mathcal{S}$ is a certain Sobolev norm and dist is our fixed right G-invariant metric on G.

See e.g. [64, §2.4] and references there for (7.1); we note that κ_X is an absolute constant if Γ is a congruence subgroup, see [17, 19, 51].

Periodic orbits

The techniques developed in [71] can also be used to prove an effective density theorem for periodic orbits of H.

Let us first recall the following non-divergence result: there exists some $\eta_X > 0$ so that for every periodic orbit Y, we have

$$\mu_Y(X_{\eta_X}) \geq 0.9 \tag{7.2}$$

where μ_Y denotes the H-invariant probability measure on Y, see e.g. [71, Lemma 3.6].

Theorem 7.3 ([71]). *Let $Y \subset X$ be a periodic H-orbit in X. Then for every $x \in X_{\mathrm{vol}(Y)^{-\kappa}}$ we have*

$$\operatorname{dist}_X(x, Y) \leq C\,\mathrm{vol}(Y)^{-\kappa}.$$

Where $\kappa \geq \kappa_X^2/L$ (for an absolute constant L) and C depends explicitly on κ_X, $\mathrm{vol}(X)$, and the minimum of the injectivity radius of points in X_{η_X}. If Γ is congruence, κ is absolute.

If Γ is an arithmetic lattice, Theorem 7.3 is a rather special case of the results we discussed in §5. Note however that Theorem 7.3 does *not* require Γ to be arithmetic — recall that arithmeticity of Γ was only used in the first step of the proof of Theorem 7.2. In particular, unlike [31, 32], Theorem 7.3 does not rely on property (τ).

We also draw the reader's attention to the use of Margulis functions in establishing isolation properties for periodic (or more generally intermediate) orbits in [41] and [83].

We end this exposition with the following application of Theorem 7.3.

Totally geodesic planes in hybrid manifolds

Gromov and Piatetski-Shapiro [56] constructed examples of non-arithmetic hyperbolic manifolds by gluing together pieces of non-commensurable arithmetic manifolds. Let Γ_1 and Γ_2 be two torsion free lattices in $\mathrm{Isom}(\mathbb{H}^3)$ — recall that $\mathrm{I}som(\mathbb{H}^3)$ is an index 2 subgroup of $\mathrm{O}(3,1)$ and that $\mathrm{SL}_2(\mathbb{C})$ is locally isomorphic to $\mathrm{O}(3,1)$. Let $M_i = \mathbb{H}^3/\Gamma_i$. Assume further that for $i = 1, 2$, there exists 3-dimensional submanifolds with boundary $N_i \subset M_i$ so that

- The Zariski closure of $\pi_1(N_i) \subset \Gamma_i$ contains $\mathrm{O}(3,1)^\circ$ where $\mathrm{O}(3,1)^\circ$ is the connected component of the identity in $\mathrm{O}(3,1)$.
- Every connected component of ∂N_i is a totally geodesic embedded surface in M_i which separates M_i.
- ∂N_1 and ∂N_2 are isometric.

Let M be the manifold obtained by gluing N_1 and N_2 using the isometry between ∂N_1 and ∂N_2. Then M carries a complete hyperbolic metric; thus, we consider $\pi_1(M)$ as a lattice in $\mathrm{O}(3,1)$. Let $\Gamma' = \pi_1(M) \cap \mathrm{O}(3,1)^\circ$, and let Γ denote the inverse image of Γ' in $G = \mathrm{SL}_2(\mathbb{C})$.

If Γ_1 and Γ_2 are arithmetic and non-commensurable, then M is non-arithmetic, i.e., Γ is a non-arithmetic lattice in G. A totally geodesic plane in M lifts to a periodic orbit of $H = \mathrm{SL}_2(\mathbb{R})$ in $X = G/\Gamma$.

Theorem 7.4. *Let M be a hyperbolic 3-manifold obtained by gluing the pieces N_1 and N_2 from non-commensurable arithmetic manifolds along $\Sigma = \partial N_1 = \partial N_2$ as described above. The number of totally geodesic planes in M is at most*

$$L\Big(\mathrm{area}(\Sigma)\,\mathrm{vol}(X)\eta_X^{-1}\kappa_X^{-1}\Big)^{L/\kappa_X^2}$$

where L is absolute and $X = G/\Gamma$ is as above.

In qualitative form, this finiteness theorem was proved by Fisher, Lafont, Miller, and Stover [44, Thm. 1.4], see also [4, §12].

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