EFFECTIVE EQUIDISTRIBUTION FOR SOME ONE PARAMETER UNIPOTENT FLOWS

E. LINDENSTRAUSS, A. MOHAMMADI, AND Z. WANG

Abstract. We prove effective equidistribution theorems, with polynomial error rate, for orbits of the unipotent subgroups of $\mathrm{SL}_2(\mathbb{R})$ in arithmetic quotients of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

The proof is based on the use of a Margulis function, tools from incidence geometry, and the spectral gap of the ambient space.

Contents

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1. Introduction

A landmark result of Ratner [Rat91b] states that if G is a Lie group, Γ a lattice in G and if u_t is a one-parameter Ad-unipotent subgroup of G, then for **any** $x \in G/\Gamma$ the orbit $u_t.x$ is equidistributed in a periodic orbit of some subgroup $L < G$ that contains both the one parameter group u_t and the initial point x. We say an orbit $L.x$ of a group L in some space X is periodic if the stabilizer of x in L is a lattice in L, equivalently that the stabilizer of x in L is discrete and $L.x$ supports a unique L-invariant probability measure $m_{L.x}$; and $u_t.x$ is equidistributed in $L.x$ in the sense that

$$\frac{1}{T}\int_0^T f(u_t.x)dt \to \int f dm_{L.x} \qquad \text{for any } f \in C_0(G/\Gamma). \tag{1.1}$$

In order to prove this equidistribution result, Ratner first classified the u_t-invariant probability measures on G/Γ [Rat90, Rat91a]; the proof also uses the non-divergence properties of unipotent flows established by Dani and Margulis [Mar71, Dan84, Dan86].

In this paper we prove a quantitative equidistribution result for orbits of a one parameter unipotent group on quotients G/Γ where G is either $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ with a polynomial error rate, which is the first quantitative equidistribution statement for individual orbits of unipotent flows on quotients of semi-simple groups beyond the horospherical case. Our approach builds on the paper [LM21] by the first two authors, where an effective density result with a polynomial rate for orbits of a Borel subgroup of a subgroup $H \simeq \mathrm{SL}_2(\mathbb{R})$ of G was proved.

Recall that a group $N < G$ is horospheric if there is some $g \in G$ so that

$$N = \{h \in G : g^{-n}hg^n \to 1 \text{ as } n \to \infty\}.$$

For instance, the one parameter unipotent group

$$\left\{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R}\right\}$$

is horospheric in $\mathrm{SL}_2(\mathbb{R})$ as are the groups

$$\left\{\begin{pmatrix} 1 & r+is \\ 0 & 1 \end{pmatrix} : r, s \in \mathbb{R}\right\} \quad \text{and} \quad \left\{\left(\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) : r, s \in \mathbb{R}\right\}$$

in $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, respectively. The classification of invariant measures and orbit closures for horospherical flows was established prior to Ratner's work by Hedlund, Furstenberg, Dani, Veech and others, and this has been understood for some time also quantitatively since one can relate the distribution properties of individual N orbits to the ergodic theoretic properties of the action of g on G/Γ (cf. §5 for more details).

The non-horospheric case, on the other hand, is much more delicate, and proving a quantitative form of Ratner's theorem regarding equidistribution of unipotent orbits has been a major challenge. We survey below in §1.4 what was known before our work as well as some very recent developments that have taken place after these results have been announced.

To state our main results we first fix some notations. Let

$$G = \mathrm{SL}_2(\mathbb{C}) \quad \text{or} \quad G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Let $\Gamma \subset G$ be a lattice, and put $X = G/\Gamma$. We let m_X denote the G-invariant probability measure on X. Throughout the paper, we will denote by H a subgroup of G isomorphic to $\mathrm{SL}_2(\mathbb{R})$, namely

$$\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C}) \quad \text{or} \quad \{(g, g) : g \in \mathrm{SL}_2(\mathbb{R})\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

For all $t, r \in \mathbb{R}$, let a_t and u_r denote the image of

$$\begin{pmatrix} e^{t/2} & o \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},$$

in H, respectively.

We fix maximal compact subgroups $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SO}(2)\times\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Let d be the right invariant metric on G which is defined using the Killing form and the aforementioned maximal compact subgroups. This metric induces a metric d_X on X, and natural volume forms on X and its submanifolds. We define the injectivity radius of a point $x \in X$ using this metric. In the sequel, $\| \ \|$ denotes the maximum norm on $\mathrm{Mat}_2(\mathbb{C})$ or $\mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R})$ with respect to the standard basis.

Our main result is the following:

1.1. **Theorem.** *Assume Γ is an arithmetic lattice. For every $x_0 \in X$, and large enough R (depending explicitly on X and the injectivity radius of x_0), for any $T \geq R^A$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\Big|\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi) R^{-\kappa_1}$$

where $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ such that $H.x$ is periodic with $\mathrm{vol}(H.x) \leq R$, and

$$d_X(x, x_0) \leq R^A (\log T)^A T^{-1}.$$

The constants A and κ_1 are positive and depend on X but not on x_0.

Theorem 1.1 can be viewed as an effective version of [Sha96, Thm. 1.4]. Combining Theorem 1.1 and the Dani–Margulis linearization method [DM91] (cf. also Shah [Sha91]), that allows to control the amount of time a unipotent trajectory spends near invariant subvarieties of a homogeneous space, we also obtain an effective equidistribution theorem for long pieces of unipotent orbits (more precisely, we use a sharp form of the linearization method taken from [LMMS19]).

1.2. **Theorem.** *Assume Γ is an arithmetic lattice. For every $x_0 \in X$ and large enough R (depending explicitly on X), for any $T \geq R^{A_1}$, at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\Big|\frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi) R^{-\kappa_2}$$

where $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) There exists $x \in G/\Gamma$ with $\mathrm{vol}(H.x) \leq R^{A_1}$, and for every $r \in [0,T]$ there exists $g \in G$ with $\|g\| \leq R^{A_1}$ so that

$$d_X(u_s x_0, gH.x) \leq R^{A_1}\left(\frac{|s-r|}{T}\right)^{1/A_2} \qquad \text{for all } s \in [0,T].$$

(3) For every $r \in [0,T]$ and $t \in [\log R, \log T]$, the injectivity radius at $a_{-t}u_r x_0$ is at most $R^{A_1}e^{-t}$.

The constants A_1, A_2, and κ_2 are positive, and depend on X but not on x_0.

The assumption in Theorem 1.1, that Γ is arithmetic, may be relaxed. Let us say Γ has *algebraic entries* if the following is satisfied: there is a number field F, a semisimple F-group $\mathbf{G}$ of adjoint type, and a place v of F so that $F_v = \mathbb{R}$ and $\mathbf{G}(F_v)$ and G are locally isomorphic — in which case there is a surjective homomorphism from G onto the connected component of the identity in $\mathbf{G}(F_v)$ — and the image of Γ in $\mathbf{G}(F_v)$ (possibly after conjugation) is contained in $\mathbf{G}(F)$. Every arithmetic lattice has algebraic entries, but there are lattices with algebraic entries that are not arithmetic.

Note that the condition that Γ has *algebraic entries* is automatically satisfied if Γ is an irreducible lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ or if $G = \mathrm{SL}_2(\mathbb{C})$. Indeed, by arithmeticity theorems of Selberg and Margulis, irreducible lattices in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ are arithmetic [Mar91, Ch. IX]. Moreover, by local rigidity, lattices in $\mathrm{SL}_2(\mathbb{C})$ always have algebraic entries [GR70, Thm. 0.11] (see also [Sel60, Wei60, Wei64]).

1.3. **Theorem.** *Assume Γ is a lattice which has algebraic entries. For every $0 < \delta < 1/4$, every $x_0 \in X$ and large enough T (depending explicitly on X, δ and the injectivity radius of x_0) at least one of the following holds.*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\Big|\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi) T^{-\delta^2\kappa_3}$$

where $\mathcal{S}(\varphi)$ is a certain Sobolev norm.

(2) There exists $x \in X$ with

$$d_X(x, x_0) \leq T^{-1/A'},$$

satisfying the following: there are elements γ_1 and γ_2 in $\mathrm{Stab}_H(x)$ with $\|\gamma_i\| \leq T^\delta$ for $i = 1,2$ so that the group generated by $\{\gamma_1, \gamma_2\}$ is Zariski dense in H.

The constants A' and κ_3 are positive, and depend on X but not on δ and x_0.

The obstacle to effective equidistribution in Theorem 1.1 is much cleaner and simpler than in Theorem 1.2. This is not an artifact of the proof but a reflection of reality; a unipotent orbit may fail to equidistribute at the expected rate without it staying near a single period orbit of some subgroup $\{u_t\} < L < G$: one must allow a slow drift of the periodic orbit in the direction of the centralizer of u_t. Unlike the work of Shah in [Sha96], where (in particular) a non-effective version of Theorem 1.1 is proved relying on Ratner's measure classification theorem for unipotent flows, our proof goes the other way, first establishing Theorem 1.1, and then deduce Theorem 1.2 from it using a linearization and non-divergence argument.

These results have been announced in [LMW22], as well as in a series of three talks at the IAS in Princeton in February 2022[1]. The announcement [LMW22] also contains an overview of the argument; the reader may find it useful to consult [LMW22] before (or while) reading the full version.

1.4. **Background and further discussion.** Ratner's equidistribution theorem implies a corresponding orbit closure classification theorem. Answering a conjecture of Raghunathan, Ratner deduced from the equidistribution theorem a classification of orbit closures: if G is a Lie group, Γ a lattice in G, and if $H < G$ is generated by one parameter Ad-unipotent subgroups of G, then for any $x \in G/\Gamma$ one has that $\overline{H.x} = L.x$ where $H \leq L \leq G$ and $L.x$ is periodic. Important special cases of Raghunathan's conjecture were proven earlier by Margulis and by Dani and Margulis using a different more direct approach, which in particular gave a proof of a rather strong form of the longstanding Oppenheim conjecture [Mar89, DM89, DM90]. The rigidity properties of unipotent flows have had many other surprising applications to number theory, from equidistribution to counting integer points and even regarding nonvanishing of central values of L-functions, as well as many other areas. Already the cases we study here, e.g., the action of u_t on $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is of interest to some number theoretic implications (e.g. [SU15, BSZ13]).

Both because of its intrinsic interest, but especially in view of the applications, obtaining quantitative versions of equidistribution results for unipotent flows has been a well known open problem (cf. [Mar00, §1.3], in particular problem 7 there, or [Gor07, Ques. 17]).

As mentioned above, the equidistribution of orbits of horospheric groups is by now well understood, in part using the relation between studying individual orbits of horospheric groups and mixing properties of a corresponding diagonalizable group. The first work in this direction we are aware of is Sarnak [Sar81] who studied periodic orbits of the horocycle flow. Burger

[1] https://www.ias.edu/video/effective-equidistribution-some-one-parameter-unipotent-flows-polynomial-rates-i-ii

[Bur90] gave a general effective treatment for quotients of $\mathrm{SL}_2(\mathbb{R})$ (even in some infinite volume cases). In [KM96], Kleinbock and Margulis use a quantitaive equidistribution result for expanding translates of orbits of horospheric groups [KM96, Proposition 2.4.8]. More recent papers in the topic include the work of Flaminio and Forni [FF03], Strömbergsson [Str13], and Sarnak and Ubis [SU15]. Quantitative horospheric equidistribution has now been established in much greater generality e.g. by Kleinbock and Margulis in [KM12], McAdam in [McA19] and by Asaf Katz [Kat19]. Moreover a quantitative equidistribution estimate twisted by a character was proved by Venkatesh [Ven10] and further developed by Tanis and Vishe as well as Flaminio, Forni, and Tanis [TV15, FFT16]; this was generalized to a disjointness result with a general nil-system by Asaf Katz in [Kat19]. Closely related is the case of translates of periodic orbits of subgroups $L \subset G$ which are fixed by an involution by Duke, Rudnick and Sarnak, Eskin and McMullen, and Benoist and Oh in [DRS93, EM93, BO12].

Unipotent dynamics have a very different flavour when the ambient group G itself is a unipotent group (in which case the study of these flows, e.g. the classification of invariant measures, dates back to work by Leon Green, Parry and others from the late 1960s) on the one extreme and when G is a semisimple group on the other. The case when G is a skew product $G' \ltimes N$ with G' semisimple and N unipotent, with the acting group U projecting to a horospheric subgroup of G', can be viewed as intermediate between these two cases.

- Even when G is unipotent (and G/Γ a nilmanifold) the quantitative behaviour of unipotent flows has only been understood relatively recently by Green and Tao [GT12].
- In the case of quotients of the skew product $G = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$, Strombergsson [Str15] has an effective equidistribution result for one parameter unipotent orbits (which are not horospheric in G, but project to a horospheric group on $\mathrm{SL}_2(\mathbb{R})$), and this has been generalized by several authors, in particular by Wooyeon Kim [Kim21] (using a completely different argument) to $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. The case where G is a direct product $G = G' \times N$ and U projects to a horospheric subgroup of G' is discussed in Katz paper [Kat19].
- Not quite in this framework, but also somewhat of an intermediate case between the case of G semisimple and nilpotent is the study of random walks by automorphisms of the torus or nilmanifold X driven by a probability measure on Aut(X) whose support generates a group with sufficiently large Zariski closure. Here there is a quantitative equidistribution result by Bourgain, Furman, Mozes and the first named author [BFLM11], which was extended by Weikun He and de Saxce [HdS19]. Elements from this proof were used by Wooyen Kim in [Kim21].
- When G is semisimple, there have been some results regarding effective *density* of non-horospherical unipotnet flows. Specifically, for $G/\Gamma =$

$\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ and u_t is the generic one parameter unipotent subgroup a result towards effective density with a logarithmic error term was proved by Margulis and the first named author [LM14] in order to give an effective and quantitative proof of the Oppenheim Conjecture. A more general result in this direction, with iterated logarithmic rate[2], was announced by Margulis, Shah and two of us (E.L. and A.M.) with the first installment of this work appearing in [LMMS19]. An effective density result for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and u_t a one-parameter unipotent (i.e. the case we consider in this paper), with a polynomial rate, was established by the first two named authors [LM21].

- When G is semisimple, there have been some results regarding effective *equidistribution* of *special* orbits of non-horospherical groups generated by unipotents. In particular we note the work of Einsiedler, Margulis and Venkatesh [EMV09] showing that periodic orbits of semisimple subgroups H of a semisimple group G are quantitatively equidistributed in an appropriate homogeneous subspace of G/Γ if Γ is a congruence lattice and H has finite centralizer in G. Subsequently Einsiedler, Margulis, Venkatesh and the second named author by using Prasad's volume formula and a more adelic view point were able to prove such an equidistribution result for periodic orbits of maximal semisimple subgroups of G when the subgroup is allowed to vary [EMMV20] with arithemetic applications. The equidistribution of periodic orbits of semisimple groups is also closely connected to the equidistribution of Hecke points; a quantitative treatment of such equidistribution was given by Clozel, Oh and Ullmo in [COU01].

  In a different direction, but also under this general heading we note the paper of Chow and Lei Yang [CY19] which deals with expanding translates of special 1-parameter unipotent orbits, with applications to diophantine approximations.
- For G semisimple and U a nonhorospheric unipotent group there were no quantitative equidistribution results known, with any rate, before our work (certainly not for a one parameter group U; but see e.g. [Ubi17] for a related result in an "almost horospheric" situation). Our work was announced in [LMW22]. While we were working on finishing this paper Lei Yang posted a very interesting preprint treating another nonhorospheric case [Yan22] — the case of trajectories of a non-generic one-parameter unipotent group on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. That paper uses some elements common with our approach (e.g. a similar closing lemma as a starting point and a similar last stage), but the critical dimension increment phase seems to be done quite differently. We note that the case treated by Lei Yang in that paper is the same case for which Chow and Yang proved equidistribution for translates of special orbits in [CY19].

An extremely interesting analogue to unipotent flows on homogeneous spaces is given by the action of $\mathrm{SL}_2(\mathbb{R})$ and its subgroups on strata of abelian

[2] I.e. very far from the right kind of dependence which should be polynomial.

differentials. Let $g \geq 1$, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a partition of $2g-2$. Let $\mathcal{H}(\alpha)$ be the corresponding stratum of abelian differentials, i.e., the space of pairs (M, ω) where M is a compact Riemann surface with genus g and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1, \ldots, \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities and a natural area from. Let $\mathcal{H}_1(\alpha)$ be the space of unit area surfaces in $\mathcal{H}(\alpha)$. The space $\mathcal{H}(\alpha)$ admits a natural action of $\mathrm{SL}_2(\mathbb{R})$; this action preserves the unit area hyperboloid $\mathcal{H}_1(\alpha)$.

A celebrated theorem of Eskin and Mirzakhani [EM18] shows that any P-invariant ergodic measure is $\mathrm{SL}_2(\mathbb{R})$-invariant and is supported on an affine invariant manifold, where P denotes the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$. We shall refer to these measures as *affine invariant measures.* Moreover, if we define, for any interval $I \subset \mathbb{R}$ and $x \in \mathcal{H}_1(\alpha)$, the probability measure μ_I^x on $\mathcal{H}_1(\alpha)$ by

$$\mu_I^x = |I|^{-1} \int_I \delta_{u_s x} \, \mathrm{d}s,$$

then Eskin, Mirzakhani and the second named author [EMM15] showed that for any $x \in \mathcal{H}_1(\alpha)$ the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} a_t \mu_{[0,1]}^x \, \mathrm{d}t \qquad \text{exists in weak}^* \text{ sense} \tag{1.2}$$

and is equal to an ($\mathrm{SL}_2(\mathbb{R})$-invariant) affine invariant probability measure with x in its support. On the other hand, there are several results, in particular by Chaika, Smillie and B. Weiss in [CSW20], that show that an analogue of Ratner's equidistribution theorem (or our Theorem 1.2) fails to hold in this setting, for instance for some x the sequence of measure $\mu_{[0,T]}^x$ may fail to converge as $T \to \infty$, or may converge to a non-ergodic measure. However the following conjecture of Forni seems to us very plausible:

1.5. **Conjecture** ([For21, Conj. 1.4])**.** *Let $\mathcal{H}_1(\alpha)$ be the space of unit area surfaces in stratum of abelian differentials on a genus g surface whose zeros have multiplicities given by $\alpha = (\alpha_1, \ldots, \alpha_n)$, and let $x \in \mathcal{H}_1(\alpha)$. Then $\lim_{t \to \infty} a_t \mu_{[0,1]}^x$ exists in the weak* sense and is equal to an affine invariant measure with x in its support.*

Of course, once one establishes that $\lim_{t \to \infty} a_t \mu_{[0,1]}^x$ exists, the rest follows from [EMM15]. In this context again obtaining quantitative equidistribution results would be very interesting.

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2. The main steps of the proofs

As mentioned above, Theorem 1.2 is proved by combining Theorem 1.1 and the linearization techniques [DM91] in their quantitative form [LMMS19], see §16 for details. We note that the idea of using equidistribution of expanding translates of a fixed piece of a U orbit of the type $\{a_t u_s.x : 0 \le s \le 1\}$ to deduce equidistribution of a large segment of a non-translated U orbit $\{u_s.x : 0 \le s \le T\}$ is quite classical.

Let us now highlight some of the main ingredients used in the proof of Theorem 1.1. Assume that part (2) in Theorem 1.1 fails for x_0, T, and R as the proof is complete otherwise. We begin with a version of avoidance principle á la linearization techniques of Dani–Margulis albeit for random walks.

Roughly speaking, the following proposition asserts that failure of part (2) in Theorem 1.1 may be upgraded to a Diophantine estimate with a polynomial rate (whose degree is absolute) in terms of R. We will let $\mathrm{inj}(x)$ denote (our slightly modified) injectivity radius of x, see §3 and §4.1.

2.1. **Proposition.** *There exist D_0 (absolute) and C_1, s_0 (depending on X) so that the following holds. Let $R, S \ge 1$. Suppose $x_0 \in X$ is so that*

$$d_X(x_0, x) \ge (\log S)^{D_0} S^{-1}$$

for all x with $\mathrm{vol}(Hx) \le R$. Then for all

$$s \ge \max\bigl\{\log S, 2|\log(\mathrm{inj}(x_0))|\bigr\} + s_0$$

and all $0 < \eta \le 1$, we have

$$\left|\left\{r \in [0,1] \colon \begin{array}{c} \mathrm{inj}(a_s u_r x_0) \le \eta \textit{ or there is } x \textit{ with} \\ \mathrm{vol}(Hx) \le R \textit{ s.t. } d_X(a_s u_r x_0, x) \le \frac{1}{C_1 R^{D_0}} \end{array}\right\}\right| \le C_1(\eta^{1/2} + R^{-1}).$$

The proof of this proposition uses *Margulis functions* for periodic H-orbits and is completed in Appendix A, see also §4.5 for more details.

We will apply this proposition with $\eta = R^{-\star}$ where $\star$ is a small constant. In view of this proposition and the fact that part (2) in Theorem 1.1 does not hold, for all but a set with measure $\ll R^{-\star}$ of $r \in [0,1]$, the point $x_1 = a_s u_r x_0$ (where $s = \log T - C \log R$ for appropriate choice of C) satisfy

$$\mathrm{inj}(x_1) \ge \eta \quad \text{and} \quad d(x, x_1) \ge R^{-D_0} \text{ for every } x \text{ with } \mathrm{vol}(Hx) \le R. \tag{2.1}$$

Thus, in order to show that $\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r$ is within $R^{-\star}$ of $\int \varphi\,\mathrm{d}m$, it suffices to show that $\int_0^1 \varphi(a_{C\log R} u_r x_1)\,\mathrm{d}r$ is within $R^{-\star}$ of $\int \varphi\,\mathrm{d}m$ where x_1 satisfies (2.1). We will show this statement in three phases.

A closing lemma and the initial dimension. In this phase, we show that the improved Diophantine condition (2.1) for x_1 implies that points in $\{a_{\star \log R} u_r x_0 : r \in [0,1]\}$ (possibly after removing an exceptional set of measure $\ll R^{-\star}$) are *separated transversal to* H.

Let $t > 0$ be a large parameter, and fix some $e^{-0.01t} < \beta = e^{-\kappa t}$ (in our application, κ will be chosen to be $\ll 1/D_0$ where the implied constant depends on X and D_0 is as in Proposition 4.6, moreover, we will assume $\beta = \eta^2$ in that proposition).

For every $\tau \geq 0$, put

$$\mathsf{E}_\tau = \mathsf{B}^{s,H}_\beta \cdot a_\tau \cdot \{u_r : r \in [0,1]\} \subset H,$$

where $\mathsf{B}^{s,H}_\beta := \{u^-_s : |s| \leq \beta\} \cdot \{a_t : |t| \leq \beta\}$ and u^-_s is the transpose of u_s.

Let $\mathfrak{g} = \mathrm{Lie}(G)$, that is, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$. Let $\mathfrak{r} = i\mathfrak{sl}_2(\mathbb{R})$ if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{r} = \mathfrak{sl}_2(\mathbb{R}) \oplus \{0\}$ if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$. In either case $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ where $\mathfrak{h} = \mathrm{Lie}(H) \simeq \mathfrak{sl}_2(\mathbb{R})$, and both $\mathfrak{h}$ and $\mathfrak{r}$ are $\mathrm{Ad}(H)$-invariant.

Let $\tau \geq 0$ and $y \in X$. Assume that $\mathsf{h} \mapsto \mathsf{h}y$ is injective over E. For every $z \in \mathsf{E}_\tau.y$, put

$$I_\tau(z) := \big\{ w \in \mathfrak{r} : \|w\| < \mathrm{inj}(z) \text{ and } \exp(w)z \in \mathsf{E}_\tau.y \big\};$$

this is a finite subset of $\mathfrak{r}$ since E_τ is bounded — we will define $I_{\mathcal{E}}(h,z)$ for all $h \in H$ and more general sets $\mathcal{E}$ in the bootstrap phase below.

Let $0 < \alpha < 1$. Define the function $f_\tau : \mathsf{E}_\tau.y \to [1,\infty)$ as follows

$$f_\tau(z) = \begin{cases} \sum_{0 \neq w \in I_\tau(z)} \|w\|^{-\alpha} & \text{if } I_\tau(z) \neq \{0\} \\ \mathrm{inj}(z)^{-\alpha} & \text{otherwise} \end{cases}.$$

2.2. **Proposition.** *Assume Γ is arithmetic. There exists D_1 (which depends on Γ explicitly) satisfying the following. Let $D \geq D_1$ and $x \in X$. Then for all large enough t at least one of the following holds.*

(1) There is a subset $I(x) \subset [0,1]$ with $|[0,1] \setminus I(x)| \ll_X \beta^{1/4}$ such that for all $r \in I(x)$ we have the following
  - *(a) $\mathrm{inj}(a_{8t} u_r x) \geq \beta^{1/2}$.*
  - *(b) $\mathsf{h} \mapsto \mathsf{h}.a_{8t} u_r x$ is injective over E_t.*
  - *(c) For all $z \in \mathsf{E}_t.a_{8t} u_r x$, we have*

$$f_t(z) \leq e^{Dt}.$$

(2) There is $x' \in X$ such that Hx' is periodic with

$$\mathrm{vol}(Hx') \leq e^{D_1 t} \quad \textit{and} \quad d_X(x', x) \leq e^{(-D+D_1)t}.$$

This proposition will be proved in §4.7. We also refer to that section for discussions regarding the assumption that Γ is arithmetic.

For the rest of the argument, let $t = \frac{1}{D_1} \log R$, where R is as in Theorem 1.1, and let x_1 be as in (2.1). Apply Proposition 2.2 with the point x_1. Then for every $r_1 \in I(x_1)$, the conclusions in part (1) of that proposition holds for $x_2 = a_{8t}u_{r_1}x_1$. That is, $h \mapsto hx_2$ is injective over E_t and the transverse dimension of $\mathsf{E}_t.x_2$ is $\geq 1/D$ for all

$$x_2 \in \{a_{8t}u_{r_1}x_1 : r_1 \in I(x_1)\} \tag{2.2}$$

where $D = D_0D_1+2D_1$. Therefore, in order to show that $\int_0^1 \varphi(a_{C\log R}u_r x_1)\,\mathrm{d}r$ is within $R^{-\star}$ of $\int \phi$, it is enough to show a similar estimate for

$$\int_0^1 \varphi(a_{C\log R-8t}u_r x_2)\,\mathrm{d}r$$

for all x_2 as in (2.2).

Improving the dimension. Roughly speaking, Proposition 2.2 states that the set $\{a_{8t}u_r x_1 : r \in [0,1]\}$ has transversal dimension $1/D$. In this step, we will improve this dimension to reach at dimension α, close to 1.

We need some notation. Recall that $t = \frac{1}{D_1}\log R$. Let $\beta = e^{-\kappa t}$ for some small $\kappa > 0$. (More explicitly, we will fix some $0 < \varepsilon \leq 10^{-8}$ to be explicated later, and let $\kappa = 10^{-6}\varepsilon/(2D)$, $D = D_0D_1 + 2D_1$). Let

$$\mathsf{E} = \mathsf{B}_\beta^{s,H} \cdot \{u_r : |r| \leq \eta_0\}.$$

It will be more convenient to *approximate* translations

$$\{a_\bullet u_r x_0 : r \in [0,1]\}$$

with sets which are a disjoint union of local E-orbits as we now define. Let $F \subset B_{\mathfrak{r}}(0,\beta)$ be a finite set with $\#F \geq e^{t/2}$, and let $y \in X$ with $\mathrm{inj}(y) \geq \beta^{1/2}$. Put

$$\mathcal{E} = \bigcup \mathsf{E}.\{\exp(w)y : w \in F\}. \tag{2.3}$$

For every $w \in F$, we let μ_w be a measure which is absolutely continuous with respect to the pushforward of the Haar measure $m_H|_{\mathsf{E}}$ to $\mathsf{E}.\exp(w)y$ whose density satisfies certain Lipschitz condition, see §7.6 for more details. We equip $\mathcal{E}$ with the probability measure $\mu_{\mathcal{E}}$ proportional to $\sum_w \mu_w$.

Let θ be a small constant; in our application, the exact choice of θ will depending on the decay of matrix coefficients in G/Γ, see (2.8). Let

$$\alpha = 1-\theta \quad \text{and} \quad \varepsilon = \theta^2.$$

Let $\ell = 0.01\varepsilon t$, and let ν_ℓ be the probability measure on H defined by

$$\nu_\ell(\varphi) = \int_0^1 \varphi(a_\ell u_r)\,\mathrm{d}r \qquad \text{for all } \varphi \in C_c(H);$$

let $\nu_\ell^{(n)} = \nu_\ell * \cdots * \nu_\ell$ denote the n-fold convolution of ν_ℓ for all $n \in \mathbb{N}$.

The following proposition is one of main steps in the proof.

2.3. **Proposition.** *Let $x_1 \in X$, and assume that Proposition 2.2(2) does not hold for D, x_1, and t. Let*

$$J := [d_2, d_1] \cap \mathbb{N},$$

where $d_1 = 100\lceil\frac{4D-3}{2\varepsilon}\rceil$ *and* $d_2 = d_1 - \lceil 10^4\varepsilon^{-1/2}\rceil$.

Let $r_1 \in I(x_1)$, *see Proposition 2.2(1), and put* $x_2 = a_{8t}u_{r_1}x_1$. *For every* $d \in J$, *there is a collection* $\Xi_d = \{\mathcal{E}_{d,i} : 1 \le i \le N_d\}$ *of sets*

$$\mathcal{E}_{d,i} = \mathsf{E}.\{\exp(w)y_{d,i} : w \in F_{d,i}\},$$

with $F_{d,i} \subset B_{\mathfrak{r}}(0,\beta)$ *and* $\mathrm{inj}(y_{d,i}) \ge \beta^{1/2}$, *and admissible measures* $\mu_{\mathcal{E}_{d,i}}$, *see* §*7.6, so that both of the following hold:*

(1) Put $b = e^{-\sqrt{\varepsilon}t}$. *Let* $d \in J$, $1 \le i \le N_d$, *and let* $w_0 \in B_{\mathfrak{r}}(0,\beta)$. *Then for every* $w \in B_{\mathfrak{r}}(w_0, b)$ *and all* $\delta \ge e^{-t/2}$, *we have*

$$\frac{\#\big(B_{\mathfrak{r}}(w,\delta)\cap B_{\mathfrak{r}}(w_0,b)\cap F_{d,i}\big)}{\#\big(B(w_0,b)\cap F_{d,i}\big)} \le e^{\varepsilon t}(\delta/b)^{\alpha}. \tag{2.4}$$

(2) For all $s \le t$ *and all* $r \in [0,2]$, *we have*

$$\int \varphi(a_su_rz)\,\mathrm{d}\nu_\ell^{(d_1)} * \mu_{\mathsf{E}_t.x_2}(z) = \sum_{d,i} c_{d,i}\int \varphi(a_su_rz)\,\mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z) + O(\mathrm{Lip}(\varphi)\beta^{\kappa_4}) \tag{2.5}$$

where $\varphi \in C_c^\infty(X)$, $c_{d,i} \ge 0$ *and* $\sum_{d,i} c_{d,i} = 1 - O(\beta^{\kappa_4})$, $\mathrm{Lip}(\varphi)$ *is the Lipschitz norm of* φ, *and* κ_4 *and the implied constants depend on* X.

Roughly speaking, the proposition states that up to an exponentially small error, $\nu_\ell^{(d_1)} * \mu_{\mathsf{E}_t.x_1}$ may be decomposed as $\sum_{d,i} c_{d,i}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}$ where $\sum_{d,i} c_{d,i} = 1 - O(\beta^{\kappa_4})$ (see (2.5)) and for all $d \in J$ and $1 \le i \le N_d$ the dimension of $\mathcal{E}_{d,i}$ transversal to H at controlled scales is $\ge \alpha$ (see (2.4)). See Proposition 10.1 for a more precise formulation which relies on a *Modified Margulis function*. The proof of Proposition 10.1 (and hence of Proposition 2.3) will be completed in §10–12.

Using this proposition we further reduce the analysis to equidistribution of sets $\mathcal{E}$ satisfying part (1) in Proposition 2.3: Let $s = 2\sqrt{\varepsilon}t$ (note that this is much larger than $\ell = 0.01\varepsilon t$ but much smaller than t). Then

$$\int_0^1 \varphi(a_{s+d_1\ell+t}u_rx_2)\,\mathrm{d}r$$

is within $R^{-\star}$ of

$$\int_0^1\int \varphi(a_su_rz)\,\mathrm{d}\nu_\ell^{(d_1)} * \mu_{\mathsf{E}_t.x_2}(z)\,\mathrm{d}r.$$

We now use Proposition 2.3 to improve the small transversal dimension from $1/D$ to α. More precisely, Proposition 2.3 shows that

$$\int_0^1\int \varphi(a_su_rz)\,\mathrm{d}\nu_\ell^{(d_1)} * \mu_{\mathsf{E}_t.x_2}(z)\,\mathrm{d}r$$

is within $R^{-\star}$ of a convex combination of integrals of the form

$$\int_0^1 \int \varphi(a_s u_r z)\,\mathrm{d}\nu_\ell^{(n)} * \mu_{\mathcal{E}}(z)\,\mathrm{d}r \tag{2.6}$$

where $0 \le n = d_1 - d \le 10^4 \varepsilon^{-1/2}$ and $\mathcal{E} = \mathcal{E}_{d,i}$ has dimension at least α transversal to H at controlled scales, see (2.4).

From large dimension to equidistribution. In this final step of the argument, we will show that (2.6) equidistributes so long as θ (recall that $\alpha = 1 - \theta$) is chosen carefully.

Let begin with the following quantitative decay of correlations for the ambient space X: There exists $0 < \kappa_0 \le 1$ so that

$$\left|\int \varphi(gx)\psi(x)\,\mathrm{d}m_X - \int \varphi\,\mathrm{d}m_X \int \psi\,\mathrm{d}m_X\right| \ll \mathcal{S}(\varphi)\mathcal{S}(\psi)e^{-\kappa_0 d(e,g)} \tag{2.7}$$

for all $\varphi, \psi \in C_c^\infty(X) + \mathbb{C}\cdot 1$, where m_X is the G-invariant probability measure on X and d is our fixed right G-invariant metric on G. See, e.g., [KM96, §2.4] and references there for (2.7); we note that κ_0 is absolute if Γ is a congruence subgroup. This is known in much greater generality, but the cases relevant to our paper are due to Selberg and Jacquet-Langlands [Sel65, JL70].

The quantitative decay of correlation can be used to establish quantitative results regarding the equidistribution of translates of pieces of an N-orbit. Specifically we employ the results in [KM96], but there is rich literature around the subject; a more complete list can be found in §1.4.

Now let $\xi : [0,1] \to \mathfrak{r}$ be a smooth non-constant curve. Then using the quantitative results regarding equidistribution of translates of pieces of an N-orbit such as [KM96], one can show that for every $x \in X$,

$$a_\tau\big\{u_r \exp(\xi(s)).x : r, s \in [0,1]\big\}$$

is equidistributed in X as $\tau \to \infty$ (with a rate which is polynomial in $e^{-\tau}$). The key point in the deduction of this equidistribution result from the equidistribution of shifted N orbits is that conjugation by a_τ moves $u_r \exp(\xi(s))$ to the direction of N, hence the above average essentially reduces to an average on a N orbit.

Roughly speaking, the following proposition states that one may replace the curve $\{\xi(s) : s \in [0,1]\}$ with a measure on $\mathfrak{r}$ so long as the measure has dimension $\ge 1 - \theta$, for an appropriate choice of θ depending on κ_0.

The precise formulation is the following.

2.4. **Proposition.** *For any $\theta > 0$ and $c > 0$ there is a κ_5 so that the following holds: Let $0 < b_0 < 10^{-6}$, and let $F \subset B_{\mathfrak{r}}(0, b_0)$ be a finite set satisfying*

$$\frac{\#(F \cap B_{\mathfrak{r}}(0,\delta))}{\#F} \le b_1^{-c}\big(\delta/b_0\big)^{1-\theta} \quad \text{for all } \delta \ge b_1$$

where $b_1 < b_0^{10}$.

Then for all $x \in X$ *with* $\mathrm{inj}(x) \geq b_0^{1/20}$*, all* $|\log(b_0)| \leq \tau \leq \frac{1}{10}|\log(b_1)|$*, and every* $\varphi \in C_c^\infty(X)$*, we have*

$$\Big|\int_0^1 \frac{1}{\#F}\sum_{w\in F}\varphi(a_\tau u_r \exp(w)x)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \ll_X \mathcal{S}(\varphi)\max\Big((b_1/b_0)^{\kappa_5}, b_1^{-2c}e^{2\tau\theta}b_0^{\kappa_0^2/M}\Big),$$

where $\mathcal{S}(\varphi)$ *is a certain Sobolev norm and* M *an absolute constant.*

The proof of this proposition is significantly more delicate than that of the "toy version" of a shifted curve, and relies on an adaptation of a projection theorem due to Käenmäki, Orponen, and Venieri [KOV17], based on the works of Wolff [Wol00], Schlag [Sch03], and [Zah12a], in conjunction with a sparse equidistribution argument due to Venkatesh [Ven10]. These elements also played a crucial role in previous work by E.L. and A.M. [LM21] regarding quantitative density for the action of AU on the spaces we consider here. A slightly modified statement and the proof are given in §13, see in particular Proposition 13.1.

We now use this proposition and outline the last step in the proof of Theorem 1.1: Using the above notation, fix θ and ε as follows

$$0 < \theta < 10^{-8}\kappa_0^2/M \quad\text{and}\quad \varepsilon = \theta^2. \tag{2.8}$$

Recall that $s = 2\sqrt{\varepsilon}t$. In view of (2.6), it now suffices to show that

$$\int_0^1\int \varphi(a_s u_r z)\,\mathrm{d}\nu_\ell^{(n)} * \mu_{\mathcal{E}}(z)\,\mathrm{d}r$$

is within $R^{-\star}$ of $\int\varphi\,\mathrm{d}m_X$ for all $\mathcal{E}$ and n as above. We will use Proposition 2.4 to show this. First note that

$$\int_0^1\int \varphi(a_s u_r z)\,\mathrm{d}\nu_\ell^{(n)} * \mu_{\mathcal{E}}(z)\,\mathrm{d}r$$

is within $R^{-\star}$ of

$$\int\int_0^1 \varphi(a_{s+n\ell}u_r z)\,\mathrm{d}r\,\mathrm{d}\,\mu_{\mathcal{E}}(z).$$

Moreover, we have

$$2\sqrt{\varepsilon}t \leq s + n\ell \leq 2\sqrt{\varepsilon}t + \frac{10^4\ell}{\sqrt{\varepsilon}} = 102\sqrt{\varepsilon}t;$$

in view of our choice of θ the right most term in the above series of inequalities is $\leq (10^{-5}\kappa_0^2/M)t$. Thus, Proposition 2.4, applied with $\theta = \sqrt{\varepsilon} = 1-\alpha$, $c = 2\varepsilon$, $b_0 = e^{-\sqrt{\varepsilon}t}$, $b_1 = e^{-t/2}$, and $\tau = s + n\ell$, gives

$$\Big|\iint \varphi(a_{s+n\ell}u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z)\,\mathrm{d}r - \int\varphi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)e^{-\star t} = \mathcal{S}(\varphi)R^{-\star} \tag{2.9}$$

where the implied constants depend on X.

Note that the total time required for these three phases is $s + d_1\ell + 9t$ which in view of the choices of s, ℓ and t is indeed a (large) constant times $\log R$. Theorem 1.1 follows.

3. Notation and preliminary results

Throughout the paper

$$G = \mathrm{SL}_2(\mathbb{C}) \quad \text{or} \quad G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Let $\Gamma \subset G$ be a lattice, and put $X = G/\Gamma$.

Let $A = \{a_t : t \in \mathbb{R}\} \subset H$. Let $U \subset N$ denote the group of upper triangular unipotent matrices in $H \subset G$, respectively. More explicitly, if $G = \mathrm{SL}_2(\mathbb{C})$, then

$$N = \left\{ n(r,s) = \begin{pmatrix} 1 & r+is \\ 0 & 1 \end{pmatrix} : (r,s) \in \mathbb{R}^2 \right\}$$

and $U = \{n(r,0) : r \in \mathbb{R}\}$; note that $n(r,0) = u_r$ for $r \in \mathbb{R}$. Let

$$V = \{n(0,s) = v_s : s \in \mathbb{R}\};$$

if $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, then

$$N = \left\{ n(r,s) = \left(\begin{pmatrix} 1 & r+s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \right) : (r,s) \in \mathbb{R}^2 \right\}$$

and $U = \{n(r,0) : r \in \mathbb{R}\}$. As before, $n(r,0) = u_r$ for $r \in \mathbb{R}$. Let

$$V = \{n(0,s) = v_s : s \in \mathbb{R}\}.$$

In both cases, we have $N = UV$. Let us denote the transpose of U by U^- and its elements by u_r^-.

Lie algebras and norms. Let $|\ |$ denote the usual absolute value on $\mathbb{C}$ (and on $\mathbb{R}$). Let $\|\ \|$ denotes the maximum norm on $\mathrm{Mat}_2(\mathbb{C})$ and $\mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R})$, with respect to the standard basis.

Let $\mathfrak{g} = \mathrm{Lie}(G)$, that is, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$. We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ where $\mathfrak{h} = \mathrm{Lie}(H) \simeq \mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{r} = i\mathfrak{sl}_2(\mathbb{R})$ if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{r} = \mathfrak{sl}_2(\mathbb{R}) \oplus \{0\}$ if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$.

Note that $\mathfrak{r}$ is a *Lie algebra* in the case $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, but not when $G = \mathrm{SL}_2(\mathbb{C})$.

Throughout the paper, we will use the uniform notation

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

for elements $w \in \mathfrak{r}$, where $w_{ij} \in i\mathbb{R}$ if $G = \mathrm{SL}_2(\mathbb{C})$ and $w_{ij} \in \mathbb{R}$ if $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

We fix a norm on $\mathfrak{h}$ by taking the maximum norm where the coordinates are given by $\mathrm{Lie}(U)$, $\mathrm{Lie}(U^-)$, and $\mathrm{Lie}(A)$; similarly fix a norm on $\mathfrak{r}$. By taking maximum of these two norms we get a norm on $\mathfrak{g}$. These norms will also be denoted by $\|\ \|$.

Let $C_2 \geq 1$ be so that

$$\|hw\| \leq C_2\|w\| \text{ for all } \|h - I\| \leq 2 \text{ and all } w \in \mathfrak{g}. \tag{3.1}$$

For all $\beta > 0$, we define

$$\mathsf{B}^H_\beta := \{u^-_s : |s| \leq \beta\} \cdot \{a_t : |t| \leq \beta\} \cdot \{u_r : |r| \leq \beta\} \tag{3.2}$$

for all $0 < \beta < 1$. Note that for all $h_i \in (\mathsf{B}^H_\beta)^{\pm 1}$, $i = 1, \ldots, 5$, we have

$$h_1 \cdots h_5 \in \mathsf{B}^H_{100\beta}. \tag{3.3}$$

We also define $\mathsf{B}^G_\beta := \mathsf{B}^H_\beta \cdot \exp(B_{\mathfrak{r}}(0, \beta))$ where $B_{\mathfrak{r}}(0, \beta)$ denotes the ball of radius β in $\mathfrak{r}$ with respect to $\| \|$.

Similarly, using $| \ |$ we define B^L_δ for $\delta > 0$ and $L = U^\pm, A, AU, H, N$. Given an open subset $\mathsf{B} \subset L$, and $\delta > 0$, $\partial_\delta \mathsf{B} = \{\mathsf{h} \in \mathsf{B} : \mathsf{B}^L_\delta.\mathsf{h} \not\subset \mathsf{B}\}$.

We deviate slightly from the notation in the introduction, and define the injectivity radius of $x \in X$ using B^G_β instead of the metric d on G. Put

$$\mathrm{inj}(x) = \min\bigl\{0.01, \sup\bigl\{\beta : \ g \mapsto gx \text{ is injective on } \mathsf{B}^G_{100\beta}\bigr\}\bigr\}. \tag{3.4}$$

Taking a further minimum if necessary, we always assume that the injectivity radius of x defined using the metric d dominates $\mathrm{inj}(x)$.

For every $\eta > 0$, let

$$X_\eta = \bigl\{x \in X : \mathrm{inj}(x) \geq \eta\bigr\}.$$

The set $\mathsf{E}_{\eta,t,\beta}$. For all $\eta, t, \beta > 0$, set

$$\mathsf{E}_{\eta,t,\beta} := \mathsf{B}^{s,H}_\beta \cdot a_t \cdot \{u_r : r \in [0, \eta]\} \subset H. \tag{3.5}$$

Then $m_H(\mathsf{E}_{\eta,t,\beta}) \asymp \eta\beta^2 e^t$ where m_H denotes our fixed Haar measure on H.

Throughout the paper, the notation $\mathsf{E}_{\eta,t,\beta}$ will be used only for $\eta, t, \beta > 0$ which satisfy $e^{-0.01t} < \beta \leq \eta^2$ even if this is not explicitly mentioned.

For all $\eta, \beta, m > 0$, put

$$\mathsf{Q}^H_{\eta,\beta,m} = \bigl\{u^-_s : |s| \leq \beta e^{-m}\bigr\} \cdot \{a_t : |t| \leq \beta\} \cdot \bigl\{u_r : |r| \leq \eta\bigr\}. \tag{3.6}$$

Roughly speaking, $\mathsf{Q}^H_{\eta,\beta,m}$ is a *small thickening* of the (β, η)-neighborhood of the identity in AU. We write $\mathsf{Q}^H_{\beta,m}$ for $\mathsf{Q}^H_{\beta,\beta,m}$.

The following lemma will also be used in the sequel.

3.1. **Lemma** ([LM21], Lemma 2.3)**.** *(1) Let $m \geq 1$, and let $0 < \eta, \beta < 0.1$. Then*

$$\Bigl(\bigl(\mathsf{Q}^H_{0.01\eta,0.01\beta,m}\bigr)^{\pm 1}\Bigr)^3 \subset \mathsf{Q}^H_{\eta,\beta,m}.$$

(2) For all $0 \leq \beta, \eta \leq 1$, $t, m > 0$, and all $|r| \leq 2$, we have

$$\bigl(\mathsf{Q}^H_{\eta,\beta^2,m}\bigr)^{\pm 1} \cdot a_m u_r \mathsf{E}_{\eta',t,\beta'} \subset a_m u_r \mathsf{E}_{\eta,t,\beta}, \tag{3.7}$$

where $\eta' = \eta(1 - 100e^{-t})$ and $\beta' = \beta(1 - 100\beta)$.

Constants and the $\star$-notation. In our analysis, the dependence of the exponents on Γ are via the application of results in §5, see (5.1), and §4.7.

We will use the notation $A \asymp B$ when the ratio between the two lies in $[C^{-1}, C]$ for some constant $C \geq 1$ which depends at most on G and Γ in general. We write $A \ll B^\star$ (resp. $A \ll B$) to mean that $A \leq CB^\kappa$ (resp. $A \leq CB$) for some constant $C > 0$ depending on G and Γ, and $\kappa > 0$ which follows the above convention about exponents.

Commutation relations. We also record the following two lemmas.

3.2. **Lemma** ([LM21], Lemma 2.1). *There exist absolute constants β_0 and C_3 so that the following holds. Let $0 < \beta \leq \beta_0$, and let $w_1, w_2 \in B_{\mathfrak{r}}(0, \beta)$. There are $h \in H$ and $w \in \mathfrak{r}$ which satisfy*

$$\tfrac{2}{3}\|w_1 - w_2\| \leq \|w\| \leq \tfrac{3}{2}\|w_1 - w_2\| \quad \text{and} \quad \|h - I\| \leq C_3\beta\|w\|$$

so that $\exp(w_1)\exp(-w_2) = h\exp(w)$. *More precisely,*

$$\|w - (w_1 - w_2)\| \leq C_3\beta\|w_1 - w_2\|$$

3.3. **Lemma** ([LM21], Lemma 2.2). *There exists β_0 so that the following holds for all $0 < \beta \leq \beta_0$. Let $x \in X_{10\beta}$ and $w \in B_{\mathfrak{r}}(0, \beta)$. If there are $h, h' \in \mathsf{B}^H_{2\beta}$ so that $\exp(w')hx = h'\exp(w)x$, then*

$$h' = h \quad \text{and} \quad w' = \mathrm{Ad}(h)w.$$

Moreover, we have $\|w'\| \leq 2\|w\|$.

4. Avoidance principles in homogeneous spaces

In this section we will collect statements concerning avoidance principles for unipotent flows and random walks on homogeneous spaces.

4.1. **Nondivergence results.** This subsection, is devoted to non-divergence results for unipotent flows. The results in this section are known to the experts and were also proved in details in [LM21, §3].

The results of this subsection are trivial when Γ a uniform lattice.

4.2. **Proposition** (Prop. 3.1,[LM21]). *There exist $C_4 \geq 1$ with the following property. Let $0 < \delta, \varepsilon < 1$ and $x \in X$. Let $I \subset [-10, 10]$ be an interval with $|I| \geq \delta$. Then*

$$\left|\left\{r \in I : \mathrm{inj}(a_t u_r x) < \varepsilon^2\right\}\right| < C_4\varepsilon|I|$$

so long as $t \geq |\log(\delta^2\mathrm{inj}(x))| + C_4$.

The following is a direct corollary of Proposition 4.2.

4.3. **Proposition** (Prop. 3.4,[LM21]). *There exists $0 < \eta_X < 1$, depending on X, so that the following holds. Let $0 < \eta < 1$ and let $x \in X$. Let $I \subset \mathbb{R}$ be an interval of length at least η. Then*

$$|\{r \in I : a_t u_r x \in X_{\eta_X}\}| \geq 0.9|I|$$

for all $t \geq |\log(\eta_X^2\mathrm{inj}(x))| + C_4$.

Proof. Apply Proposition 4.2 with $\varepsilon = 0.1 C_4{}^{-1}$. The claim thus holds with $\eta_X = \varepsilon^2$. ☐

The subsets $X_{\rm cpt}$ and $\mathfrak{S}_{\rm cpt}$. If X is compact, let $X_{\rm cpt} = X$; otherwise, let $X_{\rm cpt} = \{gx : x \in X_{\eta_X}, \|g - I\| \le 2\}$ where X_{η_X} is given by Proposition 4.3. Note that by [LM21, Lemma 3.6], we have

$$\mu_{Hx}(X_{\rm cpt}) > 0.9 \tag{4.1}$$

for every periodic orbit Hx.

We also fix once and for all a compact subset with piecewise smooth boundary $\mathfrak{S}_{\rm cpt} \subset G$ which projects onto $X_{\rm cpt}$.

More generally, we have the following lemma which is a consequence of reduction theory. In this form, the lemma is a spacial case of [LMMS19, Lemma 2.8].

4.4. **Lemma.** *There exist D_2 (absolute) and C_5 (depending on X) so that the following holds for all $0 < \eta \le \eta_X$. Let $g \in G$ be so that $g\Gamma \in X_\eta$. Then there is some $\gamma \in \Gamma$ so that*

$$\|g\gamma\| \le C_5 \eta^{-D_2}.$$

4.5. **Inheritance of the Diophantine property.** As it was mentioned in the outline given in §2, assuming part (2) in Theorem 1.1 does not hold, the first step in the proof is to improve this Diophantine condition. The following proposition (which was also stated in §2) is tailored for this purpose.

4.6. **Proposition.** *There exist D_0 (absolute) and C_1, s_0 (depending on X) so that the following holds. Let $R, S \ge 1$. Suppose $x_0 \in X$ is so that*

$$d_X(x_0, x) \ge (\log S)^{D_0} S^{-1}$$

for all x with $\mathrm{vol}(Hx) \le R$. Then for all

$$s \ge \max\bigl\{\log S, 2|\log(\mathrm{inj}(x_0))|\bigr\} + s_0$$

and all $0 < \eta \le 1$, we have

$$\left|\left\{ r \in [0,1] \colon \begin{array}{c} \mathrm{inj}(a_s u_r x_0) \le \eta \text{ or there is } x \text{ with} \\ \mathrm{vol}(Hx) \le R \text{ s.t. } d_X(a_s u_r x_0, x) \le \frac{1}{C_1 R^{D_0}} \end{array} \right\}\right| \le C_1(\eta^{1/2} + R^{-1}).$$

In the proof of Proposition 4.6, which is given in Appendix A, we use Margulis functions for periodic H-orbits similar to those which were used in [LM21, §9], see also [EMM15, Prop. 2.13] and the original paper [EMM98]. This will then be combined with the fact that the number of periodic H-orbits with volume $\le R$ in X is $\ll R^6$, see e.g [MO20, §10], to conclude. We also refer the reader to [ELMV09, §2] for results concerning isolation of periodic orbits.

It is also worth mentioning that even though [LMMS19, Thm. 1.4] concerns long pieces of U-orbits and Proposition 4.6 deals with translates of pieces of U-orbits, similar tools are applicable here as well. In particular, a version of Proposition 4.6 can be proved using the methods of [LMMS19].

4.7. **Closing lemma.** Let $t > 0$ be a large parameter. Fix some

$$e^{-0.01t} < \beta = \eta^2 < \eta_X^2;$$

in our application, we will let $\beta = e^{-\kappa t}$ where $\kappa \ll 1/D_0$ with D_0 as in Proposition 4.6 and the implied constant depending on X.

For every $\tau \geq 0$, put

$$\mathsf{E}_\tau = \mathsf{B}_\beta^{s,H} \cdot a_\tau \cdot \{u_r : r \in [0,1]\} \subset H.$$

If $y \in X$ is so that the map $\mathsf{h} \mapsto \mathsf{h}y$ is injective over E_τ, then $\mu_{\mathsf{E}_\tau.y}$ denotes the pushforward of the normalized Haar measure on E_τ to $\mathsf{E}_\tau.y \subset X$.

Let $\tau \geq 0$ and $y \in X$. For every $z \in \mathsf{E}_\tau.y$, put

$$I_\tau(z) := \big\{w \in \mathfrak{r} : \|w\| < \mathrm{inj}(z) \text{ and } \exp(w)z \in \mathsf{E}_\tau.y\big\};$$

this is a finite subset of $\mathfrak{r}$ since E_τ is bounded — we will define $I_{\mathcal{E}}(h,z)$ for all $h \in H$ and more general sets $\mathcal{E}$ in the bootstrap phase below.

Let $0 < \alpha < 1$. Define the function $f_\tau : \mathsf{E}_\tau.y \to [1,\infty)$ as follows

$$f_\tau(z) = \begin{cases} \sum_{0\neq w \in I_\tau(z)} \|w\|^{-\alpha} & \text{if } I_\tau(z) \neq \{0\} \\ \mathrm{inj}(z)^{-\alpha} & \text{otherwise} \end{cases}.$$

The following proposition supplies an initial dimension which we will bootstrap in the next phase. Roughly speaking, it asserts that points in $\{a_{8t}u_r x_0 : r \in [0,1]\}$ (possibly after removing an exponentially small set of exceptions) are *separated transversal to* H, unless x_0 is extremely close to a periodic H orbit.

4.8. **Proposition.** *Assume Γ is arithmetic. There exists D_1 (which depends on Γ explicitly) satisfying the following. Let $D \geq D_1$ and $x_1 \in X$. Then for all large enough t (depending on $\mathrm{inj}(x_1)$) at least one of the following holds.*

(1) There is a subset $I(x_1) \subset [0,1]$ with $|[0,1] \setminus I(x_1)| \ll_X \eta^{1/2}$ such that for all $r \in I(x_1)$ we have the following
   - *(a) $a_{8t}u_r x_1 \in X_\eta$.*
   - *(b) $\mathsf{h} \mapsto \mathsf{h}.a_{8t}u_r x_1$ is injective on E_t.*
   - *(c) For all $z \in \mathsf{E}_t.a_{8t}u_r x_1$, we have*

$$f_t(z) \leq e^{Dt}.$$

(2) There is $x \in X$ such that Hx is periodic with

$$\mathrm{vol}(Hx) \leq e^{D_1 t} \quad \textit{and} \quad d_X(x, x_1) \leq e^{(-D+D_1)t}.$$

The proof of this proposition is a minor modification of the proof of [LM21, Prop. 6.1]. The details are provided in Appendix B.

Proposition 4.8 is where the arithmeticity assumption on Γ is used. If we replace the assumption that Γ is arithmetic with the weaker requirement that Γ has algebraic entries, we get a version of this proposition where part (2) is replaced with the following.

(2') There is $x \in X$ with

$$d_X(x, x_1) \le e^{(-D+D_1)t},$$

satisfying the following: there are elements γ_1 and γ_2 in $\mathrm{Stab}_H(x)$ with $\|\gamma_i\| \le e^{D_1 t}$ for $i = 1, 2$ so that the group generated by $\{\gamma_1, \gamma_2\}$ is Zariski dense in H.

See Appendix B for more details.

5. Equidistribution of translates of horospheres

We begin by recalling the following quantitative decay of correlations for the ambient space X: There exists $0 < \kappa_0 \le 1$ so that

$$\Big|\int \varphi(gx)\psi(x)\,\mathrm{d}m_X - \int \varphi\,\mathrm{d}m_X \int \psi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)\mathcal{S}(\psi)e^{-\kappa_0 d(e,g)} \tag{5.1}$$

for all $\varphi, \psi \in C_c^\infty(X) + \mathbb{C}\cdot 1$, where m_X is the G-invariant probability measure on X and d is the right G-invariant metric on G defined on p. 3. See, e.g., [KM96, §2.4] and references there for (5.1).

Here $\mathcal{S}(\cdot)$ is a certain Sobolev norm on $C_c^\infty(X) + \mathbb{C}\cdot 1$ which is assumed to dominate $\|\cdot\|_\infty$ and the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$. Moreover, $\mathcal{S}(g.f) \ll \|g\|^\star \mathcal{S}(f)$ where the implied constants are absolute.

We note that by the works of Selberg and Jacquet-Langlands [Sel65, JL70], the constant κ_0 is absolute if Γ is a congruence subgroup, with the best known constant[3] given by Kim and Sarnak [Kim03] (this phenomenon, sometimes called *property* (τ) of congruence lattices, also holds in much greater generality).

Recall that $N = \{u_r v_s : r, s \in \mathbb{R}\}$ is a maximal unipotent subgroup of G, see §3. For $\delta_1, \delta_2 > 0$, put $B^N_{\delta_1,\delta_2} = \big\{u_r v_s : 0 \le r \le \delta_1, 0 \le s \le \delta_2\big\}$. We will denote $B^N_{1,1}$ by B^N_1. Let $\mathrm{d}n = \mathrm{d}r\,\mathrm{d}s$; in particular, $|B^N_{\delta_1,\delta_2}| = \delta_1\delta_2$.

It follows from Proposition 4.2, that for every $\varepsilon > 0$ and all $x \in X$,

$$\big|\big\{s \in [0,1] : \mathrm{inj}(a_t v_s x) < \varepsilon^2\big\}\big| < C_4\varepsilon$$

so long as $t \ge |\log(\mathrm{inj}(x))| + C_4$. Indeed Proposition 4.2 is stated with u_r instead of v_s, but the proof applies to this case as well — note that $a_t, v_s \in H'$ where $H' = gHg^{-1}$ where $g = \mathrm{diag}(i, 1)$.

5.1. **Proposition** (cf. [KM96], Prop. 2.4.8)**.** *There exists $\kappa_6 \gg \kappa_0$ (where the implied constant is absolute) so that the following holds. Let $0 < \eta, \delta \le 1$ and $x \in X_\eta$. Then for every $t \ge 4|\log\eta| + 2C_4$ we have*

$$\Big|\frac{1}{|B^N_{\delta,1}|}\int_{B^N_{\delta,1}} f(a_t n.x)\,\mathrm{d}n - \int f\,\mathrm{d}m_X\Big| \ll \mathcal{S}(f)(e^t\delta)^{-\kappa_6}$$

here $f \in C_c^\infty(X) + \mathbb{C}\cdot 1$ and the implied constant depends on X.

[3] To give a numerical value one needs to fix a normalization for d.

Proof. We may assume $e^t\delta > 1$ or else the statement holds trivially. Put $d_1 = \frac{1}{2}\log(e^t\delta)$ and

$$\begin{aligned} d_2 = t - d_1 &= \tfrac{1}{2}(t + |\log\delta|) = |\log\delta| + \tfrac{1}{2}\log(e^t\delta)\\ &\geq 2|\log\eta| + C_4, \end{aligned}$$

where we used $t \geq 4|\log\eta| + 2C_4$.

Now, for every $u_r v_s \in B_1^N$, we have

$$a_{d_1} u_r v_s a_{d_2} = a_t u_{e^{-d_2}r} v_{e^{-d_2}s};$$

moreover, for every $u_r v_s \in B_1^N$, we have

$$\frac{|u_{e^{-d_2}r} v_{e^{-d_2}s} B_{\delta,1}^N \triangle B_{\delta,1}^N|}{|B_{\delta,1}^N|} \ll (e^{d_2}\delta)^{-1} = (e^t\delta)^{-1/2}.$$

We conclude that

$$\begin{aligned} &\frac{1}{|B_{\delta,1}^N|}\int_{B_{\delta,1}^N} f(a_t n.x)\,\mathrm{d}n = \frac{1}{|B_{\delta,1}^N|}\int_{B_1^N}\mathrm{d}n_1\int_{B_{\delta,1}^N} f(a_t n_2.x)\,\mathrm{d}n_2 =\\ &\frac{1}{|B_{\delta,1}^N|}\int_{B_1^N}\int_{B_{\delta,1}^N} f(a_{d_1} n_1 a_{d_2} n_2.x)\,\mathrm{d}n_2\,\mathrm{d}n_1 + O\big((e^t\delta)^{-1/2}\mathcal{S}(f)\big). \end{aligned}$$

The above and the definition of d_1, thus, reduce the proof to showing that

$$\Big|\frac{1}{|B_{\delta,1}^N|}\int_{B_1^N}\int_{B_{\delta,1}^N} f(a_{d_1} n_1 a_{d_2} n_2.x)\,\mathrm{d}n_2\,\mathrm{d}n_1 - \int f\,\mathrm{d}m_X\Big| \ll \mathcal{S}(f)(e^t\delta)^{-\kappa_6}.$$

We now turn to the proof of the above. Let ε be a constant which will be optimized and will be chosen to be $(e^t\delta)^{-\star}$. Since $d_2 \geq 2|\log\eta| + C_4$, Proposition 4.2, applied to $u_r x$ for any $0 \leq r \leq \delta$, implies that

$$\{s \in [0,1] : \mathrm{inj}(a_{d_2} v_s u_r x) \leq \varepsilon^2\} \leq \varepsilon.$$

This in particular implies the following: Put

$$\mathsf{B} := \{n_2 \in B_{\delta,1}^N : \mathrm{inj}(a_{d_2} n_2 x) \leq \varepsilon^2\},$$

then $|B_{\delta,1}^N \setminus \mathsf{B}| \ll \varepsilon |B_{\delta,1}^N|$.

In consequence, the following holds

$$\begin{aligned} &\frac{1}{|B_{\delta,1}^N|}\int_{B_1^N}\int_{B_{\delta,1}^N} f(a_{d_1} n_1 a_{d_2} n_2.x)\,\mathrm{d}n_2\,\mathrm{d}n_1 =\\ &\qquad\qquad \frac{1}{|\mathsf{B}|}\int_{B_1^N}\int_{\mathsf{B}} f(a_{d_1} n_1 a_{d_2} n_2.x)\,\mathrm{d}n_2\,\mathrm{d}n_1 + O(\varepsilon\mathcal{S}(f)). \end{aligned}$$

This reduces the investigations to the study of

$$\frac{1}{\mathsf{B}}\int_{B_1^N}\int_{\mathsf{B}} f(a_{d_1} n_1 a_{d_2} n_2.x)\,\mathrm{d}n_2\,\mathrm{d}n_1.$$

Recall that $d_1 = \frac{1}{2}\log(e^t\delta)$. For every $n_2 \in \mathsf{B}$, we have $z_{n_2} = a_{d_2}n_2x \in X_{\varepsilon^2}$. Therefore, using e.g. [LM21, Prop. 4.1], we have

$$\Big|\int_{B_1^N} f(a_{d_1}n_1.z_{n_2})\,\mathrm{d}n_1 - \int f\,\mathrm{d}m_X\Big| \ll \varepsilon^{-\star}\mathcal{S}(f)e^{-\star d_1} = \varepsilon^{-\star}\mathcal{S}(f)(e^t\delta)^{-\star}.$$

Hence, if we choose ε to be a small negative power of $e^t\delta$, the above is $\ll \mathcal{S}(f)(e^t\delta)^{-\star}$. Averaging this over B finishes the proof. □

Using Proposition 5.1 and an argument due to Venkatesh [Ven10], we obtain the following.

5.2. **Proposition.** *There exist $\kappa_7 \gg \kappa_0^2$ so that the following holds. Let $0 \le \theta, \theta' < 1$ and $0 < b \le 0.1$. Let ρ be a probability measure on $[0,1]$ which satisfies the following: there exists $C \ge 1$ so that*

$$\rho(J) \le C b^{1-\theta} \tag{5.2}$$

for every interval J of length b.

Let $|\log b|/4 \le t \le (1-\theta')|\log b|$, $0 < \eta, \delta \le 1$. Let $x \in X_\eta$, and assume

$$|\log b| \ge 16|\log \eta| + 8C_4. \tag{5.3}$$

Then for all $f \in C_c^\infty(X) + \mathbb{C}\cdot 1$, we have

$$\Big|\frac{1}{\delta}\int_0^1\int_0^\delta f(a_tu_rv_s.x)\,\mathrm{d}r\,\mathrm{d}\rho(s) - \int f\,\mathrm{d}\mu_X\Big| \ll \mathcal{S}(f)\max\big\{(Cb^{-\theta})^{1/2}(e^t\delta)^{-\kappa_7}, b^{\theta'}\big\}. \tag{5.4}$$

where the implied constant depends on X.

Proof. We will prove this for the case $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$; the proof in the case $G = \mathrm{SL}_2(\mathbb{C})$ is similar.

Without loss of generality, we may assume $\int_X f\,\mathrm{d}\mu_X = 0$.

Let $M \in \mathbb{N}$ be so that $1/M \le b \le 1/(M-1)$. For every $1 \le j \le M$, let $I_j = \big[\frac{j-1}{M}, \frac{j}{M}\big)$; also put $s_j = \frac{2j-1}{2M}$ and $c_j = \rho(I_j)$ for all j. Since I_j's are disjoint, we have $\sum_j c_j = 1$.

For all such j, let

$$\mathsf{B}_j = \big\{u_rv_s : 0 \le r \le \delta, 0 \le s - s_j \le \tfrac{b}{4}\big\}.$$

In view of the choice of M, we have $\mathsf{B}_j \cap \mathsf{B}_{j'} = \emptyset$ for all $j \ne j'$. Let $\varphi = \sum_j (\delta b/4)^{-1}c_j\mathbb{1}_{\mathsf{B}_j}$. Then $\int_N \varphi(r,s)\,\mathrm{d}r\,\mathrm{d}s = 1$.

In view of (5.2), we have $c_j \le Cb^{1-\theta}$ for all j. This and the fact that B_j's are disjoint imply that

$$\varphi(n(z)) \le \max\{(\delta b/4)^{-1}c_j : 1 \le j \le M\} \ll Cb^{-\theta}\delta^{-1} \tag{5.5}$$

for all $n(z) \in N$; here and in what follows, $z = (r,s)$ and $\mathrm{d}z = \mathrm{d}r\,\mathrm{d}s$.

Using the fact that I_j's are disjoint, we have

$$\int_0^1\int_0^\delta f(a_tu_rv_s.x)\,\mathrm{d}r\,\mathrm{d}\rho(s) = \sum_j\int_{I_j}\int_0^\delta f(a_tu_rv_s.x)\,\mathrm{d}r\,\mathrm{d}\rho(s);$$

thus, we conclude that

$$\begin{aligned}(5.6)\qquad &\Big|\delta^{-1}\int_0^1\int_0^\delta f(a_tu_rv_s.x)\,\mathrm{d}r\,\mathrm{d}\rho(s)-\sum_j c_j\delta^{-1}\int_0^\delta f(a_tu_rv_{s_j}.x)\,\mathrm{d}r\Big|\\ &\leq\sum_j\int_{I_j}\delta^{-1}\int_0^\delta\big|f(a_tu_rv_s.x)-f(a_tu_rv_{s_j}.x)\big|\,\mathrm{d}r\,\mathrm{d}\rho(s)\ll\mathcal{S}(f)\beta^{\theta'}\end{aligned}$$

where we used the facts that $|s-s_j|\leq\beta$ and $t\leq(1-\theta')|\log\beta|$ in the last inequality.

In view of (5.6), thus, we need to bound $\sum_j\delta^{-1}\int c_jf(a_tu_rv_{s_j}x)\,\mathrm{d}r$. Similar to (5.6), we can now make the following computation.

$$\begin{aligned}(5.7)\qquad &\Big|\sum_j\delta^{-1}\int_0^\delta c_jf(a_tn(s_j,r).x)\,\mathrm{d}r-\int_N\varphi(n(z))f(a_tn(z).x)\,\mathrm{d}z\Big|\\ &\leq\sum_j\int_0^\delta(\beta\delta/4)^{-1}c_j\int_{s_j}^{s_j+\frac{\beta}{4}}\big|f(a_tn(s_j,r).x)-f(a_tn(s,r).x)\,\mathrm{d}s\big|\,\mathrm{d}r\\ &\ll\mathcal{S}(f)\beta^{\theta'}\end{aligned}$$

where again we used the facts that $|s-s_j|\leq\beta$ and $t\leq(1-\theta')|\log\beta|$.

Thus, it suffices to investigate

$$A_1=\int\varphi(n(z))f(a_tn(z).x)\,\mathrm{d}z.$$

To that end, let $N\geq 1$ be so that $\mathcal{S}(g.f)\leq\|g\|^N\mathcal{S}(f)$. Let

$$(5.8)\qquad \tau=\delta\cdot(e^t\delta)^{-1+\frac{\kappa_6}{2N}},$$

and define

$$A_2:=\tau^{-1}\int_0^\tau\int\varphi(n(z))f(a_tu_rn(z).x)\,\mathrm{d}z\,\mathrm{d}r.$$

Roughly speaking, we introduce an extra averaging in the direction of U.

For every $0\leq r\leq\tau$, we have $|(\mathsf{B}_j+r)\Delta\mathsf{B}_j|\ll|\mathsf{B}_j|\tau/\delta$. Hence,

$$\begin{aligned}\Big|\int\varphi(z)f(a_tu_rn(z).x)\,\mathrm{d}z-\int\varphi(z)f(a_tn(z).x)\,\mathrm{d}z\Big|&\\ \leq\sum_j(\beta\delta/4)^{-1}c_j\int_{(\mathsf{B}_j+r)\Delta\mathsf{B}_j}|f(a_tn(z)x)|\,\mathrm{d}z&\\ \leq\sum_j(\beta\delta/4)^{-1}c_j|\mathsf{B}_j|(\tau/\delta)\|f\|_\infty&\\ \leq\|f\|_\infty\cdot(\tau/\delta)\ll\mathcal{S}(f)\cdot(\tau/\delta);&\end{aligned}$$

we used $|\mathsf{B}_j|=\beta\delta/4$ for every j and $\sum c_j=1$, in the second to the last inequality. Averaging the above over $[0,\tau]$, we conclude that

$$(5.9)\qquad |A_1-A_2|\ll\mathcal{S}(f)\tau/\delta\leq\mathcal{S}(f)(e^t\delta)^{-1/2};$$

where we used (5.8).

In consequence, we have reduced the proof to the study of A_2 to which we now turn. By the Cauchy-Schwarz inequality, we have

$$|A_2|^2 \le \int \varphi(z)\Big(\tau^{-1}\int_0^\tau f(a_t u_r n(z).x)\,\mathrm{d}r\Big)^2\,\mathrm{d}z.$$

Now using $\Big(\tau^{-1}\int_0^\tau f(a_t u_r n(z).x)\,\mathrm{d}r\Big)^2 \ge 0$, (5.5), and the above estimate, we conclude

$$\begin{aligned}|A_2|^2 &\ll \frac{C\mathsf{b}^{-\theta}}{|B^N_{\delta,1}|}\int_{B^N_{\delta,1}}\Big(\tau^{-1}\int_0^\tau f(a_t u_r n(z).x)\,\mathrm{d}r\Big)^2\,\mathrm{d}z\\ &= \frac{1}{\tau^2}\int_0^\tau\int_0^\tau \frac{C\mathsf{b}^{-\theta}}{|B^N_{\delta,1}|}\int_{B^N_{\delta,1}} \hat f_{r_1,r_2}(a_t n(z).x)\,\mathrm{d}z\,\mathrm{d}r_1\,\mathrm{d}r_2 \qquad (5.10)\end{aligned}$$

where $B^N_{\delta,1} = \{u_r v_s : 0 \le r \le \delta, 0 \le s \le 1\}$ and for all $r_1, r_2 \in [0,\tau]$

$$\hat f_{r_1,r_2}(y) = f(a_t u(r_1)a_{-t}.y)f(a_t u(r_2)a_{-t}.y).$$

By (5.8), we have

$$\mathcal{S}(\hat f_{r_1,r_2}) \ll \mathcal{S}(f)^2(e^t\tau)^N \ll \mathcal{S}(f)^2(e^t\delta)^{\kappa_6/2}. \qquad (5.11)$$

Now since $t \ge 4|\log\eta| + 2C_4$, by Proposition 5.1, we have

$$\Big|\frac{1}{|B^N_{\delta,1}|}\int_{B^N_{\delta,1}} \hat f_{r_1,r_2}(a_t n(z)x)\,\mathrm{d}z\Big| = \int_X \hat f_{r_1,r_2}\,\mathrm{d}\mu_X + O(\mathcal{S}(\hat f_{r_1,r_2})(e^t\delta)^{-\kappa_6}).$$

Recall from (5.11) that $\mathcal{S}(\hat f_{r_1,r_2})(e^t\delta)^{-\kappa_6} \le \mathcal{S}(f)^2(e^t\delta)^{-\kappa_6/2}$. Altogether, we conclude that

$$\Big|\frac{1}{|B^N_{\delta,1}|}\int_{B^N_{\delta,1}} \hat f_{r_1,r_2}(a_t n(z)x)\,\mathrm{d}z\Big| = \int_X \hat f_{r_1,r_2}\,\mathrm{d}\mu_X + O(\mathcal{S}(f)^2(e^t\delta)^{-\kappa_6/2}). \qquad (5.12)$$

We now use estimates on the decay of matrix coefficients, (5.1), and obtain the following: If $|r_1 - r_2| > \tau\cdot(e^t\delta)^{-\frac{\kappa_6}{4N}}$, then

$$\Big|\int_X \hat f_{r_1,r_2}(x)\,\mathrm{d}\mu_X\Big| \ll \mathcal{S}(f)^2(e^t\delta)^{-\kappa_6\kappa_0/4N} \qquad (5.13)$$

where we used $e^t\tau = (e^t\delta)^{\frac{\kappa_6}{2N}}$.

Divide now the integral $\int_0^\tau\int_0^\tau$ in (5.10) into terms: one with $|r_1 - r_2| \le \tau\cdot(e^t\delta)^{-\frac{\kappa_6}{4N}}$ and the other its complement. We thus get from (5.10), (5.12), and (5.13) that

$$|A_2| \ll (C\mathsf{b}^{-\theta})^{1/2}\mathcal{S}(f)\Big((e^t\delta)^{-\kappa_6\kappa_0/4N} + (e^t\delta)^{-\kappa_6/4N}\Big)^{1/2}.$$

This, together with (5.6), (5.7), and (5.9), implies that the proposition holds with $\kappa_7 = \kappa_6\kappa_0/8N$. ∎

6. Discretized dimension

Let $0 < \alpha \leq 1$. We begin by defining a modified (and localized) α-dimensional energy for finite subsets of $\mathbb{R}^d$.

Fix some norm $\|\ \|$ on $\mathbb{R}^d$ (below we will apply this for the cases $d = 3$ and $d = 1$). Let $0 < b_0 \leq 1$, and let $\Theta \subset \{w \in \mathbb{R}^d : \|w\| < b_0\}$ be a finite set. For $\mathsf{R} \geq 1$, define $\mathcal{G}_{\Theta,\mathsf{R}} : \Theta \to (0,\infty)$ as follows: If $\#\Theta \leq \mathsf{R}$, put

$$\mathcal{G}_{\Theta,\mathsf{R}}(w) = b_0^{-\alpha}, \quad \text{for all } w \in \Theta,$$

and if $\#\Theta > \mathsf{R}$, put

$$\mathcal{G}_{\Theta,\mathsf{R}}(w) = \min\left\{ \sum_{\Theta'} \|w - w'\|^{-\alpha} : \begin{array}{c} \Theta' \subset \Theta \text{ and} \\ \#(\Theta \setminus \Theta') = \mathsf{R} \end{array} \right\}.$$

We will also use this notation for finite subsets of $\mathfrak{r}$, which as a vector space is $\simeq \mathbb{R}^3$.

6.1. A projection theorem.

We now state a projection theorem which plays a crucial role in our argument. Indeed, this theorem (as stated here) will be used in improving the dimension phase, §9–§12; a modified version of it (Theorem C.3) will also be used in the endgame phase, §13.

6.2. Theorem. *Let $0 < \alpha \leq 1$, and let $0 < \mathsf{c} < 0.01\alpha$. Let $\Upsilon \geq 1$ be large enough depending on c, and let $\Theta \subset B_{\mathfrak{r}}(0, b_0)$ be a finite set satisfying*

$$\mathcal{G}_{\Theta,\mathsf{R}}(w) \leq \Upsilon \quad \textit{for every } w \in \Theta \textit{ and some } \mathsf{R} \geq 1. \tag{6.1}$$

Consider the one-parameter family of projections $\xi_r : \mathfrak{r} \to \mathbb{R}$ given by

$$\xi_r(w) = (\mathrm{Ad}(u_r)w)_{12} = -w_{21}r^2 - 2w_{11}r + w_{12}.$$

Let $J \subset [0,1]$ be an interval with $|J| \geq 10^{-6}$. There exists a subset $J' \subset J$ with $|J \setminus J'| \leq L_1\Upsilon^{-\mathsf{c}^2}$, where $L_1 = L\mathsf{c}^{-L}$ for an absolute constant L, so that the following holds. Let $r \in J'$, then there exists a subset $\Theta_r \subset \Theta$ with

$$\#(\Theta \setminus \Theta_r) \leq L_1\Upsilon^{-\mathsf{c}^2} \cdot (\#\Theta)$$

such that the projected set $\xi_r(\Theta)$ satisfies that

$$\mathcal{G}_{\xi_r(\Theta),\mathsf{R}_1}(w) \leq \Upsilon_1 \qquad \text{for all } w \in \xi_r(\Theta_r)$$

where $\mathsf{R}_1 = \mathsf{R} + L_1\Upsilon^{7\mathsf{c}}$, $\Upsilon_1 = L_1\Upsilon^{1+8\mathsf{c}}$.

This theorem will be proved in Appendix C. We also refer to that section for references and historic comments.

6.3. Regularization lemmas.

It will be more convenient to work with finite sets which have more *regular* structure, see [BFLM11, Lemma 5.2] and [Bou10, §2]. In this section we recall this construction, tailored to the applications in our paper.

Let $t, m_0 \geq 1$ and $0 < \varepsilon < 1$ be three parameters: t is large and arbitrary, m_0 is moderate and fixed, and ε is small and fixed; in particular, our estimates are allowed to depend on m_0 and ε, but not on t. Let $e^{-0.01\varepsilon t} \leq \eta \leq 1$ and let $b_0 = e^{-\sqrt{\varepsilon}t}\eta$.

Let $F \subset B_{\mathfrak{r}}(0,1)$ with

$$e^{t/2} \leq \#F \leq e^{m_0 t}.$$

For all $w \in F$, let $F_w = B_{\mathfrak{r}}(w, b_0) \cap F$, and assume that

$$\mathcal{G}_{F_w,\mathsf{R}}(w') \leq \Upsilon \qquad \text{for all } w' \in I_w \tag{6.2}$$

where $1 \leq \mathsf{R} \leq e^{0.01\varepsilon t}$ and $\Upsilon > 0$ satisfying the following

$$\Upsilon \leq e^{(m_0+1)t}. \tag{6.3}$$

Note that there is $w \in F$ so that $\#F_w \geq e^{0.5t - 4\sqrt{\varepsilon}} > e^{9t/20}$. Thus (6.2) and the the fact that $1 \leq \mathsf{R} \leq e^{0.01\varepsilon t}$ imply that indeed, $\Upsilon \geq e^{0.4t}$.

Let $\beta = e^{-\kappa t}$ for some κ satisfying $0 < \kappa(m_0 + 1) \leq 10^{-6}\varepsilon$. Fix $\mathsf{M} \in \mathbb{N}$, large enough, so that both of the following hold

$$2^{-\mathsf{M}}(m_0 + 1) < \kappa/100 \quad \text{and} \quad 6\mathsf{M} < 2^{\kappa\mathsf{M}/100}. \tag{6.4}$$

Define $k_0 := \lfloor(-\log_2 b_0)/\mathsf{M}\rfloor$ and $k_1 := \lceil(1 + \alpha^{-1}\log_2 \Upsilon)/\mathsf{M}\rceil + 1$; note that

$$2^{(\mathsf{M}k_1 - 1)\alpha} > \Upsilon. \tag{6.5}$$

In view of (6.2) and (6.5), we have

$$\#\big(B_{\mathfrak{r}}(w, 2^{-\mathsf{M}k_1}) \cap F\big) \leq \mathsf{R} \qquad \text{for all } w \in \mathfrak{r}. \tag{6.6}$$

For every $k_0 \leq k \leq k_1$, let $\mathcal{Q}_{\mathsf{M}k}$ denote the collection of $2^{-\mathsf{M}k}$-cubes

$$\{w \in \mathfrak{r} : w_{rs} \in [\tfrac{n_{rs}}{2^{\mathsf{M}k}}, \tfrac{n_{rs}+1}{2^{\mathsf{M}k}}), r, s = 1, 2\}$$

for some trace zero $(n_{ij}) \in \mathrm{Mat}_2(\mathbb{Z})$ if $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and with the obvious modification when $G = \mathrm{SL}_2(\mathbb{C})$.

6.4. **Lemma.** *For all large enough t, we can write $F = F' \bigcup (\bigcup_{i=1}^N F_i)$ (a disjoint union) with*

$$\#F' < \beta^{1/4} \cdot (\#F) \quad \textit{and} \quad \#F_i \geq \beta^2 \cdot (\#F)$$

so that the following holds. For every i and every $k_0 - 10 \leq k \leq k_1$, there exists some τ_{ik} so that for every cube $Q \in \mathcal{Q}_{\mathsf{M}k}$ we have

$$2^{\mathsf{M}(\tau_{ik}-2)} \leq \#F_i \cap Q \leq 2^{\mathsf{M}\tau_{ik}} \quad \textit{or} \quad F_i \cap Q = \emptyset. \tag{6.7}$$

Moreover, for every i and every cube $Q \in \mathcal{Q}_{\mathsf{M}k_0}$, we have

$$\#F_i \cap Q \geq e^{-4\sqrt{\varepsilon}t} \cdot (\#F_i) \quad \textit{or} \quad F_i \cap Q = \emptyset. \tag{6.8}$$

Proof. This lemma is essentially proved in [BFLM11, Lemma 5.2]. We explicate this construction for completeness. Let us begin with a preparatory step before applying the construction in loc. cit.; this step is also present in [BFLM11, Lemma 5.2].

Claim. *We may write* $F = F'' \bigcup (\cup \hat{F}_j)$ *(disjoint union) satisfying that* $\#F'' \leq \beta^{1/2} \cdot (\#F)$ *and for each* $\hat{F}_j$*, there exists some* $w_j \in \mathfrak{r}$ *so that if* $Q, Q' \in \mathcal{Q}_{\mathsf{M}k}$ *intersect* $\hat{F}_j + w_j$ *non-trivially, the distance between* $Q \cap (\hat{F}_j + w_j)$ *and* $Q' \cap (\hat{F}_j + w_j)$ *is at least* $2^{-\mathsf{M}k - \mathsf{M}}$.

Proof of the Claim. For every $k_0 - 10 \leq k \leq k_1$, the density of

$$D_k = \big\{ w \in \mathfrak{r} : \exists r, s, \text{ such that } w_{rs} \in 2^{-\mathsf{k}}(\mathbb{Z} + [0, 2^{-\mathsf{M}}]) \big\}$$

in $\mathfrak{r}$ is $\leq 3 \times 2^{-\mathsf{M}}$. Using the definition, we conclude that the density of $D := \bigcup_k D_k$ in $\mathfrak{r}$ is $\geq 1 - (1 - 3 \times 2^{-\mathsf{M}})^{k_1 - k_0 + 1}$.

Hence there exists some w_1 so that

$$\#(F + w_1 \setminus D) \geq (1 - 3 \times 2^{-\mathsf{M}})^{k_1 - k_0 + 1} \cdot (\#F) \gg \beta^{0.1} \cdot (\#F),$$

where we used $k_1 - k_0 \leq 2(m_0 + 1)t$ and the fact that $2^{-\mathsf{M}}(m_0 + 1) \leq \kappa/100$.

Note that $F + w_1 \subset B_{\mathfrak{r}}(0, 10)$, and put

$$\hat{F}_1 := (F + w_1 \setminus D) - w_1.$$

Cover $B_{\mathfrak{r}}(0, 10)$ with dyadic cubes $\{Q_r\}$ in $\mathcal{Q}_{\mathsf{M}k_1}$, and set

$$\hat{Q}_1^r = \big((F + w_1 \setminus D) \cap Q_r\big) - w_1$$

for any r so that $(F + w_1 \setminus D) \cap Q_r \neq \emptyset$.

Assuming $\hat{F}_1, \ldots, \hat{F}_n$ are defined, repeat the above with $F \setminus (\cup_{i=1}^n \hat{F}_i)$ if this set has $\geq \beta^{1/2} \cdot (\#F)$ many elements. Each set thus obtain satisfies

$$\#\hat{F}_j \gg \beta^{0.6} \cdot (\#F).$$

In consequence, this process terminates after $N' \ll \beta^{-0.6}$ many steps and yields sets $\hat{F}_1, \ldots, \hat{F}_{N'}$. Define $\{\hat{Q}_j^r\}$ similarly for each $\hat{F}_j$.

Let $F'' = F \setminus (\bigcup \hat{F}_j)$, then $\#F'' \leq \beta^{1/2} \cdot (\#F)$. The claim follows. □

We now further subdivide the sets $\hat{F}_j$ so that the resulting sets satisfy (6.7) and (6.8). Fix some j. We will begin trimming $\hat{F}_j$ from the smallest cells, i.e., $2^{-\mathsf{M}k_1}$-cubes. In view of (6.6), $\#\hat{Q}_j^r \ll \mathsf{R}$ for all r. For $\ell \in \mathbb{N}$, let

$$\hat{F}_{j\ell} = \bigcup \{\hat{Q}_j^r : 2^{-\ell-1}\mathsf{R} \leq \#\hat{Q}_j^r \leq 2^{-\ell}\mathsf{R}\}.$$

Let $\hat{F}_j' = \bigcup_\ell \big\{\hat{F}_{j\ell} : \#\hat{F}_{j\ell} \leq \beta \cdot (\#\hat{F}_j)\big\}$.

Recall that $1 \leq \mathsf{R} \leq e^{0.01\varepsilon t}$ and $\beta = e^{-\kappa t}$. Therefore,

$$\#\big(\bigcup F_j'\big) \ll \sum \#F_j' \ll N' \cdot \beta \cdot (\#\hat{F}_j) \cdot \log \mathsf{R} < \beta^{0.3} \cdot (\#F),$$

so long as t is large enough. Put $\bar{F} = F'' \bigcup (\bigcup F_j')$, then $\#\bar{F} < 2\beta^{0.3} \cdot (\#F)$.

Thanks to this and the claim we can now apply the construction in [BFLM11, p. 246], with $\hat{F}_{j\ell}$ and dyadic cubes $2^{-\mathsf{M}k}$ with $k_0 - 10 \leq k \leq k_1$, and write

$$\hat{F}_{j\ell} = F_{j\ell}' \bigcup \big(\textstyle\bigcup_q \hat{F}_{j\ell}^q\big)$$

so that $\#F'_{j\ell} \ll \beta \cdot (\#\hat{F}_{j\ell})$, Moreover, for every q, $F^q_{j\ell}$ satisfies (6.7) and

$$\#\hat{F}^q_{j\ell} \gg (6\mathsf{M})^{-k_1} \cdot (\#\hat{F}_{j\ell}) \gg 2^{-\kappa \mathsf{M} k_1/10} \cdot (\#\hat{F}_{j\ell}) \gg \beta^{0.1} \cdot (\#\hat{F}_{j\ell});$$

we used $6\mathsf{M} \le 2^{\kappa \mathsf{M}/10}$, see (6.4), in the second inequality, and used the definitions of k_1 and β together with (6.3) in the last inequality.

Recall now that $\#\hat{F}_{j\ell} \ge \beta \cdot (\#\hat{F}_j) \ge \beta^{1.6} \cdot (\#F)$. Hence,

$$\#\hat{F}^q_{j\ell} \ge \beta^2 \cdot (\#F)$$

if we assume t is large enough to account for implied multiplicative constant.

In view of (6.7), if for some j, ℓ, q and $2^{-\mathsf{M}k_0}$ cube Q with $F^q_{j\ell} \cap Q \neq \emptyset$ we have $\#(F^q_{j\ell} \cap Q) \le e^{-4\sqrt{\varepsilon}t} \cdot (\#F^q_{j\ell})$, then (6.7), applied with k_0, implies

$$\#F^q_{j\ell} \ll e^{-\sqrt{\varepsilon}t} \cdot (\#F^q_{j\ell}),$$

which is a contradiction if t is large enough.

Finally, note that as it was done

$$\#\bigcup_{j,\ell} F'_{j\ell} \le N' \cdot \log R \cdot \beta \cdot (\#F) < \beta^{0.3} \cdot (\#F).$$

The lemma thus holds with $F' = \bar{F} \bigcup (\bigcup_{j,\ell} F'_{j\ell})$ and $\{\hat{F}^q_{j\ell} : j, \ell, q\}$. □

Recall that for all $w \in F$, we put $F_w = B_{\mathfrak{r}}(w, b_0) \cap F$. Assume now that for some $\mathsf{C} \le e^{10\varepsilon t}$ for all $w' \in F_w$, we have

$$\mathcal{G}_{F_w, \mathsf{R}}(w') \le \mathsf{C} \cdot b_0^{-\alpha} \cdot (\#F_w). \tag{6.9}$$

Since $e^t \le \#F \le e^{m_0 t}$ and $b_0 = e^{-\lfloor \sqrt{\varepsilon} t \rfloor} \eta$ where $\eta > e^{-0.01\varepsilon t}$, (6.9) implies

$$\mathcal{G}_{F_w, \mathsf{R}}(w') \le e^{(m_0 + 2\sqrt{\varepsilon})t}.$$

In particular, (6.2) holds with $\Upsilon = e^{(m_0+2\sqrt{\varepsilon})t}$, and Lemma 6.4 is applicable.

6.5. **Lemma.** *Let $F = F' \bigcup (\bigcup_{i=1}^N F_i)$ be a decomposition of F as in Lemma 6.4. Then for every i and all $w \in F_i$ we have*

$$\mathcal{G}_{F_{i,w}, \mathsf{R}}(w') \le \mathsf{C}\beta^{-4} b_0^{-\alpha} \cdot (\#F_{i,w})$$

for all $w' \in F_{i,w} := F_i \cap B_{\mathfrak{r}}(w, b_0)$.

Proof. Let $k_0 \le k \le k_1$ and let $w \in F_i$. Then using (6.9) and the fact that $\mathsf{R} \le 2^{0.01\varepsilon t}$, we conclude that

$$\begin{aligned} \#\big(B\big(w, 2^{-\mathsf{M}k}\big) \cap F_i\big) &\le \#(B\big(w, 2^{-\mathsf{M}k}\big) \cap F) \\ &\le 2^{10\mathsf{M}} \mathsf{C} \cdot (2^{-\mathsf{M}k}/b_0)^{\alpha} \cdot (\#F_w). \end{aligned} \tag{6.10}$$

Let $Q_0 \in \mathcal{Q}_{\mathsf{M}k_0}$ be so that $Q_0 \cap F_i \neq \emptyset$, and let $w \in F_i$. Then $B(w, 2^{-\mathsf{M}k_0})$ can be covered by at most 8 cubes in $\mathcal{Q}_{\mathsf{M}k_0}$, moreover, it contains at least one cube in $\mathcal{Q}_{\mathsf{M}(k_0+1)}$ which also contains w. Thus by (6.7),

$$2^{-3-4\mathsf{M}}(\#Q_0 \cap F_i) \le \#F_{i,w} \le 2^{3+2\mathsf{M}}(\#Q_0 \cap F_i) \tag{6.11}$$

We claim that there exists $w_i \in F_i$ so that

$$\begin{aligned}\#F_{w_i} = \#(B_{\mathfrak{r}}(w_i, b_0) \cap F) &\leq \beta^{-3} \cdot (\#(B_{\mathfrak{r}}(w, b_0) \cap F_i) \\ &= \beta^{-3} \cdot (\#F_{i,w_i}).\end{aligned} \tag{6.12}$$

Let us assume (6.12) and finish the proof. Note that (6.10) applied with $w = w_i$, together with (6.12), implies that

$$\#\big(B_{\mathfrak{r}}\big(w_i, 2^{-\mathsf{M}k}\big) \cap F_i\big) \leq 2^{\star\mathsf{M}}\beta^{-3}\mathsf{C} \cdot (2^{-\mathsf{M}k}/b_0)^{\alpha} \cdot (\#F_{i,w_i}), \tag{6.13}$$

where we assumed t is large.

Let now $k_0 + 2 \leq k' \leq k_1$. Then

$$\#\big(B_{\mathfrak{r}}\big(w, 2^{-\mathsf{M}k'}\big) \cap F_i\big) \leq \#(Q \cap F_i)$$

where Q is a $2^{-\mathsf{M}(k'-1)}$ cube which contains $B_{\mathfrak{r}}\big(w, 2^{-\mathsf{M}k'}\big)$. Let Q' be a cube of same size which contains w_i, then using (6.7), we have

$$\#(Q \cap F_i) \leq 2^{2\mathsf{M}} \cdot (\#(Q' \cap F_i)).$$

Since $Q' \subset B_{\mathfrak{r}}(w_i, 2^{-\mathsf{M}(k'-2)})$, using (6.13) with $k = k' - 2$, we conclude that

$$\begin{aligned}\#\big(B_{\mathfrak{r}}\big(w, 2^{-\mathsf{M}k'}\big) \cap F_i\big) &\leq 2^{2\mathsf{M}}\#(B_{\mathfrak{r}}(w_i, 2^{-\mathsf{M}(k'-2)}) \cap F_i) \\ &\leq 2^{\star\mathsf{M}}\beta^{-3}\mathsf{C} \cdot (2^{-\mathsf{M}(k'-2)}/b_0)^{\alpha} \cdot (\#F_{i,w_i}).\end{aligned}$$

This and (6.11) (whic is used to replace F_{i,w_i} with $F_{i,w}$) imply that

$$\#\big(B_{\mathfrak{r}}\big(w, 2^{-\mathsf{M}k'}\big) \cap F_i\big) \leq 2^{\star\mathsf{M}}\beta^{-3}\mathsf{C} \cdot (2^{-\mathsf{M}k'}/b_0)^{\alpha} \cdot (\#F_{i,w}). \tag{6.14}$$

Since $\#(B_{\mathfrak{r}}(w, 2^{-\mathsf{M}k_1}) \cap F_i) \leq \#(B_{\mathfrak{r}}(w, 2^{-\mathsf{M}k_1}) \cap F) \leq \mathsf{R}$, see (6.6), from (6.14) we conclude that

$$\begin{aligned}\mathcal{G}_{F_{i,w},\mathsf{R}}(w) &\leq k_1 2^{\star\mathsf{M}}\beta^{-3}\mathsf{C} \cdot (b_0)^{-\alpha} \cdot (\#F_{i,w}) \\ &\leq \beta^{-4}\mathsf{C} \cdot (b_0)^{-\alpha} \cdot (\#F_{i,w}),\end{aligned}$$

so long at t is large enough. This completes the proof assuming (6.12).

We now prove (6.12). Let $\mathcal{B} = \{B_{\mathfrak{r}}(v, b_0) : v \in F_i\}$ be a covering of F_i with multiplicity $\leq K$. Then

$$\begin{aligned}\sum \#(B(v) \cap F) &\leq K \cdot \big(\#\bigcup(B(v) \cap F)\big) \leq K \cdot (\#F) \\ &\leq K\beta^{-2} \cdot (\#F_i) \leq K\beta^{-2} \sum \#(B(v) \cap F_i),\end{aligned}$$

where we write $B(v)$ for $B_{\mathfrak{r}}(v, b_0)$. We conclude that for some $w_i \in F_i$,

$$\begin{aligned}\#F_{w_i} = \#(B(w_i) \cap F) &\leq K\beta^{-2} \cdot \big(\#(B(w_i) \cap F_i)\big) \\ &\leq \beta^{-3} \cdot \big(\#(B(w_i) \cap F_i)\big) = \beta^{-3}(\#F_{i,w_i})\end{aligned}$$

as was claimed in (6.12). □

7. Boxes, complexity and the Folner property

For every $\ell > 0$, let ν_ℓ be the probability measure on H defined by

$$\nu_\ell(\varphi) = \int_0^1 \varphi(a_\ell u_r)\,\mathrm{d}r \qquad \text{for all } \varphi \in C_c(H). \tag{7.1}$$

Our goal in this section and the next is to show that $\nu_\ell^{(d)}$ (the d-fold convolution of ν_ℓ) can be approximated with a convex combination of certain natural measures supported on a finite union of local H orbits, see §7.6.

This section will lay the groundwork for this decomposition. In particular, we will prove a covering lemma, Lemma 7.1, define the notion of an admissible measure, §7.6, and prove a certain almost invariance property for a class of measures appearing in our analysis, Lemmas 7.5 and 7.7.

Covering lemmas. We will fix $0 < \eta \le 0.01\eta_X$ and $\beta = \eta^2$ throughout this section. For $m \ge 0$, we introduce the shorthand notation Q_m^H for

$$\mathsf{Q}^H_{\eta,\beta^2,m} = \big\{u_s^- : |s| \le \beta^2 e^{-m}\big\} \cdot \{a_\tau : |\tau| \le \beta^2\} \cdot U_\eta, \tag{7.2}$$

where for every $\delta > 0$, let $U_\delta = \{u_r : |r| \le \delta\}$, see (3.6).

Define $\mathsf{Q}_m^G \subset G$ by thickening Q_m^H in the transversal direction as follows:

$$\mathsf{Q}_m^G := \mathsf{Q}_m^H \cdot \exp(B_{\mathfrak{r}}(0, 2\beta^2)). \tag{7.3}$$

We begin by fixing a particular covering of $X_{2\eta}$.

7.1. **Lemma.** *For every $m \ge 0$, there exists a covering*

$$\big\{\mathsf{Q}_m^G.y_j : j \in \mathcal{J}_m, y_j \in X_{3\eta/2}\big\}$$

of $X_{2\eta}$ with multiplicity K, depending only on X. In particular, $\#\mathcal{J}_m \ll \eta^{-1}\beta^{-10}e^m$.

Proof. We first prove the following. There exists a covering

$$\big\{\big(\mathsf{B}^{s,H}_{\beta^2} \cdot U_\eta \cdot \exp(B_{\mathfrak{r}}(0,\beta^2))\big).\hat{y}_k : k \in \mathcal{K}, \hat{y}_k \in X_{2\eta}\big\}$$

of $X_{2\eta}$ with multiplicity $O(1)$ depending only on X.

Let us write $\bar{\mathsf{B}}^G_{\eta,\beta^2} = \mathsf{B}^{s,H}_{\beta^2} \cdot U_\eta \cdot \exp(B_{\mathfrak{r}}(0,\beta^2))$. Then

$$\big(\bar{\mathsf{B}}^G_{0.1\eta,0.1\beta^2}\big)^{-1} \cdot \big(\bar{\mathsf{B}}^G_{0.1\eta,0.1\beta^2}\big) \subset \big(\bar{\mathsf{B}}^G_{10\eta,10\beta^2}\big), \tag{7.4}$$

see Lemma 3.2.

Let $\{\hat{y}_k \in X_{2\eta} : k \in \mathcal{K}\}$ be maximal with the following property

$$\bar{\mathsf{B}}^G_{0.01\eta,0.01\beta^2}.\hat{y}_i \cap \bar{\mathsf{B}}^G_{0.01\eta,0.01\beta^2}.\hat{y}_j = \emptyset \quad \text{ for all } i \ne j.$$

In view of (7.4) thus $\{\bar{\mathsf{B}}^G_{\eta,\beta^2}.\hat{y}_k : k \in \mathcal{K}\}$ covers $X_{2\eta}$ with multiplicity $O(1)$. Since $m_G(\bar{\mathsf{B}}^G_{\eta,\beta^2}) \asymp \eta\beta^{10}$, we also conclude that $\mathcal{K} \ll \eta^{-1}\beta^{-10}$.

The following generalization will also be used: for any $1 \le c \le 100$,

$$\{\bar{\mathsf{B}}^G_{c\eta,c\beta^2}.\hat{y}_k : k \in \mathcal{K}\} \tag{7.5}$$

covers $X_{2\eta}$ with multiplicity $\le K_1$, depending only on X.

Let now $m \geq 0$, and recall that we write Q^H_m for $\mathsf{Q}^H_{\eta,\beta^2,m}$. Fix a subset $\mathcal{H} \subset \mathsf{Q}^H_0$ which is maximal with the following property

$$\mathsf{Q}^H_{0.01\eta,0.01\beta^2,m} h \cap \mathsf{Q}^H_{0.01\eta,0.01\beta^2,m} h' = \emptyset,$$

for all $h \neq h' \in \mathcal{H}$. Since

$$m_H(\mathsf{Q}^H_{0.01\eta,0.01\beta^2,m}) \asymp e^{-m} m_H(\mathsf{Q}^H_0),$$

we have $\#\mathcal{H} \ll e^m$ where the implied constants are absolute. Furthermore,

$$\big(\mathsf{Q}^H_{0.01\eta,0.01\beta^2,m}\big)^{\pm 1} \cdot \mathsf{Q}^H_{0.01\eta,0.01\beta^2,m} \subset \mathsf{Q}^H_{0.1\eta,0.1\beta^2,m}.$$

Thus $\{\mathsf{Q}^H_m h_j : h_j \in \mathcal{H}\}$ covers $\mathsf{Q}^H_0 = \mathsf{B}^{s,H}_{\beta^2} \cdot U_\eta$ with multiplicity $\ll K_2$.

Combining these two coverings, we obtain a covering

$$\{\mathsf{Q}^H_m h_j \exp(B_{\mathfrak{r}}(0,\beta^2)).\hat{y}_k : h_j \in \mathcal{H}, k \in \mathcal{K}\}.$$

of $X_{2\eta}$. Note further that

$$\mathsf{Q}^H_m h_j \exp(B_{\mathfrak{r}}(0,\beta^2)) = \mathsf{Q}^H_m \exp\big(\operatorname{Ad}(h_j) B_{\mathfrak{r}}(0,\beta^2)\big) h_j \subset \mathsf{Q}^G_m h_j;$$

where we used the fact that $\operatorname{Ad}(h_j)B_{\mathfrak{r}}(0,\beta^2) \subset B_{\mathfrak{r}}(0,2\beta^2)$ in the final inclusion above — this holds since $\|h_j - I\| \leq 2\beta^2$ and β is small.

Finally note that since $\hat{y}_k \in X_{2\eta}$ and $\|h_j - I\| \leq 2\beta^2$, we have $h_j\hat{y}_k \in X_{19\eta/10}$, for every j,k. Altogether, we obtain a covering

$$\{\mathsf{Q}^G_m.y_j : j \in \mathcal{J}, y_j \in X_{19\eta/10}\} = \{\mathsf{Q}^G_m.h_j\hat{y}_k : h_j \in \mathcal{H}, k \in \mathcal{K}\}$$

of $X_{2\eta}$.

We claim: the multiplicity of this covering is $\leq K_1K_2$. Suppose $z \in X$ belongs to $M > K_1K_2$ sets $\mathsf{Q}^G_m.h_j\hat{y}_k$. That is, for $i = 1,\ldots,M$, we have

$$z = \mathsf{h}_i \exp(w_i) h_{j_i} \hat{y}_{k_i} \in \mathsf{Q}^G_m.h_{j_i}\hat{y}_{k_i}.$$

Note that $\mathsf{Q}^G_m h_{j_i} \subset \bar{\mathsf{B}}^G_{10\eta,10\beta^2}$. Thus in view of (7.5) and the fact that for all $\hat{y}_k$, $g \mapsto g\hat{y}_k$ is injective over $\mathsf{B}^G_{10\eta}$, we conclude that for at least $M/K_1 > K_2$ many choices of i we have $\mathsf{h}_i \exp(w_i) h_{j_i} = \mathsf{h}\exp(w)h$. This implies

$$\mathsf{h}_i h_{j_i} \exp(\operatorname{Ad}(h_{j_i}^{-1})w_i) = \mathsf{h}h\exp(\operatorname{Ad}(h^{-1})w).$$

Since the map $(h,w) \mapsto h\exp(w)$ is injective on $\mathsf{B}^H_{100\eta} \times B_{\mathfrak{r}}(0,100\eta)$, for more than K_2 choices of i we have $\mathsf{h}_i h_{j_i} = \mathsf{h}h$. This contradicts the choice of K_2 and completes the proof. $\square$

A density function. For every $m \geq 0$, we fix a covering

$$\{\mathsf{Q}^G_m y_j : y_j \in X_{3\eta/2}, j \in \mathcal{J}_m\}$$

as in Lemma 7.1. For every $z \in X$, let $\mathsf{k}_m(z) = \#\{j : z \in \mathsf{Q}^G_m.y_j\}$. Then $1 \leq \mathsf{k}_m(z) \leq K$. Define

$$\rho_m : X \to \{1/d : d = 1,\ldots,K\} \qquad \text{by} \quad \rho_m(z) := 1/\mathsf{k}_m(z).$$

For every $j \in \mathcal{J}_m$, put

$$\rho_{m,j} = \rho_m|_{\mathsf{Q}^G_m.y_j}.$$

Note that $\sum_j \rho_{m,j}(z) = 1$ for all $z \in X$.

7.2. **Boxes and complexity.** Let $\mathsf{prd} : \mathbb{R}^3 \to H$ be the map

$$\mathsf{prd}(s, \tau, r) = u_s^- a_\tau u_r.$$

A subset $\mathsf{D} \subset H$ will be called a *box* if there exist intervals $I^\bullet \subset \mathbb{R}$ (for $\bullet = \pm, 0$) so that

$$\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+).$$

We say $\Xi \subset H$ has complexity bounded by L (or at most L) if $\Xi = \bigcup_1^L \Xi_i$ where each Ξ_i is a box.

For every interval $I \subset \mathbb{R}$, let $\partial I = \partial_{100\eta|I|} I$ (recall that $\eta = \beta^{1/2}$), and put $\mathring{I} = I \setminus \partial I$. Given a box $\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+)$, we let

$$\mathring{\mathsf{D}} = \mathsf{mul}\big(\mathring{I}^- \times \mathring{I}^0 \times \mathring{I}^+\big) \quad \text{and} \tag{7.6a}$$

$$\partial \mathsf{D} = \mathsf{D} \setminus \mathring{\mathsf{D}}. \tag{7.6b}$$

More generally, if $\mathsf{D} = \mathsf{prd}(I^- \times I^0 \times I^+)$ is a box, and $\Xi \subset \mathsf{D}$ has complexity bounded by L, we define $\partial\Xi := \bigcup \partial\Xi_i$ and

$$\mathring{\Xi}_{\mathsf{D}} := \bigcup \mathring{\Xi}_i \tag{7.7}$$

where the union is taken over those i so that $\Xi_i = \mathsf{prd}(I_i^- \times I_i^0 \times I_i^+)$ with $|I_i^\bullet| \geq 100\eta|I^\bullet|$ for $\bullet = \pm, 0$.

7.3. **Lemma.** *There exists K' depending only on X so that the following holds. Let $j \in \mathcal{J}_m$ and $w \in B_{\mathfrak{r}}(0, 2\beta^2)$. Then for every $1 \leq \mathsf{k} \leq K$, there is $\Xi^{\mathsf{k}} = \Xi^{\mathsf{k}}(j, w) \subset \mathsf{Q}_m^H$ with complexity at most K' so that*

$$\rho_{m,j}(z) = 1/\mathsf{k} \quad \text{for all } z \in \Xi^{\mathsf{k}}.\exp(w)y_j \text{ and}$$

$$\big|\{z \in \mathsf{Q}_m^H.\exp(w)y_j : \rho_{m,j}(z) = 1/\mathsf{k}\} \setminus \big(\Xi^{\mathsf{k}}.\exp(w)y_j\big)\big| \ll \eta|\mathsf{Q}_m^H|$$

where the implied constant depends only on X.

Proof. We will use that $(h, v) \mapsto h\exp(v)y$ is injective over $\mathsf{B}_{10\eta}^H \times B_{\mathfrak{r}}(0, 10\eta)$ for all $y \in X_\eta$, and that

$$(\mathsf{Q}_m^H)^{\pm 1} \cdot (\mathsf{Q}_m^H)^{\pm 1} \cdot (\mathsf{Q}_m^H)^{\pm 1} \subset \mathsf{Q}_{10\eta, 10\beta^2, m}^H \quad \text{for all } m \geq 0.$$

Let $\mathcal{Y}_j = \{y_{k_i} : \mathsf{Q}_m^G.y_j \cap \mathsf{Q}_m^G.y_{k_i}\} \neq \emptyset$. We now find the local H-leaves in $\mathsf{Q}_m^G.y_{k_i}$ ($y_{k_i} \in \mathcal{Y}_j$) which intersect $\mathsf{Q}_m^H.\exp(w)y_j$. Let

$$\mathcal{Y}_j^w = \big\{(w_i, y_{k_i}) \in B_{\mathfrak{r}}(0, 2\beta^2) \times \mathcal{Y}_j : \big(\mathsf{Q}_m^H.\exp(w)y_j\big) \cap \big(\mathsf{Q}_m^H.\exp(w_i)y_{k_i}\big) \neq \emptyset\big\}.$$

Note that if $w_i, w_i' \in B_{\mathfrak{r}}(0, 2\beta^2)$ are so that $\mathsf{h}\exp(w)y_j = \bar{\mathsf{h}}\exp(w_i)y_{k_i}$ and $\mathsf{h}'\exp(w)y_j = \bar{\mathsf{h}}'\exp(w_i')y_{k_i}$. Then

$$\mathsf{h}^{-1}\bar{\mathsf{h}}\exp(w_i)y_{k_i} = \mathsf{h}'^{-1}\bar{\mathsf{h}}'\exp(w_i')y_{k_i},$$

which implies $w_i = w_i'$. Thus $\#\mathcal{Y}_j^w = n \leq \#\mathcal{Y}_j \leq K$.

For every $(w_i, y_{k_i}) \in \mathcal{Y}_j^w$, let $\mathsf{h}_i \in \mathsf{B} = (\mathsf{Q}_m^H)^{-1} \cdot (\mathsf{Q}_m^H)$ be so that

$$\exp(w_i)y_{k_i} = \mathsf{h}_i \exp(w)y_j.$$

Let us list these elements as $\{\mathsf{h}_{cd}\}$ where $1 \le c \le l$ and for every such c we have $1 \le d \le n_c$, moreover, $\mathsf{h}_{c_1 d_1} = \mathsf{h}_{c_1 d_2}$ and if and only if $c_1 = c_2$.

Let $\mathcal{N}_{\mathsf{k}}$ denote the set of $L \subset \{1, \ldots, l\}$ so that $\sum_{c \in L} n_c = \mathsf{k}$. Then

$$z \in \mathsf{Q}_m^H . \exp(w) y_j$$

satisfies $\rho_{m,j}(z) = 1/\mathsf{k}$ if and only if there exists an $L \in \mathcal{N}_{\mathsf{k}}$ so that

$$z \in \mathsf{Q}_m^H \mathsf{h}_{cd} . \exp(w) y_j$$

for all $c \in L$ and all $1 \le d \le n_c$, and $z \notin \mathsf{Q}_m^H \mathsf{h}_{cd} . \exp(w) y_j$ for any (c,d) with $c \notin L$. Therefore, $\{z \in \mathsf{Q}_m^H . \exp(w) y_j : \rho_{m,j}(z) = 1/\mathsf{k}\}$ is the image under the map $g \mapsto g \exp(w) y_j$ of the set

$$\bigcup_{L \in \mathcal{N}_{\mathsf{k}}} \Big(\textstyle\bigcap_{c \in L} (\mathsf{Q}_m^H \cap \mathsf{Q}_m^H \mathsf{h}_{cd}) \Big) \bigcap \Big(\textstyle\bigcap_{c \notin L} (\mathsf{Q}_m^H \setminus \mathsf{Q}_m^H \mathsf{h}_{cd}) \Big). \tag{7.8}$$

We now study the set appearing in (7.8). Let us begin with the following computation. Suppose $h \in H$ can be written as $h = u^-_{s_0} a_{\tau_0} u_{r_0}$. Then

$$u^-_s a_\tau u_r h = u_{\hat{s}} a_{\hat{\tau}} u_{\hat{r}}$$

where $(\hat{s}, \hat{\tau}, \hat{r})$ are given by

$$\begin{aligned}
\hat{r} &= \hat{r}_h(r) = \frac{r}{e^{\tau_0}(1 + r s_0)} + r_0 = r + r_0 + \tilde{r}_h(r) r, \\
\hat{\tau} &= \hat{\tau}_h(r, \tau) = \tau + \tau_0 + \tfrac{1}{2} \log(1 + r s_0) = \tau + \tau_0 + \tilde{\tau}_h(r) r, \\
\hat{s} &= \hat{s}_h(r, \tau, s) = s + \frac{s_0}{e^{\tau}(1 + r s_0)} = s + s_0 + \tilde{s}_{h,1}(r) r + \tilde{s}_{h,2}(r, \tau) \tau,
\end{aligned} \tag{7.9}$$

so long as these parameters are defined (which is always the case near the identity).

Apply the above with $u^-_s a_\tau u_r \in \mathsf{Q}_m^H$ and $h = \mathsf{h}_{cd}$ with $1 \le c \le l$. Then $|s_0| \le 10 e^{-m} \beta^2$ and $|\tau_0| \le 10 \beta^2$, see (7.2), and the functions $\tilde{r}_h$, $\tilde{\tau}_h$, $\tilde{s}_{h,1}$, and $\tilde{s}_{h,2}$ are analytic functions satisfying the following

$$\begin{aligned}
&|\tilde{r}_h(r)| \le 10 |\tau_0| \le 100 \beta^2, \\
&|\tilde{\tau}_h(r)| \le 10 |s_0| \le 100 e^{-m} \beta^2, \\
&|\tilde{s}_{h,1}(r, \tau)|, |\tilde{s}_{h,2}(r, \tau)| \le 10 |s_0| \le 100 e^{-m} \beta^2.
\end{aligned}$$

Therefore, there exists a box $\Xi_{cd} \subset \mathsf{Q}_m^H \mathsf{h}_{cd}$ so that

$$|\mathsf{Q}_m^H \mathsf{h}_{cd} \setminus \Xi_{cd}| \ll \eta |\mathsf{Q}_m^H|.$$

Repeat this for all $c \in L$ and all $1 \le d \le n_c$; let $\Xi(L) = \bigcap_L (\Xi_{cd} \cap \mathsf{Q}_m^H)$. Then

$$\big| \big(\textstyle\bigcap_L (\mathsf{Q}_m^H \mathsf{h}_{cd} \cap \mathsf{Q}_m^H) \big) \setminus \Xi(L) \big| \ll \eta |\mathsf{Q}_m^H|.$$

Similarly, there is $\Xi(L^{\complement})$ of complexity $\ll 1$ so that

$$\big| \big(\textstyle\bigcap_{L^{\complement}} (\mathsf{Q}_m^H \setminus \mathsf{Q}_m^H \mathsf{h}_{cd}) \big) \setminus \Xi(L^{\complement}) \big| \ll \eta |\mathsf{Q}_m^H|.$$

The claim in the lemma thus holds with $\Xi^{\mathsf{k}} = \bigcup_{\mathcal{N}_{\mathsf{k}}} \big(\Xi(L) \cap \Xi(L^{\complement}) \big)$. $\square$

Thickening in the stable direction. We now record two lemmas whose proofs are essentially based on almost invariance (under small translations) of the measures in question, and on commutation relations in H. Let σ denotes the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$, where as before,

$$\mathsf{B}^{s,H}_{\delta} = \{u^-_s : |s| \leq \delta\} \cdot \{a_\tau : |\tau| \leq \delta\}$$

for all $\delta > 0$.

We will write $V = m_{U^-A}(\mathsf{B}^{s,H}_{\beta})$ where m_{U^-A} denotes the left invariant measure. Recall also the definition of ν_t from (7.1):

$$\nu_t(\varphi) = \int_0^1 \varphi(a_t u_r)\, \mathrm{d}r \qquad \text{for all } \varphi \in C_c(H).$$

We fixed $0 < \eta \leq 0.01\eta_X$ and $\beta = \eta^2$. In the discussion below, we will work with ν_t with large enough t so that $e^{-t} \leq \beta^2$.

Let us begin with the following lemma.

7.4. **Lemma.** *Let $x \in X$. Let $t_1, t_2 > 0$, and assume that $e^{-t_1} \leq \beta^2$. Put $\mu = \sigma * \nu_{t_2} * \sigma * \nu_{t_1}$. For every $\varphi \in C_c^\infty(X)$, we have*

$$\Big|\int \varphi(hx)\, \mathrm{d}\nu_{t_2+t_1}(h) - \int \varphi(hx)\, \mathrm{d}\mu(h)\Big| \ll \beta\, \mathrm{Lip}(\varphi)$$

where the implied constant is absolute.

Proof. Let us recall the the following: for $c, d > 0$, $a_d \mathsf{B}^{s,H}_c a_{-d} \subset \mathsf{B}^{s,H}_c$ and $u_r a_d = a_d u_{e^{-d}r}$. Moreover, for every $r \in [0,1]$ and $\mathsf{h} \in \mathsf{B}^{s,H}_c$, we have $u_r \mathsf{h} = \mathsf{h}' u_{r'}$ where $\mathsf{h}' \in \mathsf{B}^{s,H}_{10c}$ and $|r'| \leq 2$. Altogether, we conclude that for every $\mathsf{h} \in \mathsf{B}^{s,H}_{\beta+100\beta^2}$ and $r \in [0,1]$ we have

$$a_{t_2} u_r \mathsf{h} a_{t_1} = \mathsf{h}' a_{t_1+t_2} u_{e^{-t_1}r'}$$

where $|r'| \leq 2$. Since $\big|[0,1] \triangle (e^{-t}r' + [0,1])\big| \ll \beta$, we conclude that

$$\Big|\int \varphi(hx)\, \mathrm{d}\nu_{t_2+t_1}(h) - \int \varphi(hx)\, \mathrm{d}\nu_{t_2} * \sigma * \nu_{t_1}(h)\Big| \ll \beta\, \mathrm{Lip}(\varphi).$$

The lemma follows. □

7.5. **Lemma.** *Let $x \in X$ and $t > 0$. Assume that $e^{-t} \leq \beta^2$ and that $h \mapsto hx$ is injective on $\mathsf{B}^{s,H}_{\beta} \cdot a_t \cdot U_1$. Let $j \in \mathcal{J}_0$ and $w \in B_{\mathfrak{r}}(0, 2\beta^2)$ be so that*

$$\mathsf{Q}^H_0 . \exp(w) y_j \subset \mathrm{supp}(\sigma * \nu_t * \delta_x) \cap \mathsf{Q}^G_0 . y_j.$$

Put $\bar\mu_{j,w} = (\sigma * \nu_t * \delta_x)|_{\mathsf{Q}^H_0 . \exp(w) y_j}$ and put

$$\mathrm{d}\mu_{j,w}(z) = \rho_{0,j}(z)\, \mathrm{d}\bar\mu_{j,w}(z).$$

Then for all $\varphi \in C_c^\infty(X)$, all $\mathsf{d} \geq 0$, and $|r_1|, |r_2| \leq 2$ with $|r_1 - r_2| \leq c\beta$,

$$\Big|\int \varphi(a_{\mathsf{d}} u_{r_1} z)\, \mathrm{d}\mu_{j,w}(z) - \int \varphi(a_{\mathsf{d}} u_{r_2} z)\, \mathrm{d}\mu_{j,w}(z)\Big| \ll \eta\, \mathrm{Lip}(\varphi) \mu_{j,w}(X)$$

where the implied constant depends on X and c.

Proof. Write $r_2 = r_1 + r'$ where $|r'| \le c\beta$, and let $\mathsf{h}u_s \in \mathsf{Q}_0^H = \mathsf{B}^{s,H}_{\beta^2} U_\eta$. Then

$$u_{r'}\mathsf{h}u_s = \mathsf{h}\mathsf{h}'u_{s+r''} \quad \text{where } |r''| \le 10c\beta \text{ and } \|\mathsf{h}' - I\| \ll \beta^3, \tag{7.10}$$

see (7.9).

Write $\mathsf{Q}_0^H . \exp(w)y_j = \bigcup_{\mathsf{k}=1}^K \{z \in \mathsf{Q}_0^H . \exp(w)y_j : \rho_{0,j}(z) = 1/\mathsf{k}\}$, and let

$$\Xi^{\mathsf{k}} . \exp(w)y_j \subset \{z \in \mathsf{Q}_0^H . \exp(w)y_j : \rho_{0,j}(z) = 1/\mathsf{k}\}$$

be as in Lemma 7.3. By that lemma, there are collections of intervals $\mathcal{J}^- = \{J^- \subset [-\beta^2, \beta^2]\}$, $\mathcal{J}^0 = \{J^0 \subset [-\beta^2, \beta^2]\}$, and $\mathcal{J}^+ = \{J^+ \subset [-\eta, \eta]\}$ with $\#\mathcal{J}^\bullet \le K'$, and $\mathcal{J} \subset \mathcal{J}^- \times \mathcal{J}^0 \times \mathcal{J}^+$ so that

$$\Xi^{\mathsf{k}} = \bigcup_{\mathcal{J}} \mathsf{prd}(J^- \times J^0 \times J^+),$$

where $\mathsf{prd}(s, \tau, r) = u_s^- a_\tau u_r$.

Let $\mathring{\Xi}^{\mathsf{k}}$ denote $\mathring{\Xi}^{\mathsf{k}}_{\mathsf{Q}_0^H}$, see (7.7). We will write $\Xi^{\mathsf{k}}_{j,w}$ and $\mathring{\Xi}^{\mathsf{k}}_{j,w}$ for $\Xi^{\mathsf{k}} . \exp(w)y_j$ and $\mathring{\Xi}^{\mathsf{k}} . \exp(w)y_j$, respectively. Using (7.10) and the definition of $\mathring{\Xi}^{\mathsf{k}}$, we conclude that

$$u_{r'}\mathring{\Xi}^{\mathsf{k}}_{j,w} \subset \Xi^{\mathsf{k}}_{j,w} \tag{7.11}$$

so long as β is small enough compared to c, see §7.2.

Recall now that

$$\operatorname{supp}(\sigma * \nu_t) = \mathsf{B}^{s,H}_{\beta + 100\beta^2} \cdot a_t \cdot \{u_r : r \in [0, 1]\}$$

and that $V = m_{U^-A}(\mathsf{B}^{s,H}_{\beta+100\beta^2})$, where m_{U^-A} is the left invariant measure. For $|s|, |\tau| \le \beta + 100\beta^2$ and $r \in [0, 1]$,

$$\mathrm{d}\sigma * \nu_t(u_s^- a_{\tau+t} u_r) = \frac{e^\tau}{V}\, \mathrm{d}s\, \mathrm{d}\tau\, \mathrm{d}r. \tag{7.12}$$

Note also that $\mathsf{Q}_0^H . \exp(w)y_j \subset \operatorname{supp}(\sigma * \nu_t * \delta_x) \cap \mathsf{Q}_0^G . y_j$. Thus the definition of $\bar{\mu}_{j,w}$, and the fact $1/K \le \rho_{0,j} \le 1$, imply that

$$\mu_{j,w}(\Xi^{\mathsf{k}}_{j,w} \setminus \mathring{\Xi}^{\mathsf{k}}_{j,w}) \ll \eta \mu_{j,w}(X). \tag{7.13}$$

Using (7.13), Lemma 7.3 and the definition of $\mu_{j,w}$ again, we have

$$\Big| \int \varphi(a_{\mathsf{d}} u_{r_i} z)\, \mathrm{d}\mu_{j,w}(z) - \sum_{\mathcal{N}} \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}} u_{r_i} z)\, \mathrm{d}\mu_{j,w}(z) \Big| \ll \eta \operatorname{Lip}(\varphi) \mu_{j,w}(X),$$

for $i = 1, 2$, where $\mathcal{N} = \{1 \le \mathsf{k} \le K : \mathring{\Xi}^{\mathsf{k}} \ne \emptyset\}$.

In view of this, and since $r_2 = r_1 + r'$, we need to estimate the following

$$\Big| \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}} u_{r_1} z)\, \mathrm{d}\mu_{j,w}(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}} u_{r_1} u_{r'} z)\, \mathrm{d}\mu_{j,w}(z) \Big| \tag{7.14}$$

for all $\mathsf{k} \in \mathcal{N}$.

Recall that $\mathrm{d}\mu_{j,w} = \rho_{0,j}\, \mathrm{d}\bar{\mu}_{j,w}$. Thus (7.14) may be written as

$$\Big| \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}} u_{r_1} z) \rho_{0,j}(z)\, \mathrm{d}\bar{\mu}_{j,w}(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}} u_{r_1} u_{r'} z) \rho_{0,j}(z)\, \mathrm{d}\bar{\mu}_{j,w}(z) \Big|.$$

In view of (7.11), $\rho_{0,j}(z) = \mathsf{k}$ and $\rho_{0,j}(u_{r'}z) = \mathsf{k}$ for all $z \in \mathring{\Xi}^{\mathsf{k}}_{j,w}$. Recall also that $h \mapsto hx$ is injective on $\operatorname{supp}(\sigma * \nu_t) \subset \mathsf{B}^{s,H}_{\beta} \cdot a_t \cdot U_1$. Thus, $\mathrm{d}\bar{\mu}_{j,w}$ is the restriction to $\mathsf{Q}^H_0.\exp(w)y_j$ of the pushforward of the measure $\frac{e^{\tau}}{V}\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}r$ under the map $h \mapsto hx$. Moreover, by (7.13) and (7.11), we have $\bar{\mu}_{j,w}(u_{r'}\mathring{\Xi}^{\mathsf{k}}_{j,w} \triangle \mathring{\Xi}^{\mathsf{k}}_{j,w}) \ll \eta\mu_{j,i}(X)$. Altogether, we conclude that

$$\Big|\int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}}u_{r_1}z)\,\mathrm{d}\mu_{j,w}(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,w}} \varphi(a_{\mathsf{d}}u_{r_1}u_{r'}z)\,\mathrm{d}\mu_{j,w}(z)\Big| \ll \eta\|\varphi\|_{\infty}\mu_{j,w}(X).$$

The proof is complete. □

7.6. **The set $\mathcal{E}$ and the measure $\mu_{\mathcal{E}}$.** Recall that $0 < \eta \le 0.01\eta_X$ and $\beta = \eta^2$. Define

$$\mathsf{E} = \mathsf{B}^{s,H}_{\beta} \cdot \{u_r : |r| \le \eta\}, \tag{7.15}$$

where $\mathsf{B}^{s,H}_{\beta} := \{u^-_s : |s| \le \beta\} \cdot \{a_t : |t| \le \beta\}$ for all $\beta > 0$.

Let $F \subset B_{\mathfrak{r}}(0,\beta)$ be a finite set, and let $y \in X_{2\eta}$. Then $\exp(w)y \in X_\eta$ for all $w \in F$, moreover $\mathsf{h} \mapsto \mathsf{h}\exp(w)y$ is injective on E. For every subset $\mathsf{E}' \subset \mathcal{E}$, put

$$\mathcal{E}_{\mathsf{E}'} = \bigcup \mathsf{E}'.\{\exp(w)y : w \in F\}; \tag{7.16}$$

we will denote $\mathcal{E}_{\mathsf{E}}$ simply by $\mathcal{E}$.

Let $\lambda, M > 0$. Let $\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$. A probability measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ is said to be (λ, M)-*admissible* if

$$\mu_{\mathcal{E}} = \frac{1}{\sum_{w\in F}\mu_w(X)}\sum_{w\in F}\mu_w$$

where for every $w \in F$, μ_w is a measure on $\mathsf{E}.\exp(w)y$ satisfying that if $\mathsf{h}\exp(w)y$ is in the support of μ_w

$$\mathrm{d}\mu_w(\mathsf{h}\exp(w)y) = \lambda\varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}) \quad \text{where } 1/M \le \varrho_w(\cdot) \le M;$$

moreover, there is a subset $\mathsf{E}_w = \bigcup_{p=1}^{M}\mathsf{E}_{w,p} \subset \mathsf{E}$ so that

(1) $\mu_w\big((\mathsf{E}\setminus\mathsf{E}_w).\exp(w)y\big) \le M\beta\mu_w(\mathsf{E}.\exp(w)y)$,
(2) The complexity of $\mathsf{E}_{w,p}$ is bounded by M for all p, and
(3) $\operatorname{Lip}(\varrho_w|_{\mathsf{E}_{w,p}}) \le M$ for all p.

Using the notation in (7.7), let $(\mathring{\mathsf{E}}_w)_{\mathsf{E}} = \bigcup_p(\mathring{\mathsf{E}}_{w,p})_{\mathsf{E}}$. Put

$$\mathring{\mathcal{E}} = \bigcup_w(\mathring{\mathsf{E}}_w)_{\mathsf{E}} \quad \text{and} \quad \mathring{\mu}_{\mathcal{E}} = \mu_{\mathcal{E}}|_{\mathring{\mathcal{E}}},$$

for $\mathcal{E}$ and an admissible measure $\mu_{\mathcal{E}}$ as above.

The following lemma is an analogue of Lemma 7.5.

7.7. **Lemma.** *Let $\ell > 0$, and let $r \in [0,1]$. Assume that $e^{-\ell} \le \beta^2$. Let $\mu_{\mathcal{E}}$ be an admissible measure on $\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$ for some $F \subset B_{\mathfrak{r}}(0,\beta)$, see (7.15). Let $j \in \mathcal{J}_\ell$ and $v \in B_{\mathfrak{r}}(0, 2\beta^2)$ be so that*

$$\mathsf{Q}_\ell^H.\exp(v)y_j \subset \operatorname{supp}(a_\ell u_r \mathring{\mu}_{\mathcal{E}}) \cap \mathsf{Q}_\ell^G.y_j.$$

Put $\bar\mu_{r,j}^v = (a_\ell u_r \mathring{\mu}_{\mathcal{E}})|_{\mathsf{Q}_\ell^H.\exp(v)y_j}$, and let $\mathrm{d}\mu_{r,j}^v(z) = \rho_{\ell,j}(z)\,\mathrm{d}\bar\mu_{r,j}^v(z)$. Then for all $\varphi \in C_c^\infty(X)$, all $\mathsf{d} \ge 0$, and all $|r_1 - r_2| \le c\beta$, we have

$$\Bigl|\int \varphi(a_{\mathsf{d}} u_{r_1} z)\,\mathrm{d}\mu_{r,j}^v(z) - \int \varphi(a_{\mathsf{d}} u_{r_2} z)\,\mathrm{d}\mu_{r,j}^v(z)\Bigr| \ll \eta \operatorname{Lip}(\varphi)\mu_{r,j}^v(X)$$

where the implied constant depends on X and c.

Proof. The proof is similar to the proof of Lemma 7.5.

Since r, v, and j are fixed throughout the proof, we will denote $\mu_{r,j}^v$ and $\bar\mu_{r,j}^v$ simply by μ and $\bar\mu$.

Write $r_2 = r_1 + r'$ where $|r'| \le c\beta$. Let $\mathsf{h}u_{\hat r} \in \mathsf{Q}_\ell^H$, then

$$u_{r'}\mathsf{h}u_{\hat r} = \mathsf{h}u_s^- a_\tau u_{\hat r + r''} \quad \text{where } |r''| \ll \beta \text{ and } e^\ell |s|, |\tau| \ll e^{-\ell}\beta^2, \tag{7.17}$$

see (7.9).

Let $I^- = [-e^{-\ell}\beta^2, e^{-\ell}\beta^2]$, $I^0 = [-\beta^2, \beta^2]$, and $I^+ = [-\eta, \eta]$. As it was done in the proof of Lemma 7.5, write

$$\mathsf{Q}_\ell^H.\exp(v)y_j = \bigcup_{\mathsf{k}=1}^{K} \{z \in \mathsf{Q}_\ell^H.\exp(v)y_j : \rho_{\ell,j}(z) = 1/\mathsf{k}\},$$

and let $\Xi^{\mathsf{k}}.\exp(v)y_j \subset \{z \in \mathsf{Q}_\ell^H.\exp(v)y_j : \rho_{\ell,j}(z) = 1/\mathsf{k}\}$ be as in Lemma 7.3. There are collections of intervals $\mathcal{J}^- = \{J^- \subset [-\beta^2, \beta^2]\}$, $\mathcal{J}^0 = \{J^0 \subset [-\beta^2, \beta^2]\}$, and $\mathcal{J}^+ = \{J^+ \subset [-\eta, \eta]\}$ with $\#\mathcal{J}^\bullet \le K'$, and $\mathcal{J} \subset \mathcal{J}^- \times \mathcal{J}^0 \times \mathcal{J}^+$ so that

$$\Xi^{\mathsf{k}} = \bigcup_{\mathcal{J}} \mathsf{prd}(J^- \times J^0 \times J^+),$$

where $\mathsf{prd}(s,\tau,r) = u_s^- a_\tau u_r$.

Let $\mathring{\Xi}^{\mathsf{k}}$ denote $\mathring{\Xi}^{\mathsf{k}}_{\mathsf{Q}_\ell^H}$, see (7.7). We will write $\Xi^{\mathsf{k}}_{j,v}$ and $\mathring{\Xi}^{\mathsf{k}}_{j,v}$ for $\Xi^{\mathsf{k}}.\exp(v)y_j$ and $\mathring{\Xi}^{\mathsf{k}}.\exp(v)y_j$, respectively. Using (7.17) and the definition of $\mathring{\Xi}^{\mathsf{k}}$, we conclude that

$$u_{r'}\mathring{\Xi}^{\mathsf{k}}_{j,v} \subset \Xi^{\mathsf{k}}_{j,v} \tag{7.18}$$

so long as β is small enough compared to c, see §7.2.

In view of the definitions of $\bar\mu$ and μ, there exists some w and p so that $\bar\mu$ is the restriction of the measure

$$a_\ell u_r \mu_w|_{\mathring{\mathsf{E}}_{w,p}.\exp(w)y}$$

to $\mathsf{Q}_\ell^H.\exp(v)y_j$. Note that $a_\ell u_r \mu_w|_{\mathring{\mathsf{E}}_{w,p}.\exp(w)y}$ is supported on $a_\ell u_r \mathsf{E}.\exp(w)y$, moreover, for every $\mathsf{h} \in \mathring{\mathsf{E}}_{w,p}$, we have

$$\mathrm{d}\mu_w(\mathsf{h}\exp(w)y)| = \lambda \varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}), \tag{7.19}$$

and $\mathrm{Lip}(\varrho_w|_{\mathring{\mathsf{E}}_{w,p}}) \leq M$.

Recall that $1 \ll \rho_{\ell,j}, \varrho \ll 1$. In view of the definitions of $\bar{\mu}$ and μ, thus, the above implies

$$\mu(\Xi^{\mathsf{k}}_{j,v} \setminus \mathring{\Xi}^{\mathsf{k}}_{j,v}) \ll \eta\mu(X) \tag{7.20}$$

the implied constant depends on λ, M, and X (via K and K').

Using (7.20), Lemma 7.3, and the definition of μ again, we have

$$\int \varphi(a_{\mathsf{d}} u_{r_i} z)\,\mathrm{d}\mu(z) = \sum_{\mathcal{N}} \int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_i} z)\,\mathrm{d}\mu(z) + O(\eta\,\mathrm{Lip}(\varphi)\mu(X)),$$

for $i = 1, 2$, where $\mathcal{N} = \{1 \leq \mathsf{k} \leq K : \mathring{\Xi}^{\mathsf{k}} \neq \emptyset\}$.

In view of this, and since $r_2 = r_1 + r'$, we need to estimate the following

$$\Big|\int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} z)\,\mathrm{d}\mu(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} u_{r'} z)\,\mathrm{d}\mu(z)\Big| \tag{7.21}$$

for all $\mathsf{k} \in \mathcal{N}$.

Recall that $\mathrm{d}\mu = \rho_{\ell,j}\,\mathrm{d}\bar{\mu}$. Thus (7.21) may be written as

$$\Big|\int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} z)\rho_{\ell,j}(z)\,\mathrm{d}\bar{\mu}(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} u_{r'} z)\rho_{\ell,j}(z)\,\mathrm{d}\bar{\mu}(z)\Big|.$$

First note that by (7.18), $\rho_{\ell,j}(z) = \mathsf{k}$ and $\rho_{\ell,j}(u_{r'}z) = \mathsf{k}$ for all $z \in \mathring{\Xi}^{\mathsf{k}}_{j,v}$.

Now let $\mathsf{C}^{\mathsf{k}} \subset \mathsf{E}$ be so that $a_\ell u_r \mathsf{C}^{\mathsf{k}} \exp(w)y = \Xi^{\mathsf{k}}_{j,v}$; similarly, define $\mathring{\mathsf{C}}^{\mathsf{k}}$. Then

$$u_r \mathring{\mathsf{C}}^{\mathsf{k}} \exp(w)y = (a_{-\ell}\mathring{\Xi}^{\mathsf{k}} a_\ell).a_{-\ell}\exp(v)y_j, \tag{7.22}$$

similarly for C^{k} with Ξ^{k} on the right side.

In view of (7.22), (7.19), and the definition of $\bar{\mu}$, $\mathrm{d}\bar{\mu}|_{(u_{r'}\mathring{\Xi})\cap\mathring{\Xi}}$ is a constant multiple of the pushforward of $\varrho_w \cdot \mathrm{d}\mu^{\mathrm{Haar}}_w$ restricted to

$$\big((u_{e^{-\ell}r'}\mathring{\mathsf{C}}^{\mathsf{k}}) \cap \mathring{\mathsf{C}}^{\mathsf{k}}\big).\exp(w)y.$$

Thus, using (7.20) and (7.18), we conclude that $\bar{\mu}(u_{r'}\mathring{\Xi}^{\mathsf{k}}_{j,v} \triangle \mathring{\Xi}^{\mathsf{k}}_{j,v}) \ll \eta\mu(X)$. Altogether, we get

$$\Big|\int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} z)\,\mathrm{d}\mu(z) - \int_{\mathring{\Xi}^{\mathsf{k}}_{j,v}} \varphi(a_{\mathsf{d}} u_{r_1} u_{r'} z)\,\mathrm{d}\mu(z)\Big| \ll \eta\,\mathrm{Lip}(\varphi)\mu(X).$$

The proof is complete. □

8. A convex combination decomposition

Recall that for every $\ell > 0$, we defined

$$\nu_\ell(\varphi) = \int_0^1 \varphi(a_\ell u_r)\,\mathrm{d}r \qquad \text{for all } \varphi \in C_c(H). \tag{8.1}$$

In this section, we will show that $\nu_\ell^{(d)}$ (the d-fold convolution of ν_ℓ) can be approximated with a convex combination $\sum c_i \mu_{\mathcal{E}_i}$, where $\mu_{\mathcal{E}_i}$ is an admissible measure for all i, see §7.6. Since $\nu_\ell^{(d)}$ and $\nu_{d\ell}$ stay close to each other, see Lemma 7.4, we thus conclude that averages of the form appearing in Theorem 1.1 (albeit for $a_{d\ell}$) can be approximated by a convex combination of measures supported on sets which are a finite union of local H orbits. The main results are Lemma 8.4 and Lemma 8.9; the proofs are based on Lemmas 7.5 and 7.7.

The results of this section will be combined with Lemma 9.1 in the proof of Proposition 10.1; see, in particular, part (2) in that proposition.

Convex combination: the base case. Let $x \in X$, and let $t > 0$. Assume that $e^{-t} \le \beta$ and that $h \mapsto hx$ is injective on $\mathsf{E} \cdot a_t \cdot U_1$.

By Proposition 4.2, for every interval $I \subset [0,1]$ with $|I| \ge \delta$, we have

$$\big|\{r \in I : \mathrm{inj}(a_t u_r x) < \varepsilon^2\}\big| < C_4 \varepsilon |I|, \tag{8.2}$$

so long as $t \ge |\log(\delta^2 \mathrm{inj}(x))| + C_4$.

In order to deal with boundary effects, we will consider *interior* points for the supports of ν_t and σ. Let $\nu'_{t,1}$ be the restriction of ν_t to $\{a_t u_r : r \in [e^{-t}, 1-e^{-t}]\}$, note that for every $h \in \mathrm{supp}(\nu'_{t,1})$, we have $U_1.h \subset \mathrm{supp}(\nu_t)$. Applying (8.2), with $\varepsilon = (2\eta)^{1/2}$ and $I = [e^{-t}, 1-e^{-t}]$, we may write

$$\nu_t = \nu_{t,1} + \nu_{t,2}$$

where $\mathrm{supp}(\nu_{t,1}.x) \subset X_{2\eta}$, for every $h \in \mathrm{supp}(\nu_{t,1})$ we have $U_1.h \subset \mathrm{supp}(\nu_t)$, and $\nu_{t,2}(H) \ll e^{-t} \ll \eta^{1/2}$.

Recall that σ is the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$, write $\sigma = \sigma_1 + \sigma_2$ where

$$\sigma_1 = \sigma|_{\mathsf{B}^{s,H}_{\beta-100\beta^2}}.$$

Similarly, write $\nu_t = \mathring{\nu}_t + \partial\nu_t$ where $\mathrm{supp}(\mathring{\nu}_t.x) \subset X_{2\eta}$, for every $h \in \mathrm{supp}(\mathring{\nu}_t)$ we have $U_{1-100\eta}.h \subset \mathrm{supp}(\nu_t)$ and $\partial\nu_t(H) \ll \eta^{1/2}$; also write $\sigma = \mathring{\sigma} + \partial\sigma$ where $\mathring{\sigma} = \sigma|_{\mathsf{B}^{s,H}_\beta}$. Note that

$$\mathrm{supp}(\nu_{t,1}) \subset \mathrm{supp}(\mathring{\nu}_t) \quad \text{and} \quad \mathrm{supp}(\sigma_1) \subset \mathrm{supp}(\mathring{\sigma}).$$

For every $j \in \mathcal{J}_0$ and every $z \in \mathrm{supp}(\sigma_1 * \nu_{t,1}).x \cap \mathsf{Q}^G_0.y_j$, we have $z = \mathsf{h}\exp(w)y_j$ where $w \in B_{\mathfrak{r}}(0, 2\beta^2)$ and

$$\mathsf{h} \in \mathsf{Q}^H_0 = \{u^-_s a_\tau : |s|, |\tau| \le \beta^2\} \cdot U_\eta.$$

In consequence, $\mathsf{Q}^H_0.\exp(w)y_j \subset \mathrm{supp}\big((\mathring{\sigma} * \mathring{\nu}_t).x\big) \cap \mathsf{Q}^G_0.y_j$. This observation, in particular, implies that for every $j \in \mathcal{J}_0$, we have

$$((\sigma * \nu_t).x)|_{\mathsf{Q}^G_0.y_j} = \mu'_j + \sum_{i=1}^{N_j} \bar{\mu}_{j,i}$$

where for all i there exists w_i so that $\bar{\mu}_{j,i} = (\mathring{\sigma} * \mathring{\nu}_t.x)|_{\mathsf{Q}^H_0.\exp(w_i)y_j}$ and

$$\mu'_j(\mathsf{Q}^G_0.y_j) \le ((\sigma_2 * \nu_t).x)(\mathsf{Q}^G_0.y_j).$$

For all $j \in \mathcal{J}_0$, put

$$F_j = \big\{w_i : \bar\mu_{j,i} = (\mathring\sigma * \mathring\nu_t.x)|_{\mathsf{Q}_0^H.\exp(w_i)y_j}\big\}. \tag{8.3}$$

8.1. **Lemma.** *We have*

$$\#F_j \ll \beta^{-3}e^t.$$

Proof. The proof is similar to [LM21, Lemmas 6.4 and 7.5], we reproduce the argument for the convenience of the reader.

Recall from (3.4) that

$$\mathrm{inj}(z) = \min\big\{0.01, \sup\big\{\delta : \ g \mapsto gz \text{ is injective on } \mathsf{B}^G_{100\delta}\big\}\big\},$$

where for every $0 < \delta \le 0.1$ we put $\mathsf{B}^G_\delta := \mathsf{B}^H_\delta \cdot \exp(B_{\mathfrak{r}}(0,\delta))$.

Therefore, for every $z \in X_\eta$, the map $(\mathsf{h}, w) \mapsto \mathsf{h}\exp(w)z$ is injective over $\mathsf{B}^H_{4\eta} \times \exp(B_{\mathfrak{r}}(0,4\eta))$. Hence, for all distinct $w, w' \in B_{\mathfrak{r}}(0,2\eta)$, we have

$$\mathsf{B}^H_{4\eta}\exp(w)z \cap \mathsf{B}^H_{4\eta}\exp(w')z = \emptyset.$$

This, and the fact that $\mathsf{Q}_0^H.\exp(w_i)y_j \subset \mathrm{supp}(\sigma * \nu_t.x) \cap X_\eta$ for every $w_i \in F_j$, implies that

$$(\#F_j)\cdot(\beta^4\eta) \ll \beta^2 e^t.$$

We obtain $\#F_j \ll \beta^{-2}\eta^{-1}e^t \ll \beta^{-3}e^t$, as it was claimed. □

For any $j \in \mathcal{J}_0$ and $1 \le i \le N_j$, define $\mathrm{d}\mu_{j,i}(z) = \rho_{0,j}(z)\,\mathrm{d}\bar\mu_{j,i}(z)$. Altogether, we obtain

$$\sigma * \nu_t.x = \mu' + \sum_{j\in\mathcal{J}_0}\sum_{i=1}^{N_j}\mu_{j,i} \tag{8.4}$$

where $\mu'(X) \ll \eta^{1/2}$. Let

$$c_j = \sum_{i=1}^{N_j}\mu_{j,i}(X). \tag{8.5}$$

8.2. **Lemma.** *If $c_j \ge \beta^{11}$, then $\#F_j = N_j \ge \beta^9 e^t$. Moreover,*

$$\sum_{c_j \ge \beta^{11}} c_j \ge 1 - O(\eta^{1/2})$$

Proof. Recall that $\mathrm{d}\mu_{j,i}(z) = \rho_{0,j}(z)\,\mathrm{d}\bar\mu_{j,i}(z)$, where

$$\bar\mu_{j,i} = (\mathring\sigma * \mathring\nu_t.x)|_{\mathsf{Q}_0^H.\exp(w_i)y_j} \quad \text{and} \quad 1/K \le \rho_{0,j} \le 1.$$

Therefore, $c_j \asymp N_j e^{-t}\beta^{-2}\beta^4\eta = N_j e^{-t}\beta^2\eta$. Hence if $c_j \ge \beta^{11}$, we have

$$N_j \gg \beta^9 e^t$$

where we also used $0 < \eta \le 1$.

To see the second claim, recall from Lemma 7.1 that $\#\mathcal{J}_0 \ll \eta^{-1}\beta^{-10}$. Using $\beta = \eta^2$, thus, we conclude

$$\sum_{c_j<\beta^{11}} c_j \le \beta\eta^{-1} \le \eta.$$

This and the fact that $\mu'(X) \ll \eta^{1/2}$ imply the claim. □

For every j so that $c_j \ge \beta^{11}$, define

$$\mathcal{E}_j = \mathsf{E}.\{\exp(w_i)y_j : w_i \in F_j\}. \tag{8.6}$$

Let $\mu_{\mathcal{E}_j}$ be the restriction of

$$\sum_{i=1}^{N_j} \sigma * \mu_{j,i} \tag{8.7}$$

to $\mathcal{E}_j$, normalized to be a probability measure.

8.3. **Lemma.** *The measure $\mu_{\mathcal{E}_j}$ is a $(1/V, M)$-admissible measure on $\mathcal{E}_j$ where $V = m_{U^-A}(\mathsf{B}^{s,H}_{\beta+100\beta^2})$ and M depends only on X.*

Proof. For every $w_i \in F_j$, let μ_{w_i} denote the restriction of $\sigma * \mu_{j,i}$ to $\mathsf{E}.\exp(w_i)y_j$. Then $\mu_{\mathcal{E}_j} = \frac{1}{\sum_i \mu_{w_i}(X)} \sum \mu_{w_i}$. We will show that

$$\mathrm{d}\mu_{w_i} = V^{-1}\varrho_i \cdot \mathrm{d}m_H|_{\mathsf{E}.\exp(w_i)y_j}$$

where ϱ_i satisfies the desired properties for all i.

Recall that σ is the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$. Moreover, $\mu_{j,i} = \rho_{0,j} \cdot \bar{\mu}_{j,i}$ where

$$\bar{\mu}_{j,i} = (\mathring{\sigma} * \mathring{\nu}_t)|_{\mathsf{Q}^H_0.\exp(w_i)y_j}$$

and $\mathsf{Q}^H_0 = \mathsf{B}^{s,H}_{\beta^2} \cdot U_\eta$. These, together with $1/K \le \rho_{0,j} \le 1$, imply

$$\mathrm{d}\mu_{w_i} = V^{-1}\varrho_i \cdot \mathrm{d}m_H$$

where $1 \ll \varrho_i(\mathsf{h}) \ll 1$.

Let $\Xi^{\mathsf{k}}_{j,i}$ be as in the proof of Lemma 7.5 (and Lemma 7.7) applied with $v = w_i$, write $\mathring{\Xi}^{\mathsf{k}}_{j,i}$ for $(\mathring{\Xi}^{\mathsf{k}}_{j,i})_{\mathsf{Q}^H_0}$. We will show that the claim holds with

$$\mathsf{E}_{w_i} = \bigcup_{\mathsf{k}} \mathsf{E}_{w_i,\mathsf{k}} \quad \text{where} \quad \mathsf{E}_{w_i,\mathsf{k}} = \mathsf{B}^{s,H}_{\beta-100\beta^2} \cdot \mathring{\Xi}^{\mathsf{k}}_{j,i}.$$

First note that the complexity of $\mathsf{E}_{w_i,\mathsf{k}}$ is $\ll 1$ by its definition. Moreover,

$$\mu_{j,i}\big((\Xi^{\mathsf{k}}_{j,i} \setminus \mathring{\Xi}^{\mathsf{k}}_{j,i}).\exp(w_i)y_j\big) \ll \eta\mu_{j,i}(\mathsf{E}.\exp(w_i)y_j).$$

This and Lemma 7.3 imply that

$$\mu_{w_i}\big((\mathsf{E} \setminus \mathsf{E}_{w_i}).\exp(w_i)y_j\big) \ll \eta\mu_{w_i}(\mathsf{E}.\exp(w_i)y_j).$$

Finally, since $\rho_{0,j}$ is constant on $\mathring{\Xi}^{\mathsf{k}}_{j,i}$, we have $\mathrm{Lip}(\varrho_i|_{\mathsf{E}_{w_i,\mathsf{k}}}) \ll 1$. □

The following lemma is the base case of our inductive argument.

8.4. **Lemma.** *Let $x \in X$, and let $t > 0$. Assume that $e^{-t} \leq \beta$ and that $h \mapsto hx$ is injective on $\mathsf{E}\cdot a_t \cdot U_1$. Let $\{c_j\}$ and $\{\mu_{\mathcal{E}_j}\}$ be as in (8.5) and (8.7), respectively. Then for every $\varphi \in C_c^\infty(X)$, every $\mathsf{d} > 0$, and all $|s| \leq 2$,*

$$\Big|\int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}((\sigma * \nu_t).x)(z) - \sum_j c_j \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_j}(z)\Big| \ll \eta^{1/2}\,\mathrm{Lip}(\varphi)$$

where the implied constant depends only on X.

Proof. We begin with the following observation. For every $|r| \leq 2$ and all $\mathsf{h} \in \mathsf{B}^{s,H}_{\beta}$, we have $u_r \mathsf{h} = \mathsf{h}' u_{r_{\mathsf{h}}}$ where $|r_{\mathsf{h}} - r| \ll \beta |r|$ and $\mathsf{h}' \in \mathsf{B}^{s,H}_{10\beta}$, see (7.9). Moreover, $a_{\mathsf{d}} \mathsf{B}^{s,H}_{\bullet} a_{-\mathsf{d}} \subset \mathsf{B}^{s,H}_{\bullet}$. Therefore,

$$\Big|c_j \int \varphi(a_{\mathsf{d}} u_r z)\,\mathrm{d}\mu_{\mathcal{E}_j}(z) - \iint \varphi(a_{\mathsf{d}} u_{r_{\mathsf{h}}} z)\,\mathrm{d}\hat{\mu}_j(z)\,\mathrm{d}\sigma(\mathsf{h})\Big| \ll_X c_j \beta\,\mathrm{Lip}(\varphi) \tag{8.8}$$

where $\hat{\mu}_j = \sum_{i=1}^{N_j} \mu_{j,i}$.

Moreover, by Lemma 7.5 applied with r_{h} and r and $c = 2$, we have

$$\Big|\int \varphi(a_{\mathsf{d}} u_{r_{\mathsf{h}}} z)\,\mathrm{d}\hat{\mu}_j(z) - \int \varphi(a_{\mathsf{d}} u_r z)\,\mathrm{d}\hat{\mu}_j(z)\Big| \ll_X c_j \beta\,\mathrm{Lip}(\varphi). \tag{8.9}$$

In view of (8.4) and since $\sum c_j = 1 - O(\eta^{1/2})$, see Lemma 8.2, the claim follows from (8.8) and (8.9). $\square$

8.5. **Convex combination: the inductive step.** Let $x \in X$, and let t and ℓ be positive. Assume that $e^{-t}, e^{-\ell} < \beta$ and that $h \mapsto hx$ is injective on $\mathsf{E} \cdot a_t \cdot U_1$. We also assume fixed some $\mathsf{d}_0 \geq t, \ell$.

For any $n \in \mathbb{N}$, define

$$\mu_{t,\ell,n} = \nu_\ell * \cdots * \nu_\ell * \sigma * \nu_t \tag{8.10}$$

where ν_ℓ appears n-times. Put $\mu_{t,\ell,0} = \sigma * \nu_t$.

Let $n \geq 1$. Assume there are $0 \leq c_i' \leq 1$ and (λ_{n-1}, M_{n-1})-admissible measures $\{\mu_{\mathcal{E}_i'}\}$ supported on

$$\mathcal{E}_i' = \mathsf{E}.\{\exp(w_q')y_i' : w_q' \in F_i'\} \subset X_\eta$$

so that for every $0 < \mathsf{d} \leq \mathsf{d}_0$ and all $|s| \leq 2$, we have

$$\int \varphi(a_{\mathsf{d}} u_s hx)\,\mathrm{d}\mu_{t,\ell,n-1}(h) = \sum_i c_i' \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i'}(z) + O(\delta_{n-1}\,\mathrm{Lip}(\varphi)) \tag{8.11}$$

for some $0 < \delta_{n-1} \leq 1$.

Our goal in this section is to construct a collection of admissible measures $\mu_{\mathcal{E}_j}$ and constants $0 \leq c_j \leq 1$ so that (8.11) holds for $\mu_{t,\ell,n}$.

We begin with the following non-divergence result.

8.6. **Lemma.** *For every $r \in [0,1]$ we have*

$$\mu_{\mathcal{E}'_i}\big(\{z \in \mathcal{E}'_i : a_\ell u_r z \notin X_{2\eta}\}\big) \ll \eta^{1/2}$$

so long as $\ell \geq 3|\log \eta| + C_4$.

Proof. Recall that $\mathsf{E} = \mathsf{B}^{s,H}_\beta \cdot \{u_{r'} : |r'| \leq \eta\}$. We will show that for every $\mathsf{h} \in \mathsf{B}^{s,H}_\beta$ and every $w'_q \in F'_i$,

$$|\{r' \in [-\eta, \eta] : a_\ell u_r \mathsf{h} u_{r'} \exp(w'_q) y'_i \notin X_{2\eta}\}| \ll \eta^{1/2} \tag{8.12}$$

Since $\mathrm{d}\mu_{w'_q} = \lambda_{n-1} \varrho \,\mathrm{d}m_H$ and $\frac{1}{M_{n-1}} \leq \varrho \leq M_{n-1}$, (8.12) implies the lemma.

To see (8.12), note that $u_r \mathsf{h} = \mathsf{h}' u_{\hat{r}}$, for some $\mathsf{h}' \in \mathsf{B}^{s,H}_{10\beta}$ and $|\hat{r}| \leq 2$. Since $a_\ell \mathsf{B}^{s,H}_{10\beta} a_{-\ell} \subset \mathsf{B}^{s,H}_{10\beta}$, we conclude that

$$a_\ell u_r \mathsf{h} u_{r'} \exp(w'_q) y'_i \subset \mathsf{B}^{s,H}_{10\beta} a_\ell u_{\hat{r}+r'} \exp(w'_q) y'_i. \tag{8.13}$$

Apply Proposition (4.2) with $I = \hat{r} + [-\eta, \eta]$ and $\varepsilon = 3\eta$. Then

$$|\{r' \in [-\eta, \eta] : a_\ell u_{\hat{r}+r'} \exp(w'_q) y'_i \notin X_{3\eta}\}| \ll \eta^{1/2}.$$

This and (8.13) imply (8.12) and finish the proof. □

In view of this lemma, for the remainder of this section, we will assume that $\ell \geq 3|\log \eta| + C_4$.

Recall that $\mathcal{E}'_i = \mathsf{E}.\{\exp(w'_q) y'_i : w'_q \in F'_i\}$ is equipped with the admissible measure $\mu_{\mathcal{E}'_i}$. For every $w'_q \in F'_i$, let $\varrho_{w'_q}$ and $\mathsf{E}_{w'_q} = \bigcup_p \mathsf{E}_{w'_q,p}$ be as in the definition of an admissible measure, §7.6.

Using the notation in (7.7), let $\mathring{\mathsf{E}}_{w'_q} := \bigcup_p (\mathring{\mathsf{E}}_{w'_q,p})_{\mathsf{E}}$. Put

$$\mathring{\mathcal{E}}'_i = \bigcup_{w'_q} \mathring{\mathsf{E}}_{w'_q} \quad \text{and} \quad \mathring{\mu}_{\mathcal{E}'_i} = \mu_{\mathcal{E}'_i}|_{\mathring{\mathcal{E}}'_i}.$$

For every i and $r \in [0,1]$, put $\mu_{i,r} = a_\ell u_r \mu_{\mathcal{E}'_i}$. In view of the definition of $\mathring{\mu}_{\mathcal{E}'_i}$ and Lemma 8.6, we will write $\mu_{i,r} = \mu_{i,r,1} + \mu_{i,r,2}$ where $\mu_{i,r,2}(X) \ll \max\{M_{n-1}\beta, \eta^{1/2}\}$ and

$$\begin{aligned} \operatorname{supp}(\mu_{i,r,1}) &\subset \operatorname{supp}(a_\ell u_r \mathring{\mu}_{\mathcal{E}'_i}) \cap X_{2\eta} \\ &= a_\ell u_r \Big(\bigcup \mathring{\mathsf{E}}_{w'_q}.\{\exp(w'_q) y'_i : w'_q \in F'_i\}\Big) \cap X_{2\eta}, \end{aligned}$$

moreover, for every $z \in \operatorname{supp}(\mu_{i,r,1})$ there are q and p so that

$$\hat{\mathsf{Q}}^H_\ell.z \subset a_\ell u_r \mathsf{E}_{w'_q,p} \exp(w'_q) y'_i,$$

where $\hat{\mathsf{Q}}^H_\ell = \big\{u^-_s a_\tau : e^\ell |s|, |\tau| \leq 100\beta^2\big\} \cdot U_{10\eta}$.

For every $j \in \mathcal{J}_\ell$ as in Lemma 7.1 and every $z \in \operatorname{supp}(\mu_{i,r,1}) \cap \mathsf{Q}^G_\ell.y_j$, we have $z = \mathsf{h} \exp(v) y_j$ where $v \in B_{\mathfrak{r}}(0, 2\beta^2)$ and $\mathsf{h} \in \mathsf{Q}^H_\ell = \big\{u^-_s a_\tau : e^\ell |s|, |\tau| \leq$

$\beta^2\big\} \cdot U_\eta$. Thus,

$$\mathsf{Q}^H_\ell.\exp(v)y_j \subset (a_\ell u_r \mathsf{E}_{w'_q,p}\exp(w'_q)y'_i) \cap \mathsf{Q}^G_\ell.y_j \subset \operatorname{supp}(\mu_{i,r}) \cap \mathsf{Q}^G_\ell.y_j. \tag{8.14}$$

This observation, in particular, implies that for every $j \in \mathcal{J}_\ell$, we have

$$\mu_{i,r}|_{\mathsf{Q}^G_\ell.y_j} = \mu'_{i,r} + \sum_{\varsigma=1}^{N^j_{i,r}} \bar\mu^{j,\varsigma}_{i,r}$$

where for all ς there exists v_ς so that $\bar\mu^{j,\varsigma}_{i,r} = \mu_{i,r}|_{\mathsf{Q}^H_\ell.\exp(v_\varsigma)y_j}$ and

$$\mu'_{i,r}(\mathsf{Q}^G_\ell.y_j) \le \mu_{i,r,2}(\mathsf{Q}^G_\ell.y_j).$$

For all $j \in \mathcal{J}_\ell$, put

$$F^j_{i,r} = \big\{v_\varsigma : \bar\mu^{j,\varsigma}_{i,r} = (\mu_{i,r})|_{\mathsf{Q}^H_\ell.\exp(v_\varsigma)y_j}\big\}. \tag{8.15}$$

For any $j \in \mathcal{J}_\ell$ and $1 \le \varsigma \le N^j_{i,r}$, define $\mathrm{d}\hat\mu^{j,\varsigma}_{i,r}(z) = \rho_{\ell,j}(z)\,\mathrm{d}\bar\mu^{j,\varsigma}_{i,r}(z)$. Then

$$\mu_{i,r} = \mu' + \sum_{j\in\mathcal{J}_\ell}\sum_{\varsigma=1}^{N^j_{i,r}} \hat\mu^{j,\varsigma}_{i,r} \tag{8.16}$$

where $\mu'(X) \ll \max\{\eta^{1/2}, M_{n-1}\beta\}$. For all $j \in \mathcal{J}_\ell$, put

$$c^j_{i,r} = \sum_{\varsigma=1}^{N^j_{i,r}} \hat\mu^{j,\varsigma}_{i,r}(X). \tag{8.17}$$

We have the following analogue of Lemma 8.2.

8.7. **Lemma.** *Assume η is small enough compare to M_{n-1}. If $c^j_{i,r} \ge \beta^{12}e^{-\ell}$, then $\#F^j_{i,r} = N^j_{i,r} \ge \beta^8 \cdot (\#F'_i)$. Moreover,*

$$\sum_{c^j_{i,r}\ge\beta^{12}e^{-\ell}} c^j_{i,r} \ge 1 - O\big(\max\{\eta^{1/2}, M_{n-1}\beta\}\big)$$

Proof. Recall that $\mathrm{d}\hat\mu^{j,\varsigma}_{i,r}(z) = \rho_{\ell,j}(z)\,\mathrm{d}\bar\mu^{j,\varsigma}_{i,r}(z)$ where

$$\bar\mu^{j,\varsigma}_{i,r} = \mu_{i,r}|_{\mathsf{Q}^H_\ell.\exp(v_\varsigma)y_j}$$

and $1/K \le \rho_{0,j} \le 1$.

Since $\mu_{\mathcal{E}'_i}$ is admissible, see §7.6, we have $c^j_{i,r} \asymp N^j_{i,r}\big(e^{-\ell}\beta^4\eta\big) \cdot (\#F'_i)^{-1}$. Therefore, if $c^j_{i,r} \ge \beta^{12}e^{-\ell}$, then

$$N^j_{i,r} \ge \beta^8 \cdot (\#F'_i)$$

where we assume $0 < \eta \le 1$ is small enough to account for the implied constant which depends on M_{n-1}.

To see the second claim, recall from Lemma 7.1 that $\#\mathcal{J}_\ell \ll \eta^{-1}\beta^{-10}e^\ell \leq \beta^{-11}e^\ell$, therefore,

$$\sum_{c^j_{i,r}<\beta^{12}e^{-\ell}} c_j \leq \beta.$$

This and the fact that $\mu'(X) \ll \max\{\eta^{1/2}, M_{n-1}\beta\}$ imply the claim. □

Let j be so that $c^j_{i,r} \geq \beta^{12}e^{-\ell}$. Then by Lemma 8.7, we have $\#F^j_{i,r} \geq \beta^8 \cdot (\#F'_i)$. We write

$$F^j_{i,r} = \tilde{F}^j_{i,r} \bigcup \Big(\textstyle\bigcup_{m=1}^{M^j_{i,r}} F^{j,m}_{i,r}\Big)$$

where $\#\tilde{F}^j_{i,r} < \beta^9 \cdot (\#F'_i)$ and

$$\beta^9 \cdot (\#F'_i) \leq \#F^{j,m}_{i,r} \leq \beta^8 \cdot (\#F'_i) \tag{8.18}$$

for every m.

Let the notation be as in (8.16). As it was observed in the proof of Lemma 8.7, we have $\hat{\mu}^{j,\varsigma}_{i,r}(X) \asymp \hat{\mu}^{j,\varsigma'}_{i,r}(X)$ for all ς, ς'. Thus, we may write

$$\sum_{\varsigma=1}^{N^j_{i,r}} \hat{\mu}^{j,\varsigma}_{i,r} = \mu'_j + \sum_{m=1}^{M^j_{i,r}} \sum_{k=1}^{N^{j,m}_{i,r}} \mu^{j,m,k}_{i,r} \tag{8.19}$$

where $\mu'_j(X) \ll \beta c^j_{i,r}$. Note that for every k, there is some ς so that

$$\mu^{j,m,k}_{i,r} = \hat{\mu}^{j,\varsigma}_{i,r}.$$

Recall that $\mathrm{d}\hat{\mu}^{j,\varsigma}_{i,r}(z) = \rho_{\ell,j}(z)\,\mathrm{d}\bar{\mu}^{j,\varsigma}_{i,r}(z)$, we will write $\bar{\mu}^{j,m,k}_{i,r} = \bar{\mu}^{j,\varsigma}_{i,r}$.

For every $1 \leq m \leq M^j_{i,r}$, put

$$\mu^{j,m}_{i,r} := \sum_{k=1}^{N^{j,m}_{i,r}} \mu^{j,m,k}_{i,r},\ \ c^{j,m}_{i,r} := \mu^{j,m}_{i,r}(X).$$

Then (8.19) and (8.16) yield

$$\mu_{i,r} = \mu'' + \sum_{c^j_{i,r}\geq\beta^{12}e^{-\ell}} \ \sum_{m=1}^{M^j_{i,r}} \mu^{j,m}_{i,r} \tag{8.20}$$

where $\mu''(X) \ll \max\{\eta^{1/2}, M_{n-1}\beta\}$.

For every j so that $c^j_{i,r} \geq \beta^{12}e^{-\ell}$ and all $1 \leq m \leq M^j_{i,r}$, define

$$\mathcal{E}^{j,m}_{i,r} = \mathsf{E}.\{\exp(v_k)y_j : v_k \in F^{j,m}_{i,r}\}. \tag{8.21}$$

Let $\mu_{\mathcal{E}^{j,m}_{i,r}}$ be the restriction of

$$\sigma * \mu^{j,m}_{i,r} \tag{8.22}$$

to $\mathcal{E}^{j,m}_{i,r}$, normalized to be a probability measure.

We will refer to $(\mathcal{E}_{i,r}^{j,m}, \mu_{\mathcal{E}_{i,r}^{j,m}})$ as an *offspring* of $a_\ell u_r \mu_{\mathcal{E}'_i}$.

8.8. **Lemma.** *The measure $\mu_{\mathcal{E}_{i,r}^{j,m}}$ is a (λ_n, M_n)-admissible measure, where M_n depends only on X and M_{n-1}.*

Proof. The proof is similar to Lemma 8.3. Since r, i, j, and m are fixed throughout the argument, we will drop them from the notation whenever there is no confusion, e.g., we denote $\mathcal{E}'_i$ by $\mathcal{E}'$, $\mu_{i,r}^{j,m,k}$ by μ^k, and $\mathcal{E}_{i,r}^{j,m}$ by $\mathcal{E}$.

Recall that for every k, $\mathrm{d}\mu^k = \rho_{\ell,j}\,\mathrm{d}\bar{\mu}^k$ where $\bar{\mu}^k = \mu_{i,r}|_{\mathsf{Q}_\ell^H.\exp(v_k)y_j}$ and $1/K \le \rho_{\ell,j}(z) \le 1$. Also recall that there are w'_q and p so that

$$\mathrm{supp}(\bar{\mu}^k) \subset a_\ell u_r\big(\mathsf{E}_{w'_q,p}.\exp(w'_q)y'_i\big).$$

Moreover, $\varrho_{w'_q}$ (in the definition of $\mu_{w'_q}$) is M_{n-1}-Lipschitz on $\mathsf{E}_{w'_q,p}$.

For every $v_k \in F$, let μ_{v_k} denote the restriction of $\sigma * \mu^k$ to $\mathsf{E}.\exp(v_k)y_j$. Thus $\mu_{\mathcal{E}} = \frac{1}{\sum_i \mu_{v_k}(X)} \sum \mu_{v_k}$, and we have

$$\mathrm{d}\mu_{v_k}(\cdot) = \lambda_n \varrho_k(\cdot)\,\mathrm{d}m_H(\cdot).$$

We will show that ϱ_k satisfies the desired properties for all k.

Recall that $\mathsf{Q}_\ell^H = \{u_s^- : |s| \le e^{-\ell}\beta^2\} \cdot \{a_\tau : |\tau| \le \beta^2\} \cdot U_\eta$, and that σ is the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$. For every

$$\mathsf{h}\exp(v_k)y_j \in \mathsf{Q}_\ell^H.\exp(v_k)y_j = \mathrm{supp}(\bar{\mu}^k),$$

there exists a unique $\mathsf{h}' \in \mathsf{E}_{w'_q,p}$ so that $a_\ell u_r \mathsf{h}'\exp(w'_q)y'_i = \mathsf{h}\exp(v_k)y_j$. Let us define $\hat{\varrho}_k$ on Q_ℓ^H by

$$\hat{\varrho}_k(\mathsf{h}) = \rho_{\ell,j}(\mathsf{h}\exp(v_k)y_j)\varrho_{w'_q}(\mathsf{h}'\exp(w'_q)y_j)$$

We note that $\varrho_k = \sigma * \hat{\varrho}_k$. Thus $(KM_{n-1})^{-1} \ll \varrho_k \ll M_{n-1}$.

For every $1 \le \mathsf{f} \le K$, let $\Xi^{\mathsf{f}}_{j,i}$ be as in the proof of Lemma 7.5 (and Lemma 7.7) applied with $v = v_k$, and write $\mathring{\Xi}^{\mathsf{f}}_{j,k}$ for $(\Xi^{\mathsf{f}}_{j,k})_{\mathsf{Q}_\ell^H}$. In particular, $\rho_{\ell,j}$ equals $1/\mathsf{f}$ on $\mathring{\Xi}^{\mathsf{f}}_{j,k}$. We will show that the claim holds with

$$\mathsf{E}_{v_k} = \bigcup_d \mathsf{E}_{v_k,\mathsf{f}} \quad \text{where} \quad \mathsf{E}_{v_k,\mathsf{f}} = \mathsf{B}^{s,H}_{\beta-100\beta^2} \cdot \mathring{\Xi}^{\mathsf{f}}_{j,k}.$$

To see this note that the complexity of $\mathsf{E}_{v_k,\mathsf{f}}$ is $\ll 1$ by its definition. Moreover, $\rho_{\ell,j}$ is constant on $\mathring{\Xi}^{\mathsf{f}}_{j,k}$. Thus in order to control $\mathrm{Lip}(\varrho_k)$ on $\mathsf{E}_{v_k,\mathsf{f}}$, we may drop $\rho_{\ell,j}$ from the definition of $\hat{\varrho}_k$ above. Now $u_{r'}a_\ell u_r = a_\ell u_{r+e^{-\ell}r'}$, $\mathrm{Lip}(\varrho_{w'_q}|\mathsf{E}_{w'_q,p}) \le M_{n-1}$, furthermore,

$$\mathsf{B}^{s,H}_{\beta-100\beta^2} \subset \mathrm{supp}(\sigma) \setminus \partial_{100\beta^2}\,\mathrm{supp}(\sigma).$$

Altogether, we conclude that $\mathrm{Lip}(\sigma * \hat{\varrho}_k) \ll M_{n-1}$ on $\mathsf{E}_{v_k,\mathsf{f}}$ for every f.

The proof is complete. □

8.9. **Lemma.** *Let $x \in X$, and let ℓ and t be positive. Assume that $e^{-\ell}, e^{-t} < \beta$ and that $h \mapsto hx$ is injective on $\mathsf{E} \cdot a_t \cdot U_1$.*

Suppose that for every i, we have fixed $L_i \subset [0,1]$ with $|[0,1] \setminus L_i| \le \delta$, and let $\{r_{i,q} : q = 1, \ldots, N_i\}$ be a maximal $e^{-3\mathsf{d}_0}$-separated subset of L_i. Let $\varphi \in C_c^\infty(X)$, $0 < \mathsf{d} \le \mathsf{d}_0 - \ell$, and $|s| \le 2$. Then for every $r_{i,q}$ we have

$$\Big|\int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}(a_\ell u_{r_{i,q}} \mu_{\mathcal{E}'_i})(z) - \sum c_{i,r_{i,q}}^{j,m} \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i^{j,m}}(z)\Big| \ll \max\big\{\eta^{1/2}, M_{n-1}\beta, \beta\big\}\operatorname{Lip}(\varphi), \tag{8.23}$$

where $\sum = \sum_j \sum_m$. Moreover, we have

$$\Big|\int \varphi(a_{\mathsf{d}} u_s hx)\,\mathrm{d}\,\mu_{t,\ell,n}(h) - \sum c_{i,r_{i,q}}^{j,m} \int \varphi(a_{\mathsf{d}} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_{i,r_{i,q}}^{j,m}}(z)\Big| \ll \max\big\{\eta^{1/2}, M_{n-1}\beta, \delta, \delta_{n-1}\big\}\operatorname{Lip}(\varphi), \tag{8.24}$$

where $\sum = \sum_i \sum_q \sum_j \sum_m$.

The implied constants depend only on X and M_{n-1}.

Proof. The proof is similar to the proof of Lemma 8.4. Indeed loc. cit. will be used as case $n = 0$ in our inductive proof of this lemma.

We will first reduce (8.24) to (8.23):

$$\begin{aligned}\int \varphi(a_{\mathsf{d}} u_s hx)\,\mathrm{d}\mu_{t,\ell,n}(h) &= \iint \varphi(a_{\mathsf{d}} u_s a_\ell u_r hx)\,\mathrm{d}\mu_{t,\ell,n-1}(h)\,\mathrm{d}r\\ &= \iint \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} hx)\,\mathrm{d}\mu_{t,\ell,n-1}(h)\,\mathrm{d}r\\ &= \sum_i c'_i \iint \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} z)\,\mathrm{d}\mu_{\mathcal{E}'_i}(z)\,\mathrm{d}r + O(\delta_{n-1}\operatorname{Lip}(\varphi));\end{aligned}$$

in the last equality we used (8.11), and $0 < d + \ell \le \mathsf{d}_0$ and $|r + se^{-\ell}| \le 2$.

Since $|[0,1] \setminus L_i| \le \delta$ and $\{r_{i,q} : q = 1, \ldots, N_i\} \subset L_i$ is a maximal $e^{-3\mathsf{d}_0}$-separated subset, we have

$$\sum_i c'_i \iint \varphi(a_{\mathsf{d}+\ell} u_{r+se^{-\ell}} z)\,\mathrm{d}\mu_{\mathcal{E}'_i}(z)\,\mathrm{d}r = \sum_i \sum_q \int \varphi(a_{\mathsf{d}+\ell} u_{r_{i,q}+se^{-\ell}} z)\,\mathrm{d}\mu_{\mathcal{E}'_i}(z) + O\big(\max\{\delta, \beta\}\operatorname{Lip}(\varphi)\big),$$

where we again used $\mathsf{d} + \ell \le \mathsf{d}_0$.

In view of this, let us fix some i and q, and investigate

$$\int \varphi(a_{\mathsf{d}+\ell} u_{r_{i,q}+se^{-\ell}} z)\,\mathrm{d}\mu_{\mathcal{E}'_i}(z) = \int \varphi(a_{\mathsf{d}} u_s a_\ell u_{r_{i,q}} z)\,\mathrm{d}\mu_{\mathcal{E}'_i}(z),$$

which also completes the reduction of (8.24) to (8.23).

For simplicity, let us write $r = r_{i,q}$. Using (8.16), we have

$$\int \varphi(a_{\mathsf{d}} u_s a_\ell u_r z) \mathrm{d}\mu_{\mathcal{E}'_i}(z) = \sum_j \int \varphi(a_{\mathsf{d}} u_s z) \mathrm{d}\Big(\sum_{\varsigma} \hat{\mu}_{i,r}^{j,\varsigma}\Big)(z) + O(\beta \operatorname{Lip}(\varphi)).$$

In view of (8.20), see also Lemma 8.7, it suffices to consider j's so that $c_{i,r} \geq \beta^{12} e^{-\ell}$, we will however need to add

$$O\big(\max\{\eta^{1/2}, M_{n-1}\beta\} \operatorname{Lip}(\varphi)\big)$$

to the error. Moreover, using (8.19), we may replace $\sum_\varsigma \hat{\mu}_{i,r}^{j,\varsigma}$ with $\sum_m \mu_{i,r}^{j,m}$. Fix one such $j \in \mathcal{J}_\ell$ and let $1 \leq m \leq M_{i,r}^j$. Then $\mu_{i,r}^{j,m} = \sum_k \mu_{i,r}^{j,m,k}$.

We now compare

$$\int \varphi(a_{\mathsf{d}} u_s z) \mathrm{d}\Big(\sum_k \mu_{i,r}^{j,m,k}\Big)(z)$$

with $\int \varphi(a_d u_s z)\, \mathrm{d}\mu_{\mathcal{E}_{i,r}^{j,m}}(z)$. Recall from (8.22) that

$$\int c_{i,r}^j \varphi(a_{\mathsf{d}} u_s z)\, \mathrm{d}\mu_{\mathcal{E}_{i,r}^{j,m}}(z) = \sum_k \iint \varphi(a_{\mathsf{d}} u_s \mathsf{h} z)\, \mathrm{d}\mu_{i,r}^{j,m,k}(z)\, \mathrm{d}\sigma^s(\mathsf{h}).$$

For every $\mathsf{h} \in \mathsf{B}_\beta^{s,H}$ and all $|s| \leq 2$, we have $u_s \mathsf{h} = \mathsf{h}' u_{s+s_{\mathsf{h}}}$ where $|s_{\mathsf{h}}| \ll \beta$ and $\mathsf{h}' \in \mathsf{B}_{10\beta}^{s,H}$, moreover, $a_d \mathsf{B}_{10\beta}^{s,H} a_{-d} \subset \mathsf{B}_{10\beta}^{s,H}$ for all $d > 0$. Therefore, for every k and all $\mathsf{h} \in \mathsf{B}_\beta^{s,H}$, we have

$$\Big|\int \varphi(a_{\mathsf{d}} u_s \mathsf{h} z)\, \mathrm{d}\mu_{i,r}^{j,m,k}(z) - \int \varphi(a_{\mathsf{d}} u_{s+s_{\mathsf{h}}} z)\, \mathrm{d}\mu_{i,r}^{j,m,k}(z)\Big| \ll \beta \operatorname{Lip}(\varphi) \mu_{i,r}^{j,m,k}(X).$$

Finally by Lemma 7.7, we have

$$\Big|\int \varphi(a_{\mathsf{d}} u_{s+s_{\mathsf{h}}} z)\, \mathrm{d}\mu_{i,r}^{j,m,k}(z) - \int \varphi(a_{\mathsf{d}} u_{s+s_{\mathsf{h}}} z)\, \mathrm{d}\mu_{i,r}^{j,m,k}(z)\Big| \\ \ll M_{n-1}\beta \operatorname{Lip}(\varphi) \mu_{i,r}^{j,m,k}(X)$$

which completes the proof. □

9. Margulis functions and Incidence geometry

In this section, we will prove Lemma 9.1 which is one of the main ingredients in the proof of Proposition 10.1, see also Proposition 2.3.

The set $\mathcal{E}$ and the measure $\mu_{\mathcal{E}}$. Let $0 < \eta \leq 0.01\eta_X$ and $\beta = \eta^2$. Recall that

$$\mathsf{E} = \mathsf{B}_\beta^{s,H} \cdot \{u_r : |r| \leq \eta\}$$

where $\mathsf{B}_\beta^{s,H} := \{u_s^- : |s| \leq \beta\} \cdot \{a_t : |t| \leq \beta\}$.

Let $F \subset B_{\mathfrak{r}}(0,\beta)$ be a finite set, and let $y \in X_{2\eta}$. Then $\exp(w)y \in X_\eta$ for all $w \in F$, moreover, $\mathsf{h} \mapsto \mathsf{h}\exp(w)y$ is injective over E. For every subset $\mathsf{E}' \subset \mathcal{E}$, put

$$\mathcal{E}_{\mathsf{E}'} = \bigcup \mathsf{E}'.\{\exp(w)y : w \in F\}; \tag{9.1}$$

we will denote $\mathcal{E}_{\mathsf{E}}$ by $\mathcal{E}$. Throughout this section, we will assume fixed an admissible measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ whose definition we now recall from §7.6.

Let $\lambda, M > 0$. A probability measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ is said to be (λ, M)-admissible if

$$\mu_{\mathcal{E}} = \frac{1}{\sum_{w\in F}\mu_w(X)}\sum_{w\in F}\mu_w$$

where for every $w \in F$, μ_w is a measure on $\mathsf{E}.\exp(w)y$ satisfying that

$$\mathrm{d}\mu_w(\mathsf{h}\exp(w)y) = \lambda\varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}) \quad \text{where } 1/M \le \varrho_w(\bullet) \le M; \tag{9.2}$$

moreover, there is a subset $\mathsf{E}_w = \bigcup_{p=1}^M \mathsf{E}_{w,p} \subset \mathsf{E}$ so that

(1) $\mu_w\big((\mathsf{E}\setminus\mathsf{E}_w).\exp(w)y\big) \le M\beta\mu_w(\mathsf{E}.\exp(w)y)$,
(2) The complexity of $\mathsf{E}_{w,p}$ is bounded by M for all p, and
(3) $\mathrm{Lip}(\varrho_w|_{\mathsf{E}_{w,p}}) \le M$ for all p.

Regularity of $\mathcal{E}$. Let $0 < \delta \le \mathrm{inj}(z)$ for all $z \in \mathcal{E}$. We will say $\mathcal{E}$ is (c,δ)-regular if for all $w \in F$

$$\#(F \cap B_{\mathfrak{r}}(w, \delta/100)) \ge c \cdot \big(\#(F \cap B_{\mathfrak{r}}(w,\delta))\big), \tag{9.3}$$

see §6.3 where similar (and finer) regularity properties are discussed.

Our goal is to show that the discretized dimension of $\mathcal{E}$ at controlled scales will improve under a certain random walk. We begin by defining a function which encodes this discretized transversal dimension.

Let $0 < b \le 1/10$. For every $(h,z) \in H \times \mathcal{E}$, define

$$I_{\mathcal{E},b}(h,z) := \big\{w \in \mathfrak{r} : \|w\| < b\,\mathrm{inj}(hz),\ \exp(w)hz \in h\mathcal{E}.x\big\}. \tag{9.4}$$

Note that $I_{\mathcal{E},b}(h,z)$ contains 0 for all $z \in \mathcal{E}$. Moreover, since E is bounded, $I_{\mathcal{E},b}(h,z)$ is a finite set for all $(h,z) \in H \times \mathcal{E}$.

Fix some $0 < \alpha < 1$. For every $\mathsf{R} \ge 1$, define the modified and localized Margulis function $f_{\mathcal{E},b,\mathsf{R}} : H \times \mathcal{E} \to [1,\infty)$ as follows: if $\#I_{\mathcal{E},b}(h,z) \le \mathsf{R}$, put

$$f_{\mathcal{E},b,\mathsf{R}}(h,z) = (b\,\mathrm{inj}(hz))^{-\alpha};$$

and if $\#I_{\mathcal{E},b}(h,z) > \mathsf{R}$, put

$$f_{\mathcal{E},b,\mathsf{R}}(h,z) = \min\left\{\sum_{w\in I}\|w\|^{-\alpha} : \begin{array}{c} I \subset I_{\mathcal{E},b}(h,z) \text{ and} \\ \#(I_{\mathcal{E},b}(h,z)\setminus I) = \mathsf{R} \end{array}\right\}.$$

Let us also define $\psi_{\mathcal{E},b}$ on $H \times \mathcal{E}$ by

$$\psi_{\mathcal{E},b}(h,z) := (b\,\mathrm{inj}(hz))^{-\alpha}\cdot\big(\#I_{\mathcal{E},b}(h,z)\big). \tag{9.5}$$

If $\mathsf{E}' \subset \mathsf{E}$, we define $I_{\mathcal{E}_{\mathsf{E}'},b}$, $\psi_{\mathcal{E}_{\mathsf{E}'},b}$, and $f_{\mathcal{E}_{\mathsf{E}'},b,\mathsf{R}}$ accordingly.

Recall also the definition of $\mathcal{G}$ from §6. Let $0 < b_0 \leq 1$, and let $I \subset B_{\mathfrak{r}}(0, b_0)$. For $\mathsf{R} \geq 1$, define $\mathcal{G}_{I,\mathsf{R}} : I \to (0, \infty)$ as follows: If $\#I \leq \mathsf{R}$, put

$$\mathcal{G}_{I,\mathsf{R}}(w) = b_0^{-\alpha}, \quad \text{for all } w \in I,$$

and if $\#I > \mathsf{R}$, put

$$\mathcal{G}_{I,\mathsf{R}}(w) = \min \left\{ \sum_{I'} \|w - w'\|^{-\alpha} : \begin{array}{c} I' \subset I \text{ and} \\ \#(I \setminus I') = \mathsf{R} \end{array} \right\}.$$

Fix a small parameter $0 < \varepsilon < 1$, and let $0 < \kappa \leq \varepsilon/10^6$. Throughout the section, we assume

$$e^{-\varepsilon t/10^6} \leq \beta \quad \text{and} \quad \ell = 0.01\varepsilon t.$$

We will also use the following notation:

$$\partial_{\delta_1,\delta_2}\mathsf{E} = \left(\partial_{\delta_1}\mathsf{B}^{s,H}_{\beta}\right) \cdot (\partial_{\delta_2}\{u_r : |r| \leq \eta\}), \qquad \text{for } \delta_1, \delta_2 > 0;$$

we denote $\partial_{\delta,\delta}\mathsf{E}$ simply by $\partial_\delta\mathsf{E}$.

The following is the main result of this section.

9.1. **Lemma.** *Let $F \subset B_{\mathfrak{r}}(0, \beta)$ be a finite set with $\#F \geq e^{9t/10}$. Assume that F satisfies (9.3) with $\delta = \frac{1}{10}\mathrm{inj}(y)b$ and some $c \geq e^{-\kappa^2 t/4}$.*

Let $\mathcal{E} = \bigcup \mathsf{E}.\{\exp(w)y : w \in F\}$, and put

$$\hat{\mathcal{E}} = \bigcup \hat{\mathsf{E}}.\{\exp(w)y : w \in F\}$$

where $\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$.

Assume that for some $\Upsilon \geq 1$ (large enough depending on κ) some $1 \leq \mathsf{R} \leq e^{\varepsilon t/100}$, and for $b = e^{-\sqrt{\varepsilon}t}$, we have

$$f_{\mathcal{E},b,\mathsf{R}}(e, z) \leq \Upsilon, \qquad \text{for all } z \in \mathcal{E}. \tag{9.6}$$

There exists $L_{\mu_\mathcal{E}} \subset [0, 1]$ with

$$|[0, 1] \setminus L_{\mu_\mathcal{E}}| \ll e^{-\kappa^2 t/4}$$

and for every $r \in L_{\mu_\mathcal{E}}$, there exists a subset $\mathcal{E}_r \subset \hat{\mathcal{E}}$ with

$$\mu_\mathcal{E}(\mathcal{E} \setminus \mathcal{E}_r) \ll e^{-\kappa^2 t/64}$$

so that the following holds. For every $z \in \mathcal{E}_r$ we have

$$f_{\hat{\mathcal{E}},b,\mathsf{R}_1}(a_\ell u_r, z) \leq 200e^{-\alpha\ell}L_1\Upsilon^{1+8\kappa} + 200e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(a_\ell u_r, z)$$

where $L_1 = L\kappa^{-L}$ and $\mathsf{R}_1 = \mathsf{R} + L_1\Upsilon^\kappa$, see Theorem 6.2.

The proof of this lemma relies on Theorem 6.2 and will be completed in some steps. We begin with the following lemma.

9.2. **Lemma.** *Assume (9.6) holds. Let*

$$\mathcal{E}' = \bigcup \mathsf{E}'.\{\exp(w)y : w \in F\}$$

where $\mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{5b}\mathsf{E}}$. *Let* $m \in \mathbb{N}$. *Put* $z = \mathsf{h}\exp(w_z)y \in \mathcal{E}'$, *and let* $I_z := I_{\mathcal{E}',mb}(e,z)$. *Then*

$$\mathcal{G}_{I_z,\mathsf{R}}(w) \leq (2+6m^4)\Upsilon \qquad \textit{for every } w \in I_z,$$

where $\mathcal{G}$ *is defined as above with* $b_0 = mb\,\mathrm{inj}(z)$.

Proof. Let $w \in I_z$, then $z' := \exp(w)z \in \mathcal{E}'$. We will estimate $\mathcal{G}_{I_z,\mathsf{R}}(w)$ in terms of $f_{\mathcal{E},b,\mathsf{R}}(e,z')$.

Note that for every $v \in I_z$, there exists some $w_v \in F$ and some $\mathsf{h}_v \in \mathsf{E}'$ so that $\exp(v)z = \mathsf{h}_v\exp(w_v)y$. Thus

$$\begin{aligned} \mathsf{h}_v\exp(w_v)y &= \exp(v)z \\ &= \exp(v)\exp(-w)z' = \mathsf{h}'\exp(w_v')z' \end{aligned} \tag{9.7}$$

where $\|\mathsf{h}' - I\| \ll b^2$ and $\frac{1}{2}\|v-w\| \leq \|w_v'\| \leq 2\|v-w\|$, see Lemma 3.2.

Since $\mathsf{h}_v \in \mathsf{E}'$, we conclude from (9.7) that

$$\exp(w_v')z' = \mathsf{h}'^{-1}\mathsf{h}_v\exp(w_v)y \in \mathcal{E}$$

where we used $\mathsf{h}_v \in \mathsf{E}'$ and $\|\mathsf{h}' - I\| \ll b^2$. We emphasize that we can only guarantee $\exp(w_v')z'$ belongs to $\mathcal{E}$ and not necessarily to $\mathcal{E}' \subset \mathcal{E}$.

Note that, $v \mapsto w_v'$ is one-to-one. Moreover,

$$\text{if } \|v-w\| < \frac{1}{2}b\,\mathrm{inj}(z'), \text{ then } w_v' \in I_{\mathcal{E},b}(e,z'), \tag{9.8}$$

since in that case we have $\|w_v'\| < b\,\mathrm{inj}(z')$.

Let $\{w_1 = w, w_2, \ldots, w_N\} \subset I_z$ be a maximal $b/4$ separated subset; then $N \leq m^4$. Arguing as above with all w_i, we also conclude that

$$I_z \subset \bigcup_{i=1}^{N} I_{\mathcal{E},b}(e,z_i), \quad \text{for some } \{z_1,\ldots,z_N\} \subset \mathcal{E}. \tag{9.9}$$

Since $b = e^{-\sqrt{\varepsilon}t}$ and $\#F \geq e^{0.9t}$, we have $\sup_{\hat{z}\in\mathcal{E}} \#I_{\mathcal{E},b}(e,\hat{z}) \geq e^{0.8t}$. Therefore, (9.6) and the fact that $0 \leq \mathsf{R} \leq e^{0.01t}$ imply

$$2\Upsilon \geq \sup_{\hat{z}\in\mathcal{E}} (b\,\mathrm{inj}(\hat{z}))^{-\alpha} \cdot \big(\#I_{\mathcal{E},b}(e,\hat{z})\big) \tag{9.10}$$

Recall now that $0.9\,\mathrm{inj}(y) \leq \mathrm{inj}(\hat{z}) \leq 1.1\,\mathrm{inj}(y)$ for all $\hat{z} \in \mathcal{E}$. Therefore, (9.9) and (9.10) imply that

$$\begin{aligned} b\,\mathrm{inj}(z')^{-\alpha}\cdot(\max\{1,\#I_z\}) &\leq \tfrac{3}{2}\sum b\,\mathrm{inj}(z_i)^{-\alpha}\cdot(\max\{1,\#I_{\mathcal{E},b}(e,z_i)\}) \\ &\leq 3m^4\Upsilon. \end{aligned} \tag{9.11}$$

We now consider two cases: If $\#I_{\mathcal{E},b}(e,z') \leq \mathsf{R}$, then (9.8) implies that $\#\{v \in I_z : \|v-w\| < \frac{1}{2}b\,\mathrm{inj}(z')\} \leq \mathsf{R}$. Hence, using (9.11), we get

$$\mathcal{G}_{I_z,\mathsf{R}}(w) \leq 2(b\,\mathrm{inj}(z'))^{-\alpha}\cdot(\max\{1,\#I_z\}) \leq 6m^4\Upsilon$$

This completes the proof in this case.

Thus, let us assume $\#I_{\mathcal{E},b}(e,z') > \mathsf{R}$, and let $I' \subset I_{\mathcal{E},b}(e,z')$ be so that

$$\sum_{w'\in I'} \|w'\|^{-\alpha} = f_{\mathcal{E},b,\mathsf{R}}(e,z') \le \Upsilon.$$

Let $I = \{v \in I_z : \|v - w\| < \frac{1}{2}b\,\mathrm{inj}(z') \text{ and } w'_v \notin I'\}$. Since $v \mapsto w'_v$ is a one-to-one map from I into $I_{\mathcal{E},b}(e,z') \setminus I'$, see (9.8), we have $\#I \le \mathsf{R}$. Therefore,

$$\begin{aligned}\mathcal{G}_{I_z,\mathsf{R}}(w) &\le \sum_{v\in I_z\setminus I} \|v-w\|^{-\alpha} \le 2\sum_{v\in I_z\setminus I}\|w'_v\|^{-\alpha}\\ &\le 2\sum_{w'\in I'}\|w'\|^{-\alpha} + 2(b\,\mathrm{inj}(z'))^{-\alpha}\cdot(\max\{1,\#I_z\})\\ &\le (2+6m^4)\Upsilon,\end{aligned}$$

where we used $\frac{1}{2}\|v-w\| \le \|w'_v\|$ in the second inequality, the definition of I in the third inequality, and (9.11) in the final inequality.

This completes the proof of this case and of the lemma. □

Let us also record the following two lemma whose proof is essentially included in the argument at the beginning of the proof of Lemma 9.2.

9.3. **Lemma.** *Let $\hat{\mathcal{E}} \subset \mathcal{E}'$ be as above. Let $0 < m \le 100$, $z \in \hat{\mathcal{E}}$, and $\delta \le mb\,\mathrm{inj}(z)$. Write $z = \mathsf{h}_z \exp(w_z)y$ where $\mathsf{h}_z \in \hat{\mathsf{E}}$ and $w_z \in F$. Then*

$$\begin{aligned}\#\big(F\cap B_{\mathfrak{r}}(w_z,\delta/2)\big) &\le \#\big(I_{\mathcal{E}',mb}(e,z)\cap B_{\mathfrak{r}}(e,\delta)\big)\\ &\le \#\big(F\cap B_{\mathfrak{r}}(w_z,2\delta)\big).\end{aligned} \tag{9.12}$$

Proof. Let us write $I_z = I_{\mathcal{E}',mb}(e,z)$. We will first show: there is an injective map from $I_z \cap B_{\mathfrak{r}}(0,\delta)$ into $F\cap B_{\mathfrak{r}}(w_z,2\delta)$. For every $v \in I_z\cap B_{\mathfrak{r}}(0,\delta)$, there are $w_v \in F$ and $\mathsf{h}_v \in \mathsf{E}'$ so that $\exp(v)z = \mathsf{h}_v\exp(w_v)y$. Thus

$$\begin{aligned}\mathsf{h}_v\exp(w_v)y &= \exp(v)z\\ &= \exp(v)\mathsf{h}_z\exp(w_z)y = \mathsf{h}_z\exp(\mathrm{Ad}(\mathsf{h}_z^{-1})v)\exp(w_z)y\\ &= \mathsf{h}'\exp(w'_v)y\end{aligned}$$

where $\le \|w'_v - w_z\| \le \frac{3}{2}\|\mathrm{Ad}(\mathsf{h}_z^{-1})v\| < 2\|v\|$, see Lemma 3.2. Since the map $(h,w)\mapsto h\exp(w)y$ is injective on $\mathsf{B}^G_{10\eta}$, we conclude that $w_v = w'_v$. Thus $v\mapsto w_v$ is an injection from $I_z\cap B_{\mathfrak{r}}(0,\delta)$ into $F\cap B_{\mathfrak{r}}(w_z,2\delta)$.

The other direction is similar, let $w\in F\cap B_{\mathfrak{r}}(w_z,\delta/2)$. Then

$$\begin{aligned}\exp(w)y &= \exp(w)\exp(-w_z)\exp(w_z)y\\ &= \exp(w)\exp(-w_z)\mathsf{h}_z^{-1}z = \mathsf{h}'\exp(v'_w)\mathsf{h}_z^{-1}z\\ &= \mathsf{h}'\mathsf{h}_z^{-1}\exp(\mathrm{Ad}(\mathsf{h}_z)v'_w)z\end{aligned}$$

where $\|\mathsf{h}' - I\| \ll \eta\|w-w_z\|$ and $\|\mathrm{Ad}(\mathsf{h}_z)v'_w\| < 2\|w-w_z\|$, see Lemma 3.2.

Put $v_w = \mathrm{Ad}(\mathsf{h}_z)v'_w$. Then the above implies

$$\exp(v_w)z = \mathsf{h}_z\mathsf{h}'^{-1}\exp(w)y.$$

Since $\|\mathsf{h}' - I\| \ll \eta\|w - w_z\| \ll b\eta\mathrm{inj}(z)$ and $\mathsf{h}_z \in \hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$, we conclude

$$\mathsf{h}_z\mathsf{h}'^{-1} \in \mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{5b}\mathsf{E}}.$$

Hence $\exp(v_w)z \in \mathcal{E}'$. Moreover, we have $\|v_w\| \le 2\|w - w_z\| < \delta$. These imply that $v_w \in I_z \cap B_{\mathfrak{r}}(e, \delta)$. Altogether, $w \mapsto v_w$ is an injection from $F \cap B_{\mathfrak{r}}(w_z, \delta/2)$ into $I_z \cap B_{\mathfrak{r}}(e, \delta)$. The proof is complete. □

Let us also record the following lemma for later use

9.4. **Lemma.** *Assume (9.6) holds. Let $m \in \mathbb{N}$. For any $w \in F$, put $F_w = B_{\mathfrak{r}}(w, mb\,\mathrm{inj}(y)) \cap F$. Then*

$$\mathcal{G}_{F_w,\mathsf{R}}(w') \le (2 + 6(4m)^4)\Upsilon \qquad \textit{for every } w' \in F_w.$$

Proof. Let $w' \in F_w$ and put $z' = \exp(w')y$. Then $z' \in \hat{\mathcal{E}}$, and as it was done in the proof of Lemma 9.3, for every $w' \ne \hat{w} \in F_w$ we have

$$\begin{aligned}\exp(\hat{w})y &= \exp(\hat{w})\exp(-w')\exp(w')y\\ &= \exp(\hat{w})\exp(-w')\mathsf{h}_{w'}^{-1}z' = \bar{\mathsf{h}}\exp(v'_{\hat{w}})\mathsf{h}_{w'}^{-1}z'\\ &= \bar{\mathsf{h}}\mathsf{h}_{w'}^{-1}\exp(\mathrm{Ad}(\mathsf{h}_{w'})v'_{\hat{w}})z\end{aligned}$$

where $\|\bar{\mathsf{h}} - I\| \ll \eta\|\hat{w} - w'\|$ and $\|\mathrm{Ad}(\mathsf{h}_{w'})v'_{\hat{w}}\| < 2\|\hat{w} - w'\|$, see Lemma 3.2.

Put $v_{\hat{w}} = \mathrm{Ad}(\mathsf{h}_{w'})v'_{\hat{w}}$. Then, as in Lemma 9.3, we have $v_{\hat{w}} \in I_{\mathcal{E}',4mb}(e, z')$ and the map $\hat{w} \mapsto v_{\hat{w}}$ is injective — note that $\|\hat{w} - w'\| \le 2mb\,\mathrm{inj}(y)$.

This and Lemma 9.2, imply that

$$\mathcal{G}_{F_w,\mathsf{R}}(w') \le \mathcal{G}_{I_{\mathcal{E}',4mb}(e,z'),\mathsf{R}}(0) \le (2 + 6(4m)^4)\Upsilon$$

for every $w' \in F_w$. □

Proof of Lemma 9.1. The proof will be completed in some steps.

For every $w \in \mathfrak{r}$ and all $r \in [0, 1]$, let

$$\xi_r(w) = (\mathrm{Ad}(u_r)w)_{12} = -w_{21}r^2 - 2w_{11}r + w_{12}.$$

Applying Theorem 6.2. As in Lemma 9.2, let

$$\mathcal{E}' = \bigcup \mathsf{E}'.\{\exp(w)y : w \in F\},$$

where $\mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{5b}\mathsf{E}}$. For all $z \in \mathcal{E}'$, put $I_z = I_{\mathcal{E}',b}(e, z)$. In view of Lemma 9.2, we have

$$\mathcal{G}_{I_z,\mathsf{R}}(w) \le 8\Upsilon, \qquad \text{for all } w \in I_z, \tag{9.13}$$

where $\mathcal{G}$ is defined with $b_0 = b\,\mathrm{inj}(z)$.

Apply Theorem 6.2 with I_z and $\mathsf{c} = \kappa$; let $J_z \subset [0, 1]$ be the set J' given by that theorem. In particular,

$$|[0, 1] \setminus J_z| \le L\kappa^{-L}\Upsilon^{-\kappa^2} \le e^{-\kappa^2 t/2}. \tag{9.14}$$

To see the last inequality, recall that $\#F \ge e^{0.9t}$. Combining this with (9.10) (and the discussion preceding (9.10)), $\Upsilon^{-\kappa^2} \le e^{-0.8\kappa^2 t}$. The above estimate follows if we assume t is large enough to account for the factor $L\varepsilon^{-L}$.

Returning to the argument, by Theorem 6.2, we also have that for every $r \in J_z$ there exists $I'_{z,r} \subset I_z$ with $\#(I_z \setminus I'_{z,r}) \leq e^{-\kappa^2 t/2} \cdot (\# I_z)$ so that

$$\mathcal{G}_{\xi_r(I_z),\mathsf{R}_1}(\xi_r(w)) \leq \Upsilon_1, \qquad \text{for every } w \in I'_{z,r}, \tag{9.15}$$

where $\Upsilon_1 = 10L_1\Upsilon^{1+8\kappa} \geq L_1(8\Upsilon)^{1+8\kappa}$.

The sets $L_{\mu_{\mathcal{E}}}$ and $\mathcal{E}_r$. Equip $\mathcal{E} \times [0,1]$ with $\sigma := \mu_{\mathcal{E}} \times \mathrm{Leb}$ where Leb denotes the normalized Lebesgue measure on $[0,1]$. Let

$$Y = \left\{ (z,r) \in \hat{\mathcal{E}} \times [0,1] : \frac{\#\{w \in I_z : \mathcal{G}_{\xi_r(I_z),\mathsf{R}_1}(\xi_r(w)) > \Upsilon_1\}}{\# I_z} \leq e^{-\kappa^2 t/2} \right\}.$$

where $\hat{\mathcal{E}} = \bigcup \hat{\mathsf{E}}.\{\exp(w)y : w \in F\}$ and $\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$. Then, (9.15) implies

$$\text{for all } z \in \hat{\mathcal{E}}, \text{ we have } \{(z,r) : r \in J_z\} \subset Y.$$

Recall moreover that $\mu_{\mathcal{E}}(\mathcal{E} \setminus \hat{\mathcal{E}}) \ll_M b$, see the definition of an admissible measure and in particular (9.2). We thus conclude from (9.14) that

$$\sigma(\mathcal{E} \times [0,1] \setminus Y) \ll_M b + e^{-\kappa^2 t/2} \ll_M e^{-\kappa^2 t/2}.$$

This and Fubini's theorem imply that there is a subset $L_{\mu_{\mathcal{E}}} \subset [0,1]$ with $|[0,1] \setminus L_{\mu_{\mathcal{E}}}| \ll_M e^{-\kappa^2 t/4}$ so that for all $r \in L_{\mu_{\mathcal{E}}}$, we have

$$\lambda\big(\mathcal{E} \setminus Y_r\big) \ll_M e^{-\kappa^2 t/4} \tag{9.16}$$

where $Y_r = \{z \in \hat{\mathcal{E}} : (z,r) \in Y\}$.

For every $r \in L_{\mu_{\mathcal{E}}}$, define

$$\mathcal{E}_r := \big\{ z \in \hat{\mathcal{E}} : f_{\hat{\mathcal{E}},b,\mathsf{R}_1}(a_\ell u_r, z) \leq 200 e^{-\alpha\ell}\Upsilon_1 + 200 e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(a_\ell u_r, z) \big\}.$$

We will show that

$$\mu_{\mathcal{E}}(\mathcal{E} \setminus \mathcal{E}_r) \leq e^{-\kappa^2 t/64}. \tag{9.17}$$

Note that the lemma follows from (9.17). Thus, the rest of the argument is devoted to the proof of (9.17).

Let $r \in L_{\mu_{\mathcal{E}}}$, and let $z \in Y_r$. Then $(z,r) \in Y$, and by the definition of Y, there exists a subset $I_{z,r} \subset I_z$ with $\frac{\#(I_z \setminus I_{z,r})}{\# I_z} \leq e^{-\kappa^2 t/2}$ so that for every $w \in I_{z,r}$, we have

$$\mathcal{G}_{\xi_r(I_z),\mathsf{R}_1}(\xi_r(w)) \leq \Upsilon_1. \tag{9.18}$$

Claim. *Let $\bar{\eta} = \mathrm{inj}(y)$. For all $w \in I_{z,r} \cap B_{\mathfrak{r}}(0, 0.1\bar{\eta}b)$, we have*

$$f_{\hat{\mathcal{E}},b,\mathsf{R}_1}(a_\ell u_r, \exp(w)z) \leq 200 e^{-\alpha\ell}\Upsilon_1 + 200 e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(a_\ell u_r, z).$$

Proof of the claim. Recall that $\frac{1}{2}\bar{\eta} \leq \mathrm{inj}(\bullet) \leq 2\bar{\eta}$ for all $\bullet \in \mathcal{E}$. Let $w \in I_{z,r} \cap B_{\mathfrak{r}}(0, 0.1\bar{\eta}b)$. For ease of notation, put $\hat{z} = \exp(w)z$ and $h = a_\ell u_r$.

First note that if $\# I_{\hat{\mathcal{E}},b}(h, \hat{z}) \leq \mathsf{R}_1$, there is nothing to prove. Therefore, we will assume $\# I_{\hat{\mathcal{E}},b}(h, \hat{z}) > \mathsf{R}_1$.

Let $I^{>}_{h\hat z} = \{v \in I_{\hat{\mathcal{E}},b}(h,\hat z) : \|v\| \geq 0.01e^{-2\ell} b\,\mathrm{inj}(h\hat z)\}$. Then

$$\sum_{v\in I^{>}_{h\hat z}} \|v\|^{-\alpha} \leq 100e^{2\alpha\ell}(b\,\mathrm{inj}(h\hat z))^{-\alpha}\cdot(\#I^{>}_{h\hat z}) \leq 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat z). \tag{9.19}$$

For any subset $I \subset I_{\hat{\mathcal{E}},b}(h,\hat z)$, let

$$J_I = \{v \in I_{\hat{\mathcal{E}},b}(e,\hat z) : \mathrm{Ad}(h)v \in I\},$$

and put $I^{\mathrm{new}} = I \setminus (\mathrm{Ad}(h)I_{\hat{\mathcal{E}},b}(e,\hat z))$, i.e., I^{new} is the set of vectors in I which do *not* equal $\mathrm{Ad}(h)v$ for any vector $v \in I_{\hat{\mathcal{E}},b}(e,\hat z)$.

With this notation, we have

$$\sum_{v\in I}\|v\|^{-\alpha} \leq \sum_{v\in J_I}\|\mathrm{Ad}(h)v\|^{-\alpha} + \sum_{v\in I^{\mathrm{new}}}\|v\|^{-\alpha} \tag{9.20}$$

We first estimate the contribution of the second term on the right side of (9.20). Recall that $\|\mathrm{Ad}(a_\ell u_r)^{\pm1}v\| \leq 3e^{\ell}\|v\|$ for all $v \in \mathfrak{g}$, in particular, we have $e^{-\ell}\mathrm{inj}(\hat z)/3 \leq \mathrm{inj}(h\hat z) \leq 3e^{\ell}\mathrm{inj}(\hat z)$. Thus if $v \in I^{\mathrm{new}}$, then $\|v\| \geq e^{-2\ell}\mathrm{inj}(h\hat z)b/9$. In consequence, for any I we have $I^{\mathrm{new}} \subset I^{>}_{h\hat z}$, and the second term may be controlled using (9.19).

We now turn to the first term on the right side of (9.20). The strategy is to relate this term (for an appropriate choice of I) to (9.18).

Recall that $w \in I_{z,r} \cap B_{\mathfrak{r}}(0, 0.1\bar\eta b)$ and $\hat z = \exp(w)z$. Let now

$$v \in \hat I(\hat z) := I_{\hat{\mathcal{E}},b}(e,\hat z) \cap B_{\mathfrak{r}}(0, 0.1\bar\eta b).$$

Then we have

$$\exp(v)\hat z = \exp(v)\exp(w)z = \mathsf{h}_v\exp(w_v)z.$$

We note that $\|w_v - (v+w)\| = \|(w_v - w) - v\| \ll b\|v\|$ and $\|\mathsf{h}_v\| \ll b^2$. Since $\exp(v)\hat z \in \hat{\mathcal{E}}$, this implies that $\exp(w_v)z = \mathsf{h}_v^{-1}\exp(v)\hat z \in \mathcal{E}'$. Moreover, $\|v\|, \|w\| \leq 0.1\bar\eta b$ implies that $\|w_v\| < \mathrm{inj}(z)b$. Altogether, we have $w_v \in I_z$.

The map $v \mapsto w_v$ is on-to-one from $\hat I(\hat z)$ into I_z. Moreover, $\mathrm{Ad}(h^{-1})v \in \hat I(\hat z)$ for every $v \in I_{\hat{\mathcal{E}},b}(h,\hat z) \setminus I^{>}_{h\hat z}$. Thus if $\#I_z \leq \mathsf{R}_1$, then

$$\#\big(I_{\hat{\mathcal{E}},b}(h,\hat z) \setminus I^{>}_{h\hat z}\big) \leq \mathsf{R}_1,$$

and the proof is complete thanks to (9.19).

In view of this, we let $K_w \subset I_z$ be so that $\#(I_z \setminus K_w) \leq \mathsf{R}_1$ and

$$\sum_{w'\in K_w}\|\xi_r(w) - \xi_r(w')\|^{-\alpha} \leq \Upsilon_1, \tag{9.21}$$

see (9.18).

Let $I_{\mathrm{exc}} = \{v \in \hat I(\hat z) : w_v \notin K_w\}$. Since the map $v \mapsto w_v$ is one-to-one from I_{exc} into $I_z \setminus K_w$, we have $\#I_{\mathrm{exc}} \leq \mathsf{R}_1$.

As was remarked above, if $v \in I_{\hat{\mathcal{E}},b}(h,\hat z)$ and $\mathrm{Ad}(h^{-1})v \notin I_{\hat{\mathcal{E}},b}(e,\hat z)$, then $\mathrm{Ad}(h)v \in I^{>}_{h\hat z}$. Therefore, using (9.21) and (9.19), we have

$$
\begin{aligned}
f_{\hat{\mathcal{E}},b,\mathsf{R}_1}(a_\ell u_r,\hat{z}) &\le \sum_{v\in\hat{I}(\hat{z})\setminus I_{\rm exc}} \|\operatorname{Ad}(h)v\|^{-\alpha} + 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat{z})\\
&\le 2\sum_{v\in\hat{I}(\hat{z})\setminus I_{\rm exc}} \|\operatorname{Ad}(h)(w_v-w)\|^{-\alpha} + 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat{z})\\
&\le 2\sum_{v\in\hat{I}(\hat{z})\setminus I_{\rm exc}} \|e^{\ell}(\xi_r(w_v)-\xi_r(w))\|^{-\alpha} + 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat{z})\\
&\le 2e^{-\alpha\ell}\sum_{w'\in I_z\setminus K_w} \|\xi_r(w')-\xi_r(w)\|^{-\alpha} + 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat{z})\\
&\le 2e^{-\alpha\ell}\Upsilon_1 + 100e^{2\alpha\ell}\psi_{\hat{\mathcal{E}},b}(h,\hat{z}).
\end{aligned}
$$

We used (9.19) in the first inequality. For the second inequality we used the following: $\|(w_v-w)-v\|\ll b\|v\|$, moreover, the choice $\ell=0.01\varepsilon n$ implies that $e^{-4\ell}>b$. Consequently, we have

$$\|a_\ell u_r v\| = \|a_\ell u_r(w_v-w+w')\| \ge 0.5\|a_\ell u_r(w_v-w)\|$$

where $w'=v-(w_v-w)$ and we used $\|h^{\pm1}\bullet\|\le 3e^{\ell}\|\bullet\|$ for any $\bullet\in\mathfrak{g}$. The third inequality follows from $(\operatorname{Ad}(h)\bullet)_{12}=e^{\ell}\xi_r(\bullet)$, and the last inequality is a consequence of (9.21).

The above and (9.19) complete the proof of the claim. □

Fubini's theorem and the proof of (9.17). In view of the claim, for every $z\in Y_r$ and every $w\in I_{z,r}$, we have $\exp(w)z\in\mathcal{E}_r$ so long as $\exp(w)z\in\hat{\mathcal{E}}$. We will use this to show (9.17). That is,

$$\mu_{\mathcal{E}}(\mathcal{E}\setminus\mathcal{E}_r)\le e^{-\kappa^2 t/64}, \tag{9.22}$$

which will complete the proof of the lemma.

Recall that $\bar\eta=\operatorname{inj}(y)$ and $\frac12\bar\eta\le\operatorname{inj}(\bullet)\le 2\bar\eta$ for all $\bullet\in\mathcal{E}$. Set $b':=b\bar\eta/10$. The argument is based the following: For every $z\in Y_r$, we have

$$\#\big(I_{z,r}\cap B_{\mathfrak{r}}(0,b')\big)\ge(1-e^{-\kappa^2t/4})\cdot\big(\#(I_z\cap B_{\mathfrak{r}}(0,b'))\big), \tag{9.23}$$

Let us first establish (9.23). Let $z\in Y_r$. By Lemma 9.3, we have

$$\#\big(I_z\cap B_{\mathfrak{r}}(0,b')\big)\ge\#\big(F\cap B_{\mathfrak{r}}(w_z,b'/2)\big) \tag{9.24a}$$

$$\#I_z\le\#\big(F\cap B_{\mathfrak{r}}(w_z,40b')\big). \tag{9.24b}$$

where $z=\mathsf{h}_z\exp(w_z)y$ and in (9.24b) we used $\frac12\bar\eta\le\operatorname{inj}(z)\le2\bar\eta$.

By our assumption, F satisfies (9.3) with $c\ge e^{-\kappa^2t/4}$ and $50b'$. Thus using (9.24a) and (9.24b), we have

$$
\begin{aligned}
\#\big(I_z\cap B_{\mathfrak{r}}(0,b')\big) &\ge \#\big(F\cap B_{\mathfrak{r}}(w_z,b'/2)\big)\\
&\ge c\cdot\big(\#\big(F\cap B_{\mathfrak{r}}(w_z,50b')\big)\\
&\ge c\cdot(\#I_z)\ge e^{-\kappa^2t/4}\cdot(\#I_z)
\end{aligned}
$$

Since $\#(I_z \setminus I_{z,r}) \le e^{-\kappa^2 t/2} \cdot (\# I_z)$, the above implies that

$$\#(I_z \setminus I_{z,r}) \le e^{-\kappa^2 t/2} \cdot (\# I_z) \le e^{-\kappa^2 t/4}\big(\#\big(I_z \cap B_{\mathfrak{r}}(0,b')\big)\big).$$

Altogether, we conclude

$$\#\big(I_{z,r} \cap B_{\mathfrak{r}}(0,b')\big) \ge (1 - e^{-\kappa^2 t/4}) \cdot \big(\#(I_z \cap B_{\mathfrak{r}}(0,b'))\big),$$

as was claimed in (9.23).

Put $\mathcal{E}_r^{\complement} = \mathcal{E} \setminus \mathcal{E}_r$ and assume contrary to (9.22) that

$$\mu_{\mathcal{E}}(\mathcal{E}_r^{\complement}) > e^{-\kappa^2 t/64} =: \delta.$$

We will repeatedly use properties of an admissible measure, see in particular (9.2). Recall from (9.16) that

$$\mu_{\mathcal{E}}(\mathcal{E} \setminus Y_r) \ll e^{-\kappa^2 t/4} \le \delta^8.$$

Let $F' = \big\{w \in F : \mu_w(Y_r \cap \mathsf{E}.\exp(w)y) \ge (1-\delta^4)\mu_w(\mathsf{E}.\exp(w)y)\big\}$. Then by Fubini's theorem

$$\mu_{\mathcal{E}}\Big(\bigcup_{w \notin F'} \mathsf{E}.\exp(w)y\Big) \le \delta^4.$$

Points in $\mathcal{E}$ are represented as $\mathsf{h}'\exp(v')y$, in order to utilize (9.23), however, it is more convenient to have a representation of points in $\mathcal{E}$ in the form $\exp(v)\mathsf{h}y$. To that end, for every $w \in F'$, fix a covering $\{\mathsf{B}^H_{b'}.z'\}$ of

$$\big(\mathsf{E} \setminus \partial_{20b}\mathsf{E}\big).\exp(w)y$$

with multiplicity $\le K'$ (absolute constant), and let

$$\mathcal{B}'_w := \big\{\mathsf{B}^H_{b'}.z' : \mu_w(\mathsf{B}^H_{b'}.z' \cap Y_r) \ge (1-\delta^2)\mu_w(\mathsf{B}^H_{b'}.z')\big\}.$$

Then $\mu_w\big(\bigcup\big\{\mathsf{B}^H_{b'}.z' : \mathsf{B}^H_{b'}.z' \notin \mathcal{B}'\big\}\big) \le K'\delta^2$.

Let $\mathsf{B} = \exp(B_{\mathfrak{r}}(0,b')) \cdot \mathsf{B}^H_{b'}$, and put

$$\hat{\mathcal{B}} = \{\mathsf{B}.z' : \mathsf{B}^H_{b'}.z' \in \mathcal{B}'_w, w \in F'\}.$$

Then there is $\mathcal{B} \subset \hat{\mathcal{B}}$ so that the multiplicity of $\mathcal{B}$ is $\le K$ (absolute) and

$$\mu_{\mathcal{E}}(\cup_{\mathcal{B}}\mathsf{B}.z') \ge 1 - M^2KK'\delta^2 - \delta^4 > 1 - (M^2KK'+1)\delta^2$$

where M appears in the definition of (λ, M)-admissible measure.

Recall now that $\mu_{\mathcal{E}}(\hat{\mathcal{E}}) \ge 1 - O(b) > 1 - \delta^{16}$. Therefore, if we put $\mathcal{B}_{\rm exc} = \{\mathsf{B}.z' \in \mathcal{B} : \mu_{\mathcal{E}}(\mathsf{B}.z' \cap \hat{\mathcal{E}}) \le (1-\delta^8)\mu_{\mathcal{E}}(\mathsf{B}.z')\}$, then

$$\mu_{\mathcal{E}}\Big(\bigcup_{\mathcal{B}_{\rm exc}} \mathsf{B}.z'\Big) \le 2K\delta^8,$$

provided that δ is small enough compared to M, K, and K'.

Since $\mu_{\mathcal{E}}(\mathcal{E}_r^{\complement}) > \delta$ and the multiplicity of $\mathcal{B}$ is at most K, there exists some $\mathsf{B}.z' \in \mathcal{B} \setminus \mathcal{B}_{\rm exc}$ so that

$$\mu_{\mathcal{E}}\big(\mathsf{B}.z' \cap \mathcal{E}_r^{\complement}\big) \ge \tfrac{\delta}{4K}\mu_{\mathcal{E}}(\mathsf{B}.z') \tag{9.25}$$

Other other hand, applying the claim with $\mathsf{h}z' \in \mathsf{B}^H_{b'}.z' \cap Y_r$ we have: for every $v \in I_{\mathsf{h}z',r}$, $\exp(v)\mathsf{h}z' \in \mathcal{E}_r$ so long as $\exp(v)\mathsf{h}z' \in \hat{\mathcal{E}}$. This and the fact that every point in $\mathsf{B}.z'$ can be written uniquely as $\exp(v)\mathsf{h}z'$ for some $v \in B_{\mathfrak{r}}(0, b')$ and $\mathsf{h} \in \mathsf{B}^H_{b'}$, imply

$$\mathsf{B}.z' \cap \mathcal{E}_r^{\complement} \subset \big(\mathsf{B}.z' \cap \hat{\mathcal{E}}^{\complement}\big) \bigcup \{\exp(v)\mathsf{h}z' \in \mathsf{B}.z' : \mathsf{h}z' \notin Y_r\} \bigcup \{\exp(v)\mathsf{h}z' \in \mathsf{B}.z' : v \notin I_{\mathsf{h}z',r}\}.$$

We now bound the measure of the three sets appearing on the right side of the above and obtain a contradiction with (9.25). First note that since $\mathsf{B}.z' \notin \mathcal{B}_{\mathrm{exc}}$, we have

$$\mu_{\mathcal{E}}(\mathsf{B}.z' \cap \hat{\mathcal{E}}^{\complement}) \le \delta^8 \mu_{\mathcal{E}}(\mathsf{B}.z'). \tag{9.26}$$

Moreover, since $\mathsf{B}^H_{b'}.z' \in \mathcal{B}'_w$ for some $w \in F'$, we have $\mu_w(\mathsf{B}^H_{b'}.z' \cap Y_r^{\complement}) \le \delta^2 \mu_w(\mathsf{B}^H_{b'}.z')$, hence

$$\mu_{\mathcal{E}}(\{\exp(v)\mathsf{h}z' \in \mathsf{B}.z' : \mathsf{h}z' \notin Y_r\}) \le M^2\delta^2 \mu_{\mathcal{E}}(\mathsf{B}.z'), \tag{9.27}$$

Finally, in view of (9.23), for every $\mathsf{h}z' \in \mathsf{B}^H_{b'}.z' \cap Y_r$, we have

$$\#(I_{\mathsf{h}z',r} \cap B_{\mathfrak{r}}(0, b')) \ge (1 - \delta^8) \cdot \big(\#(I_{\mathsf{h}z'} \cap B_{\mathfrak{r}}(0, b'))\big).$$

This and the definition of admissible measure again imply

$$\mu_{\mathcal{E}}(\{\exp(v)\mathsf{h}z' \in \mathsf{B}.z' : v \notin I_{\mathsf{h}z',r}\}) \le M^2\delta^8 \mu_{\mathcal{E}}(\mathsf{B}.z'). \tag{9.28}$$

Now (9.26), (9.27) and (9.28), imply that

$$\mu_{\mathcal{E}}(\mathsf{B}.z' \cap \mathcal{E}_r^{\complement}) \le \big(M^2\delta^2 + (M^2+1)\delta^8\big)\mu_{\mathcal{E}}(\mathsf{B}.z'),$$

which contradicts (9.25) provided that δ is small enough.

The proof is complete. □

10. Improving the dimension

In this section, we will state and begin the proof of Proposition 10.1. The proof is based on an inductive scheme, and relies on results in §8 and §9; it will occupy this section as well as §11 and §12.

Fix a small parameter $0 < \varepsilon < 1$ and a large parameter t for the rest of this section as well as §11 and §12 — in our applications, ε will depend on κ_0 in (2.7) and t will be chosen $\asymp \log R$ where R is as in Theorem 1.1.

Put $\ell = \varepsilon t/100$. We will also fix a parameter $0 < \kappa \le \varepsilon/10^6$, and put $\beta = e^{-\kappa t}$ and $\eta^2 = \beta$, see Proposition 10.1. We also recall that $0.9 < \alpha < 1$.

Let σ denote the uniform measure on $\mathsf{B}^{s,H}_{\beta+100\beta^2}$, where for any $\delta > 0$,

$$\mathsf{B}^{s,H}_{\delta} = \{u^-_s : |s| \le \delta\} \cdot \{a_\tau : |\tau| \le \delta\}.$$

For all $d > 0$, define ν_d by $\int \varphi \,\mathrm{d}\nu_d = \int_0^1 \varphi(a_d u_r)\,\mathrm{d}r$ for any $\varphi \in C_c(H)$. Recall from (8.10) that

$$\mu_{t,\ell,n} = \nu_\ell * \cdots * \nu_\ell * \sigma * \nu_t$$

where ν_ℓ appears n times in the above expression.

10.1. **Proposition.** *Let $x_1 \in X$, and assume that Proposition 4.8(2) does not hold for the point x_1, and parameters $D \geq 10$ and t. Let*

$$d_1 = 100\lceil \tfrac{4D-3}{2\varepsilon} \rceil, \quad d_2 = d_1 - \lceil \tfrac{10^4}{\sqrt{\varepsilon}} \rceil, \quad \text{and} \quad \kappa = 10^{-6} d_1^{-1};$$

as before, we put $\beta = e^{-\kappa t}$ and $\eta^2 = \beta$.

Let $r_1 \in I(x_1)$ and put $x_2 = a_{8t} u_{r_1} x_1$, see Proposition 4.8(1). For every $d_2 \leq d \leq d_1$, there is a collection $\Xi_d = \{\mathcal{E}_{d,i} : 1 \leq i \leq N_d\}$ of sets

$$\mathcal{E}_{d,i} = \mathsf{E}.\{\exp(w) y_{d,i} : w \in F_{d,i}\} \subset X_\eta,$$

with $F_{d,i} \subset B_{\mathfrak{r}}(0, \beta)$, and $(\lambda_{d,i}, M_{d,i})$-admissible measures $\mu_{\mathcal{E}_{d,i}}$, see §7.6, where $M_{d,i}$ depend on d_1 and X, so that both of the following hold:

(1) Let $b = e^{-\sqrt{\varepsilon} t}$. Let $d_2 \leq d \leq d_1$, and let $1 \leq i \leq N_d$. Then for all $w \in F_{d,i}$ and all $z = h \exp(w) y_{d,i} \in \mathcal{E}_{d,i}$ with $h \in \overline{\mathsf{E} \setminus \partial_{10b} \mathsf{E}}$, both of the following hold:

$$\#\big(B_{\mathfrak{r}}(w, 4b\,\mathrm{inj}(y_{d,i})) \cap F_{d,i}\big) \geq e^{-\varepsilon t} \sup_{w' \in F_{d,i}} \#\big(B_{\mathfrak{r}}(w', 4b\,\mathrm{inj}(y_{d,i})) \cap F_{d,i}\big) \tag{10.1}$$

$$f_{\mathcal{E}_{d,i}, b, \mathsf{R}}(e, z) \leq e^{\varepsilon t} \psi_{\mathcal{E}_{d,i}, b}(e, z) \quad \text{where } \mathsf{R} \leq e^{0.01 \varepsilon t} \tag{10.2}$$

(2) For every $\varphi \in C_c^\infty(X)$, all $\tau \leq d_1 \ell$ and $|s| \leq 2$, we have

$$\Big| \int \varphi(a_\tau u_s h x_2)\, \mathrm{d}\mu_{t,\ell,d_1}(h) - \sum_{d,i} c_{d,i} \int \varphi(a_\tau u_s z)\, \mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z) \Big| \ll \mathrm{Lip}(\varphi) \beta^{\kappa_4} \tag{10.3}$$

where $c_{d,i} \geq 0$ and $\sum_{d,i} c_{d,i} = 1 - O(\beta^{\kappa_4})$, $\mathrm{Lip}(\varphi)$ is the Lipschitz norm of φ, and κ_4 and the implied constants depend on X.

As it was mentioned, the proof is based on an inductive scheme. The base case relies on Proposition 4.8(1) and Lemma 8.4. Indeed, combining Proposition 4.8(1) and Lemma 8.4, the measure $(\sigma * \nu_t).x_2$ (up to an exponentially small error) can be written as $\sum c_i \mu_{\mathcal{E}_i}$ where $\mu_{\mathcal{E}_i}$ is an admissible measure for all i, and

$$f_{\mathcal{E}_i, b, 1}(e, z) \leq e^{Dt} \qquad \text{for all } i \text{ and all } z \in \mathcal{E}_i.$$

This will serve as the base case of the induction. We will then combine Lemma 8.9 and Lemma 9.1 to inductively improve this dimension while obtaining convex combinations similar to the expressions appearing in (10.3). For technical reasons, Lemma 6.4 will be applied after every step to ensure regularity of the sets F which are used to define sets $\mathcal{E}$ (again, we are allowed to drop subsets of F with exponentially small density).

We now turn to the details of the argument, beginning with some general facts. In the next three lemmas, let

$$\mathcal{E} = \mathsf{E}.\{\exp(w) y : w \in F\} \subset X_\eta$$

where $F \subset B_{\mathfrak{r}}(0, \beta)$.

10.2. **Lemma.** *Let $z \in \mathcal{E}$, and write $z = \mathsf{h}\exp(w)y$ for some $w \in F$ and $\mathsf{h} \in \mathsf{E}$. Then*

$$4\psi_{\mathcal{E},2b}(e, \exp(w)y) \geq \psi_{\mathcal{E},b}(e, z). \tag{10.4}$$

In particular, there exists some $w_0 \in F$ so that

$$4\psi_{\mathcal{E},2b}(e, \exp(w_0)y) \geq \sup_z \psi_{\mathcal{E},b}(e, z). \tag{10.5}$$

Proof. The proof is similar to the proof of Lemma 9.3. Let us write $z' = \exp(w)y$, i.e., $z = \mathsf{h}z'$. Let $v \in I_{\mathcal{E},b}(e, z)$. Then $\exp(v)z \in \mathcal{E}$, hence, there exist $\hat{w}_v \in F$ and $\hat{\mathsf{h}} \in \mathsf{E}$ so that

$$\begin{aligned}\exp(v)z &= \hat{\mathsf{h}}\exp(\hat{w}_v)y = \hat{\mathsf{h}}\exp(\hat{w}_v)\exp(-w)\exp(w)y \\ &= \hat{\mathsf{h}}\exp(\hat{w}_v)\exp(-w)z' = \hat{\mathsf{h}}\mathsf{h}_v\exp(w_v)z';\end{aligned} \tag{10.6}$$

for some $\mathsf{h}_v \in H$ and $w_v \in \mathfrak{r}$ so that

$$0.5\|\hat{w}_v - w\| \leq \|w_v\| \leq 2\|\hat{w}_v - w\| \quad \text{and} \quad \|\mathsf{h}_v - I\| \leq C_3\beta\|w_v\|, \tag{10.7}$$

see Lemma 3.2.

Using Lemma 3.3, recall that $b\,\mathrm{inj}(z) \leq 0.01\eta$, we conclude that

$$\|w_v\| \leq 2\|v\| \leq 2b\,\mathrm{inj}(z). \tag{10.8}$$

This and (10.7) imply that $\|\mathsf{h}_v - I\| \ll b\,\mathrm{inj}(z) \leq \beta^2$ where the implied constant is absolute; hence, $\mathsf{h}_v^{\pm 1} \in \mathsf{E}$. Moreover, comparing the second and the last term in (10.6), it follows that $\mathsf{h}_v\exp(w_v)z' = \exp(\hat{w}_v)y$. Since $\hat{w}_v \in F$,

$$\exp(w_v)z' = \mathsf{h}_v^{-1}\exp(\hat{w}_v)y \in \mathcal{E}.$$

We deduce that $w_v \in I_{\mathcal{E},2b}(e, z')$. Furthermore, note that the map $v \mapsto w_v$ is injective. Hence,

$$\#I_{\mathcal{E},2b}(e, z') \geq \#I_{\mathcal{E},b}(e, z). \tag{10.9}$$

Recall now that $0.5\mathrm{inj}(z') \leq \mathrm{inj}(z) \leq 2\mathrm{inj}(z')$, and

$$\psi_{\mathcal{E},b}(h, z) = \big(\#I_{\mathcal{E},b}(h, z)\big) \cdot (b\,\mathrm{inj}(hz))^{-\alpha},$$

see (9.5). Therefore, (10.4) follows from (10.9).

To see the second claim, let $\hat{z}$ be so that $\sup_z \psi_{\mathcal{E},b}(e, z) = \psi_{\mathcal{E},b}(e, \hat{z})$. By the definition of $\mathcal{E}$, there exists some $w \in F$ and $\mathsf{h} \in \mathsf{E}$ so that $\hat{z} = \mathsf{h}\exp(w)y$. The claim thus follows from (10.4). □

Cubes and the function ψ. Recall that $\mathcal{E} = \{\exp(w)y : w \in F\} \subset X_\eta$. For a parameter M and every $k \in \mathbb{N}$, we let $\mathcal{Q}_{\mathsf{M}k}$ denote the collection of $2^{-\mathsf{M}k}$-cubes, see §6.3. Let $k_0 \in \mathbb{N}$ be so that

$$2^{-k_0-1} \leq b\,\mathrm{inj}(y) < 2^{-k_0}.$$

10.3. **Lemma.** *Let $k_1 > k_0$ be an integer, and assume that for every integer $k_0 - 10 \le k \le k_1$, there exists $\tau_k > 0$ so that, for all $Q \in \mathcal{Q}_{\mathsf{M}k}$*

$$\text{either} \quad 2^{\mathsf{M}(\tau_k - 2)} \le \#(F \cap Q) \le 2^{\mathsf{M}\tau_k} \quad \text{or} \quad F \cap Q = \emptyset. \tag{10.10}$$

Let $z = h\exp(w)y \in \mathcal{E}$ where $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$. Then

$$C_6^{-1} \sup_{w' \in F} \psi_{\mathcal{E},b}(e, \exp(w')y) \le \psi_{\mathcal{E},b}(e,z) \le C_6 \sup_{w' \in F} \psi_{\mathcal{E},b}(e, \exp(w')y)$$

where C_6 depends on M and the dimension.

Furthermore,

$$\psi_{\mathcal{E},b}(e,z) \le C_6 \sup_{w' \in F} \psi_{\mathcal{E},b}(e, \exp(w')y)$$

holds true for all $z \in \mathcal{E}$.

Proof. The upper bound is a consequence of Lemma 10.2. Indeed by that lemma, we have

$$\psi_{\mathcal{E},b}(e,z) \le 4 \sup_{w'} \psi_{\mathcal{E},2b}(e, \exp(w')y).$$

To replace $2b$ with b, note that (10.10) and the definition of ψ imply

$$\sup_{w'} \psi_{\mathcal{E},2b}(e, \exp(w')y) \ll \sup_{w'} \psi_{\mathcal{E},b}(e, \exp(w')y)$$

where the implied constant depends on M and the dimension. The upper bound estimate for $\psi_{\mathcal{E},b}(e,z)$ follows.

As the proof shows, we did not use the condition on h for this bound, thus the final claim follows.

We now turn to the proof of the lower bound. Since $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$, Lemma 9.3 applied with z, w and $\delta = b\mathrm{inj}(z)$, implies

$$\#\big(F \cap B_{\mathfrak{r}}(w, b\mathrm{inj}(z)/2)\big) \le \#I_{\mathcal{E},b}(e,z)$$

This and the definition of ψ yield the following:

$$\begin{aligned} \psi_{\mathcal{E},b}(e,z) &= (\#I_{\mathcal{E},b}(e,z)) \cdot (b\,\mathrm{inj}(z))^{-\alpha} \\ &\ge (\#F \cap B_{\mathfrak{r}}(w', b\,\mathrm{inj}(z)/2)) \cdot (b\,\mathrm{inj}(z))^{-\alpha} \\ &\gg \sup_{w'}(\#F \cap B_{\mathfrak{r}}(w', 4b\,\mathrm{inj}(z))) \cdot (b\,\mathrm{inj}(z))^{-\alpha}, \end{aligned} \tag{10.11}$$

where we used (10.10) in the last inequality.

Note that for all $w' \in F$, we have $\mathrm{inj}(z)/2 \le \mathrm{inj}(\exp(w')y) \le 2\mathrm{inj}(z)$. Moreover, $I_{\mathcal{E},b}(e, \exp(w')y) = I_{\mathcal{E}',b}(e, \exp(w')y)$ where

$$\mathcal{E}' = \big(\overline{\mathsf{E} \setminus \partial_{5b}\mathsf{E}}\big) \cdot \{\exp(w'')y : w'' \in F\}.$$

Thus (10.11) and Lemma 9.3, applied with $\delta = b\,\mathrm{inj}(\exp(w')y)$, imply

$$\psi_{\mathcal{E},b}(e,z) \gg \sup_{w'} \psi_{\mathcal{E},b}(e, \exp(w')y).$$

The proof is complete. □

We also record the following lemma which is similar to Lemma 8.1.

10.4. **Lemma.** *There exists $C_7 > 0$ so that the following holds. Let $0 < b \le \beta^6$. Then for every $m \in \mathbb{N}$ with $e^m \le b^{-1/2}$, every $|r| \le 2$, and every $z \in \mathcal{E} \subset X_\eta$, we have*

$$\psi_{\mathcal{E},b}(a_m u_r, z) \le C_7 \eta^{-3} e^{4m} \cdot \big(\sup_{z'} \psi_{\mathcal{E},b}(e, z')\big).$$

Proof. Let $z \in \mathcal{E}$, and let $w \in I_{\mathcal{E},b}(a_m u_r, z)$. Then $\exp(w) a_m u_r z \in a_m u_r \mathcal{E}$ which implies $\exp(\mathrm{Ad}(a_{-m} u_{-r}) w) z \in \mathcal{E}$. Moreover, we have

$$\|\,\mathrm{Ad}(a_{-m} u_{-r}) w\| \le 100 e^m \mathrm{inj}(a_m u_r z) b \le 100 e^m b =: b'.$$

Since $\mathrm{inj}(z) \ge \eta$, we get that $\mathrm{inj}(z) b'/\eta \ge b'$, hence

$$\mathrm{Ad}(a_{-m} u_{-r}) w \in I_{\mathcal{E}, b'/\eta}(e, z).$$

This and the fact that $e^m b \le b^{1/2} \le \beta^3$ imply: $w \mapsto \mathrm{Ad}(a_{-m} u_{-r}) w$ is an injection map from $I_{\mathcal{E},b}(a_m u_r, z)$ into $I_{\mathcal{E},b'/\eta}(e, z)$.

Now arguing as in the proof of Lemma 10.2, with b replaced by $b'/\eta \le \beta^2$, we conclude that

$$\# I_{\mathcal{E},b'/\eta}(e, z) \le \#\big(F \cap B_{\mathfrak{r}}(w_z, 2b'/\eta)\big),$$

for some $w_z \in F$. Note moreover that $B_{\mathfrak{r}}(w_z, b'/\eta)$ may be covered with $\ll \eta^{-3} e^{3m}$ boxes of the form $B_{\mathfrak{r}}(w_i, b/2)$; thus

$$\begin{aligned} \# I_{\mathcal{E},b}(a_m u_r, z) \le \# I_{\mathcal{E},b'/\eta}(e, z) &\le \#\big(F \cap B_{\mathfrak{r}}(w_z, 2b'/\eta)\big) \\ &\ll \eta^{-3} e^{3m} \cdot \sup_{w'} \#\big(F \cap B_{\mathfrak{r}}(w', b/2)\big) \\ &\ll \eta^{-3} e^{3m} \cdot \big(\sup_{z'} \# I_{\mathcal{E},b}(e, z')\big), \end{aligned}$$

see also Lemma 9.3 for the last inequality.

Since $\mathrm{inj}(a_m u_r z) \gg e^{-m} \mathrm{inj}(z)$,

$$\psi_{\mathcal{E},b}(h, z) = (\mathrm{inj}(hz) b)^{-\alpha} \cdot \big(\max\{\# I_{\mathcal{E},b}(h, z), 1\}\big),$$

and $0 < \alpha \le 1$, the lemma follows. □

10.5. **The dimension improvement lemma.** As it was done before, let $\kappa = 10^{-6} d_1 \le \varepsilon / 10^6$. Suppose

$$\mathcal{E}_{\rm old} = \mathsf{E}.\{\exp(w) y_0 : w \in F_{\rm old}\}$$

satisfies the conditions in Lemma 9.1. That is, $F_{\rm old} \subset B_{\mathfrak{r}}(0, \beta)$ is finite with $\# F_{\rm old} \ge e^{9t/10}$, and

$$\#\big(F_{\rm old} \cap B_{\mathfrak{r}}(w, b\,\mathrm{inj}(y_0)/10^3)\big) \ge e^{-\kappa^2 t/4} \cdot \big(\#(F_{\rm old} \cap B_{\mathfrak{r}}(w, b\,\mathrm{inj}(y_0)/10))\big). \tag{10.12}$$

Moreover, for all $z \in \mathcal{E}_{\rm old}$, we have

$$f_{\mathcal{E}_{\rm old}, b, \mathsf{R}}(e, z) \le \Upsilon, \tag{10.13}$$

where $\Upsilon \ge 1$, $1 \le \mathsf{R} \le e^{\varepsilon t/100}$ and $b = e^{-\sqrt{\varepsilon} t}$.

Let $\mu_{\mathcal{E}_{\rm old}}$ be an admissible measure on $\mathcal{E}_{\rm old}$. By Lemma 9.1, there exists $L_{\mu_{\mathcal{E}_{\rm old}}} \subset [0,1]$ with

$$\left|[0,1] \setminus L_{\mu_{\mathcal{E}_{\rm old}}}\right| \ll e^{-\kappa^2 t/4},$$

and for every $r \in L_{\mu_{\mathcal{E}_{\rm old}}}$, there exists a subset

$$\mathcal{E}_{{\rm old},r} \subset \hat{\mathcal{E}}_{\rm old} = \bigcup \hat{\mathsf{E}}.\{\exp(w)y_0 : w \in F\}, \qquad (\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}) \tag{10.14}$$

satisfying $\mu_{\mathcal{E}_{\rm old}}(\mathcal{E}_{\rm old} \setminus \mathcal{E}_{{\rm old},r}) \ll e^{-\kappa^2 t/64}$ and the following: for all $z' \in \mathcal{E}_{{\rm old},r}$,

$$f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z') \le 200 e^{-\alpha\ell} L_1 \Upsilon^{1+8\kappa} + 200 e^{2\alpha\ell} \psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r, z'); \tag{10.15}$$

where $L_1 = L\kappa^{-L}$ and $\mathsf{R}_1 = \mathsf{R} + L_1\Upsilon^\kappa$, and we assume Υ is large enough compared to κ, see also Theorem 6.2.

Let us put $\hat{\hat{\mathsf{E}}} = \overline{\mathsf{E} \setminus \partial_{10^3\beta^2}\mathsf{E}}$, and define

$$\hat{\hat{\mathcal{E}}}_{\rm old} = \hat{\hat{\mathsf{E}}}.\{\exp(w)y_0 : w \in F_{\rm old}\}. \tag{10.16}$$

The following lemma is an important ingredient in the proof of Lemma 10.7; the latter will be applied in every step of our inductive argument. Roughly speaking, Lemma 10.6 states that for $r \in L_{\mu_{\mathcal{E}_{\rm old}}}$, offsprings of $a_\ell u_r \mathcal{E}_{\rm old}$ (see §8.5) have improved coarse dimension, possibly after slight trimming.

Let us recall the notation

$$\mathsf{Q}_\ell^H = \{u_s^- : |s| \le e^{-\ell}\beta^2\} \cdot \{a_\tau : |\tau| \le \beta^2\} \cdot U_\eta.$$

10.6. **Lemma.** *With the above notation, let $r \in L_{\mu_{\mathcal{E}_{\rm old}}}$. Let $(\mathcal{E}', \mu_{\mathcal{E}'})$,*

$$\mathcal{E}' = \mathsf{E}.\{\exp(w)y : w \in F'\} \subset X_\eta,$$

be an offspring of $a_\ell u_r \mu_{\mathcal{E}_{\rm old}}$, see (8.21) and (8.22). Recall from (8.14) that

$$\mathsf{Q}_\ell^H.\exp(w)y \subset a_\ell u_r.\mathcal{E}_{\rm old} \quad \text{for all } w \in F',$$

Let $F \subset F'$ satisfy that for all $w \in F$, we have

$$\mathsf{Q}_\ell^H.\exp(w)y \cap \big(a_\ell u_r.(\mathcal{E}_{{\rm old},r} \cap \hat{\hat{\mathcal{E}}}_{\rm old})\big) \ne \emptyset, \tag{10.17}$$

and put $\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$ and $\mu_{\mathcal{E}} = \frac{1}{\mu_{\mathcal{E}'}(\mathcal{E})}\mu_{\mathcal{E}'}|_{\mathcal{E}}$.

Then for every $z = h\exp(w)y \in \mathcal{E}$ (where $h \in \mathsf{E}$ and $w \in F$), we have

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le 2 f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z_0) + 10\psi_{\mathcal{E},b}(e,z) \tag{10.18}$$

where $z_0 \in \mathcal{E}_{{\rm old},r} \cap \hat{\hat{\mathcal{E}}}_{\rm old}$ is so that $a_\ell u_r z_0 = \mathsf{h}_0 \exp(w)y$ for some $\mathsf{h}_0 \in \mathsf{Q}_\ell^H$.

Proof. Note that

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le \sum_I \|v\|^{-\alpha} + 10\psi_{\mathcal{E},b}(e,z) \tag{10.19}$$

for every $I \subset \{v \in I_{\mathcal{E},b}(e,z) : \|v\| \le 0.1b\,{\rm inj}(z)\}$ with $\#(I_{\mathcal{E},b}(e,z) \setminus I) \le \mathsf{R}_1$. We will relate the first term on the right side of (10.19) to

$$f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z_0).$$

Let us begin with the following computation. Let $w \neq w_1 \in F$, and let $z_1 \in \mathcal{E}_{\mathrm{old},r} \cap \hat{\hat{\mathcal{E}}}_{\mathrm{old}}$ and $\mathsf{h}_1 \in \mathsf{Q}^H_\ell$ be so that $\mathsf{h}_1 \exp(w_1)y = a_\ell u_r z_1$. Then

$$\begin{aligned} a_\ell u_r z_1 &= \mathsf{h}_1 \exp(w_1)y = \mathsf{h}_1 \exp(w_1)\exp(-w)\mathsf{h}_0^{-1} a_\ell u_r z_0 \\ &= \mathsf{h}_1 \mathsf{h}_0^{-1} \exp(\mathrm{Ad}(\mathsf{h}_0)w_1)\exp(-\mathrm{Ad}(\mathsf{h}_0)w) a_\ell u_r z_0 \\ &= \mathsf{h}_1\mathsf{h}_0^{-1}\hat{\mathsf{h}}\exp(\hat{w}) a_\ell u_r z_0 \end{aligned} \tag{10.20}$$

where $\hat{\mathsf{h}} \in H$ and $\hat{w} \in \mathfrak{r}$, moreover, by Lemma 3.2, we have

$$\|\hat{\mathsf{h}} - I\| \le C_3 \beta \|\hat{w}\| \quad \text{and} \tag{10.21a}$$

$$0.5\|\mathrm{Ad}(\mathsf{h}_0)(w - w_1)\| \le \|\hat{w}\| \le 2\|\mathrm{Ad}(\mathsf{h}_0)(w - w_1)\|. \tag{10.21b}$$

Let $v \in I_{\mathcal{E},b}(e, z)$. Then $z, \exp(v)z \in \mathcal{E}$, and we have

$$z = h\exp(w)y = h\mathsf{h}_0^{-1} a_\ell u_r z_0 = \bar{h} a_\ell u_r z_0,$$

where $\bar{h} \in \mathsf{B}^H_{1.1\eta}$, recall that $z_0 \in \mathcal{E}_{\mathrm{old},r} \cap \hat{\hat{\mathcal{E}}}_{\mathrm{old}}$. Similarly, since $\exp(v)z \in \mathcal{E}$, there exist $w_v \in F$ and $z_v \in \mathcal{E}_{\mathrm{old},r} \cap \hat{\hat{\mathcal{E}}}_{\mathrm{old}}$ so that

$$\exp(v)z = h'\exp(w_v)y \quad \text{and} \quad \mathsf{h}_v \exp(w_v)y = a_\ell u_r z_v.$$

Thus, $\exp(v)z = \bar{h}_v a_\ell u_r z_v$ where $z_v \in \mathcal{E}_{\mathrm{old},r} \cap \hat{\hat{\mathcal{E}}}_{\mathrm{old}}$ and $\bar{h} \in \mathsf{B}^H_{1.1\eta}$. Hence

$$\begin{aligned} a_\ell u_r z_v = \bar{h}_v^{-1}\exp(v)z &= \bar{h}_v^{-1}\exp(v)\bar{h} a_\ell u_r z_0 \\ &= \bar{h}_v^{-1}\bar{h}\exp(\mathrm{Ad}(\bar{h}^{-1})v) a_\ell u_r z_0 \end{aligned} \tag{10.22}$$

Applying (10.20) with $w_1 = w_v$ and $\mathsf{h}_1 = \mathsf{h}_v$, we get that

$$a_\ell u_r z_v = \mathsf{h}_v \mathsf{h}_0^{-1} \hat{\mathsf{h}} \exp(\hat{w}_v) a_\ell u_r z_0 \tag{10.23}$$

where $\hat{\mathsf{h}}$ and $\hat{w}_v$ satisfy (10.21a) and (10.21b), and $\mathsf{h}_0, \mathsf{h}_v \in \mathsf{Q}^H_\ell$.

Since $(\hat{h}, \hat{w}) \mapsto \hat{h}\exp(\hat{w}) a_\ell u_r z_0$ is injective over $\mathsf{B}^H_{10\eta} \times B_{\mathfrak{r}}(0, 10\eta)$, we conclude from (10.23) and (10.22) that $\hat{w}_v = \mathrm{Ad}(\bar{h}^{-1})v$. In particular,

$$\|\hat{w}_v\| \le 2\|v\|. \tag{10.24}$$

Moreover, the elements $\{z_v : v \in I_{\mathcal{E},b}(e, z)\}$ belong to different local H-orbits, thus $v \mapsto \hat{w}_v$ is well-defined and one-to-one.

Recall that $\mathcal{E} \subset X_\eta$. Assume now that $\|v\| \le b\,\mathrm{inj}(z)/10$, then $\|\hat{w}_v\| \le b\,\mathrm{inj}(z)/5$. This estimate and (10.21a) imply that

$$\|\hat{\mathsf{h}}_v - I\| \le C_3\beta\|\hat{w}_v\| \ll b\beta \le \beta^2 e^{-\ell};$$

recall that $b \le e^{-\sqrt{\varepsilon}t}$ and $e^{-\ell}, \beta \ge e^{-0.01\varepsilon t}$.

In view of the definition of $\hat{\hat{\mathcal{E}}}_{\mathrm{old}}$ in (10.16), we have

$$z_v \in \mathcal{E}_{\mathrm{old},r} \cap \hat{\hat{\mathcal{E}}}_{\mathrm{old}} \quad \text{implies} \quad \mathsf{B}^H_{100\beta^2}.z_v \subset \hat{\mathcal{E}}_{\mathrm{old}}.$$

Moreover, $\mathsf{h}_0, \mathsf{h}_v \in \mathsf{Q}^H_\ell$ and $\|\hat{\mathsf{h}}_v - I\| \le \beta^2 e^{-\ell}$. Therefore,

$$\hat{\mathsf{h}}^{-1}\mathsf{h}_0\mathsf{h}_v^{-1} a_\ell u_r z_v \in a_\ell u_r \hat{\mathcal{E}}_{\mathrm{old}},$$

see (3.7). This and (10.23) yield

$$\exp(\hat{w}_v)a_\ell u_r z_0 = \hat{\mathsf{h}}^{-1}\mathsf{h}_0\mathsf{h}_v^{-1}a_\ell u_r z_v \in a_\ell u_r\hat{\mathcal{E}}_{\mathrm{old}}.$$

This and $\|\hat{w}_v\| \le b\,\mathrm{inj}(z)/5 < b\,\mathrm{inj}(a_\ell u_r z_0)$ imply $\hat{w}_v \in I_{\hat{\mathcal{E}}_{\mathrm{old}},b}(a_\ell u_r, z_0)$.

Let now $J \subset I_{\hat{\mathcal{E}}_{\mathrm{old}},b}(a_\ell u_r, z_0)$ be a subset so that

$$\#I_{\hat{\mathcal{E}}_{\mathrm{old}},b}(a_\ell u_r, z_0) \setminus J = \mathsf{R}_1 \quad\text{ and }\quad f_{\hat{\mathcal{E}}_{\mathrm{old}},b,\mathsf{R}_1}(a_\ell u_r, z_0) = \sum_{\hat{w}\in J}\|\hat{w}\|^{-\alpha}.$$

Put $I_J = \{v \in I_{\mathcal{E},b}(e,z) : \|v\| \le 0.1b\,\mathrm{inj}(z), \hat{w}_v \notin J\}$. Since $v \mapsto \hat{w}_v$ is a one-to-one map from I_J into $I_{\hat{\mathcal{E}}_{\mathrm{old}},b}(a_\ell u_r, z_0) \setminus J$, we have $\#I_J \le \mathsf{R}_1$. Applying (10.19) with

$$I = \{v \in I_{\mathcal{E},b}(e,z) : \|v\| \le 0.1b\} \setminus I_J,$$

and using (10.24), we conclude

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le 2f_{\hat{\mathcal{E}}_{\mathrm{old}},b,\mathsf{R}_1}(a_\ell u_r, z_0) + 10\psi_{\mathcal{E},b}(e,z),$$

as it was claimed in the lemma. □

Recall that $d_1 = 100\lceil(4D-3)/2\varepsilon\rceil$, $\kappa = 10^{-6}d_1^{-1}$, and $\ell = 0.01\varepsilon t$, see Proposition 10.1. From this point to the end of this section, we will assume

$$\Upsilon^\kappa \le e^{\ell/100}. \tag{10.25}$$

Moreover, we assume that t is large enough so that

$$L_1 = L\kappa^{-L} < e^{\ell/100} \tag{10.26}$$

— this amounts to $t \gg |\log\varepsilon|/\varepsilon$, later we will choose ε to depend only on κ_0 in (5.1). We will also assume that $0.9 < \alpha < 1$.

The following lemma combines the results in this section, and will be applied in every step of our inductive proof of Proposition 10.1.

10.7. **Lemma.** *Let the notation be as in Lemma 10.6. In particular,*

$$L_1 = L\kappa^{-L} \quad\textit{and}\quad \mathsf{R}_1 = \mathsf{R} + L_1\Upsilon^\kappa.$$

Assume further that (10.10) (with some parameter M) holds true for F_{old}.

Let $w_0 \in F_{\mathrm{old}}$ be so that

$$\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\exp(w_0)y_0) = \sup_{w'}\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\exp(w')y_0).$$

Then we have the following.

(1) If $\Upsilon \ge e^{\varepsilon t/2}\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\exp(w_0)y_0)$, then

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le e^{-0.6\ell}\Upsilon + 10\psi_{\mathcal{E},b}(e,z) \quad\textit{for all } z \in \mathcal{E}. \tag{10.27}$$

(2) If $\Upsilon < e^{\varepsilon t/2}\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\exp(w_0)y_0)$, then both of the following hold

(a) For every $\hat{z} = \hat{h}\exp(\hat{w})y_0 \in \mathcal{E}_{\mathrm{old}}$ with $\hat{h} \in \overline{\mathsf{E}\setminus\partial_{10b}\mathsf{E}}$, we have

$$f_{\mathcal{E}_{\mathrm{old}},b,\mathsf{R}}(e,\hat{z}) \le e^{\varepsilon t/2}\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\exp(w_0)y_0) \le C_6 e^{\varepsilon t/2}\psi_{\mathcal{E}_{\mathrm{old}},b}(e,\hat{z}) \tag{10.28}$$

where C_6 is as in Lemma 10.3 (which depends on M).

(b) For every $z \in \mathcal{E}$, we have

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le e^{-0.6\ell}\big(e^{\varepsilon t/2}\cdot \psi_{\mathcal{E}_{\rm old},b}(e,\exp(w_0)y_0)\big)+10\psi_{\mathcal{E},b}(e,z). \tag{10.29}$$

Proof. Since (10.10) holds true for $F_{\rm old}$, Lemma 10.3 is applicable with $\mathcal{E}_{\rm old}$; we will utilize that lemma several times in the course of the proof.

Let $z = h\exp(w)y \in \mathcal{E}$, and let $z' \in \mathcal{E}_{{\rm old},r} \cap \hat{\hat{\mathcal{E}}}_{\rm old}$ be so that $a_\ell u_r z' = \mathsf{h}\exp(w)y$ for some $\mathsf{h} \in \mathsf{Q}^H_\ell$. By Lemma 10.6, we have

$$f_{\mathcal{E},b,\mathsf{R}_1}(e,z) \le 2f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z')+10\psi_{\mathcal{E},b}(e,z). \tag{10.30}$$

Moreover, since $z' \in \mathcal{E}_{{\rm old},r}$, we conclude from (10.15) that

$$f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z') \le 200e^{-\alpha\ell}L_1\Upsilon^{1+8\kappa}+200e^{2\alpha\ell}\psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r,z'). \tag{10.31}$$

We give initial bounds for the two terms on the right side of (10.31). In view of (10.25) and (10.26), we have

$$200e^{-\alpha\ell}L_1\Upsilon^{1+8\kappa} \le e^{-0.7\ell}\Upsilon, \tag{10.32}$$

where we also used $0.9 < \alpha < 1$ and assumed $\ell = \varepsilon t/100$ is large enough to account for the factor 200.

As for the second term, using the fact that $\hat{\mathcal{E}}_{\rm old} \subset \mathcal{E}_{\rm old}$, we obtain

$$\begin{aligned}200e^{2\alpha\ell}\psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r,z') &\le 200e^{2\alpha\ell}\psi_{\mathcal{E}_{\rm old},b}(a_\ell u_r,z')\\ &\le 200C_7\eta^{-3}e^{6\ell}\cdot\sup_{z''}\psi_{\mathcal{E}_{\rm old},b}(e,z'')\\ &\le e^{\varepsilon t/10}\cdot\sup_{w'}\psi_{\mathcal{E}_{\rm old},b}(e,\exp(w')y_0);\end{aligned} \tag{10.33}$$

we used Lemma 10.4 in the second inequality and used (the final claim in) Lemma 10.3 to replace $\sup_{z''}$ by $\sup_w$, we also used $\eta > e^{-0.01\ell}$ and assumed t is large to account for the constants C_6 and $200C_7$.

We now begin the proof of the estimates in the lemma. Let us first assume

$$\Upsilon \ge e^{\varepsilon t/2}\cdot\psi_{\mathcal{E}_{\rm old},b}(e,\exp(w_0)y_0), \tag{10.34}$$

where $\psi_{\mathcal{E}_{\rm old},b}(e,\exp(w_0)y_0) = \sup_{w'}\psi_{\mathcal{E}_{\rm old},b}(e,\exp(w')y_0)$, as in the statement of the lemma. Then (10.33) and (10.34) imply that

$$\begin{aligned}200e^{2\alpha\ell}\psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r,z') &\le e^{\varepsilon t/10}\cdot\sup_{w'}\psi_{\mathcal{E}_{\rm old},b}(e,\exp(w')y_0)\\ &\le e^{\varepsilon t/10}\cdot(e^{-\varepsilon t/2}\Upsilon) \le e^{-\ell}\Upsilon\end{aligned} \tag{10.35}$$

where we used $\ell = \varepsilon t/100$.

Thus, combining (10.30), (10.31), (10.32), and (10.35), one gets

$$\begin{aligned}f_{\mathcal{E},b,\mathsf{R}_1}(e,z) &\le 2f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r,z')+10\psi_{\mathcal{E},b}(e,z)\\ &\le e^{-0.7\ell}\Upsilon+e^{-\ell}\Upsilon+10\psi_{\mathcal{E},b}(e,z)\\ &\le e^{-0.6\ell}\Upsilon+10\psi_{\mathcal{E},b}(e,z).\end{aligned}$$

This establishes part (1).

Let us now turn to the proof of part (2). Therefore, we assume

$$\Upsilon < e^{\varepsilon t/2} \cdot \psi_{\mathcal{E}_{\rm old},b}(e, \exp(w_0)y_0). \tag{10.36}$$

First note that by Lemma 10.3, if $\hat{z} = \hat{h}\exp(\hat{w})y_0 \in \mathcal{E}_{\rm old}$ where $\hat{h} \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$,

$$C_6{}^{-1}\psi_{\mathcal{E}_{\rm old},b}(e, \exp(w_0)y_0) \le \psi_{\mathcal{E}_{\rm old},b}(e,\hat{z}) \le C_6\psi_{\mathcal{E}_{\rm old},b}(e, \exp(w_0)y_0). \tag{10.37}$$

We conclude that

$$\begin{aligned} f_{\mathcal{E}_{\rm old},b,\mathsf{R}}(e,\hat{z}) \le \Upsilon &\le e^{\varepsilon t/2} \cdot \psi_{\mathcal{E}_{\rm old},b}(e, \exp(w_0)y_0) \\ &\le C_6 e^{\varepsilon t/2} \cdot \psi_{\mathcal{E}_{\rm old},b}(e,\hat{z}), \end{aligned}$$

where we used (10.13) in the first inequality, used (10.36) in the second inequality, and used (10.37) in the final inequality. This gives (10.28).

We now turn to the proof of (10.29). Recall from (10.32) and (10.33),

$$\begin{aligned} f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z') &\le 200e^{-\alpha\ell}L_1\Upsilon^{1+8\kappa} + 200e^{2\alpha\ell}\psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r, z') \\ &\le e^{-0.7\ell}\Upsilon + e^{\varepsilon t/10}\psi_{\hat{\mathcal{E}}_{\rm old},b}(e, \exp(w_0)y_0). \end{aligned}$$

In view of (10.36) and since $\ell = \varepsilon t/100$, we have

$$e^{-0.7\ell}\Upsilon + e^{\varepsilon t/10}\psi_{\hat{\mathcal{E}}_{\rm old},b}(e, \exp(w_0)y) \le e^{-0.6\ell}\big(e^{\varepsilon t/2}\cdot \psi_{\mathcal{E}_{{\rm old},b}}(e, \exp(w_0)y_0)\big).$$

Finally, using (10.30) and the above, we conclude that

$$\begin{aligned} f_{\mathcal{E},b,\mathsf{R}_1}(e,z) &\le 2f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z') + 10\psi_{\mathcal{E},b}(e,z) \\ &\le e^{-0.6\ell}\big(e^{\varepsilon t/2}\cdot \psi_{\mathcal{E}_{{\rm old},b}}(e, \exp(w_0)y_0)\big) + 10\psi_{\mathcal{E},b}(e,z). \end{aligned}$$

The proof is complete. □

11. An inductive construction

As it was mentioned, the proof of Proposition 10.1 is based on an inductive construction. We will carry out this construction in this section and complete the proof of Proposition 10.1 in the next section.

Recall that $0 < \varepsilon < 1$ is a small parameter (in our application, ε will depend on κ_7, see (13.1)) and $t > 1$ is a large parameter (which will be chosen to be $\asymp \log R$ where R is as in Theorem 1.1). Recall also that

$$\kappa = 10^{-6}d_1^{-1} \le 10^{-6}\varepsilon, \tag{11.1}$$

where $d_1 = 100\lceil(4D-3)/(2\varepsilon)\rceil$, see Proposition 10.1.

Set $b = e^{-\sqrt{\varepsilon}t}$, $\beta = e^{-\kappa t}$, and $\eta^2 = \beta$.

From now until the end of §12, we fix some M so that

$$2^{-\mathsf{M}}(D+1) < \kappa/100 \quad \text{and} \quad 6\mathsf{M} < 2^{\kappa\mathsf{M}/100}. \tag{11.2}$$

That is, conditions in (6.4) are satisfied with $\kappa = 10^{-6}d_1^{-1}$ and $m_0 = D$; note that $\kappa(D+1) \le 10^{-6}\varepsilon$. In particular, Lemma 6.4 is applicable with M and any $F \subset B_{\mathfrak{r}}(0,\beta)$ satisfying $e^{t/2} \le \#F \le e^{2t}$ and (6.2) with $\Upsilon \le e^{(D+1)t}$. This lemma will be applied, several times, in this section.

11.1. **Consequences of Proposition 4.8.** Let x_1, t, and D be as in Proposition 10.1. By our assumption, Proposition 4.8(1) holds for these choices. Recall that $x_2 = a_{8t}u_{r_1}x_1$ where $r_1 \in I(x_1)$. Then the map $h \mapsto hx_2$ is injective over $\mathsf{B}^{s,H}_\beta \cdot a_t \cdot U_1$, see Proposition 4.8(1). In particular, Lemma 8.4 may be applied with x_2, and yields the following: for every $\varphi \in C_c^\infty(X)$, every $\tau > 0$, and all $|s| \le 2$,

$$\Big|\int \varphi(a_\tau u_s h x_2)\,\mathrm{d}(\sigma * \nu_t)(h) - \sum_i c_i \int \varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\Big| \ll \beta\,\mathrm{Lip}(\varphi) \tag{11.3}$$

where the implied constant depends only on X.

Recall from (8.6) that $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y_i : w \in F_i\}$ where $y_i \in X_{3\eta/2}$. In particular, $\mathcal{E}_i \subset X_\eta$. Recall also from Lemma 8.1 and Lemma 8.2 that

$$\beta^9 e^t \le \#F_i \le \beta^{-3} e^t. \tag{11.4}$$

Moreover, in view of the definition of $\mathcal{E}_i$ and Proposition 4.8(1), we have

$$f_{\mathcal{E}_i,b,1}(e,z) \le e^{Dt} \tag{11.5}$$

for all $z \in \mathcal{E}_i$.

11.2. **Regular tree decomposition of F_i.** We will decompose F_i into subsets which are homogeneous in all *relevant* scales. First note that in view of (11.5) and Lemma 9.4 applied with $m = 4$, we have

$$\mathcal{G}_{F_{i,w},\mathsf{R}}(w') \le 10^6 e^{Dt} \qquad \text{for every } w' \in F_{i,w} \tag{11.6}$$

where for all $w \in F_i$, we put $F_{i,w} = F_i \cap B_{\mathfrak{r}}(w, 4b\mathrm{inj}(y_i))$.

Let $k_1 > k_{i,0}$ be positive integers defined as follows:

$$2^{k_{i,0}} < (b\,\mathrm{inj}(y_i))^{-1} \le 2^{k_{i,0}+1} \quad \text{and} \quad 2^{k_1} < 10^6 e^{Dt} \le 2^{k_1+1}. \tag{11.7}$$

Let M be as above, see (11.2). For every i as above, apply Lemma 6.4 to F_i. Then we can write

$$F_i = F_i' \bigcup (\textstyle\bigcup_\varsigma F_i^\varsigma) \tag{11.8}$$

where $\#F_i' \le \beta^{1/4} \cdot (\#F_i)$. Furthermore, for every i and ς we have

$$\beta^{11} e^t \le \beta^2 \cdot (\#F_i) \le \#F_i^\varsigma \le \#F_i \le \beta^{-3} e^t, \tag{11.9}$$

(where we used (11.4)), and for every $k_{i,0} - 10 \le k \le k_1$, there exists some τ_{ik}^ς so that for all $Q \in \mathcal{Q}_{\mathsf{M}k}$ we have

$$\text{either} \quad 2^{\mathsf{M}(\tau_{ik}^\varsigma - 2)} \le \#F_i^\varsigma \cap Q \le 2^{\mathsf{M}\tau_{ik}^\varsigma} \quad \text{or} \quad F_i^\varsigma \cap Q = \emptyset. \tag{11.10}$$

11.3. **Initial dimension.** Put $\mathcal{E}_i^\varsigma = \mathsf{E}.\{\exp(w)y_i : w \in F_i^\varsigma\}$ for all i and ς. Then both of the following hold

(1) Let $z = h\exp(w) \in \mathcal{E}_i^\varsigma$ where $h \in \overline{\mathsf{E}\setminus\partial_{10b}\mathsf{E}}$, then

$$
\begin{aligned}
C_6{}^{-1}\sup_{w'\in F_i^\varsigma}\psi_{\mathcal{E}_i^\varsigma,b}(e,\exp(w')y) &\le \psi_{\mathcal{E}_i^\varsigma,b}(e,z)\\
&\le C_6\sup_{w'\in F_i^\varsigma}\psi_{\mathcal{E}_i^\varsigma,b}(e,\exp(w')y).
\end{aligned}
\tag{11.11}
$$

(2) For all $z \in \mathcal{E}_i^\varsigma$, we have

$$
f_{\mathcal{E}_i^\varsigma,b,0}(e,z) \le e^{Dt}. \tag{11.12}
$$

Note that (11.11) is a consequence of Lemma 10.3, and (11.12) follows from (11.5) since $\mathcal{E}_i^\varsigma \subset \mathcal{E}_i$. We also note that the second inequality in (11.11) holds true for all $z \in \mathcal{E}_i^\varsigma$, see Lemma 10.3.

With this notation, (11.3) may be rewritten as follows: for all $\tau > 0$ and $|s| \le 2$, we have

$$
\Big|\int \varphi(a_\tau u_s h x_2)\,\mathrm{d}\,\mu_{t,\ell,0}(h) - \sum_i\sum_\varsigma c_{i,\varsigma}\int \varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_i^\varsigma}(z)\Big| \ll \beta\,\mathrm{Lip}(\varphi), \tag{11.13}
$$

here $c_{i,\varsigma} = c_i\mu_{\mathcal{E}_i}(\mathcal{E}_i^\varsigma)$; $\mu_{\mathcal{E}_i^\varsigma}$ denotes $\mu_{\mathcal{E}_i}|_{\mathcal{E}_i^\varsigma}$ normalized to be a probability measure; for any integer $n \ge 0$, we put $\mu_{t,\ell,n} = \nu_\ell * \cdots * \nu_\ell * \sigma * \nu_t$ where ν_ℓ appears n-times; and the implied constant depends only on X.

For notational convenience, let us write

$$
\{(\mathcal{E}_i^\varsigma, \mu_{\mathcal{E}_i^\varsigma}) : i,\varsigma\} = \{(\mathcal{E}_\zeta, \mu_\zeta) : \zeta \in \mathcal{Z}\}, \tag{11.14}
$$

for an index set $\mathcal{Z}$.

11.4. **Random walk trajectories: one step.** Beginning with $\mathcal{E}_{\zeta_0}$ for some $\zeta_0 \in \mathcal{Z}$ as above, we will use Lemma 8.9 to construct sets $\mathcal{E}$. Then Lemma 10.6 implies that the estimate on the corresponding Margulis function exponentially improves after each step.

Let us begin by fixing some notation. Let $\zeta_0 \in \mathcal{Z}$ be as above. Put

$$
\mathsf{A}_0^{\zeta_0} = \{\zeta_0\},
$$

and recall $\big(\mathcal{E}_{\zeta_0}, \mu_{\mathcal{E}_{\zeta_0}}\big)$ from above. Using an inductive construction, we will define $\mathsf{A}_n^{\zeta_0}$ and $(\mathcal{E}(\Xi), \mu_{\mathcal{E}(\Xi)})$ for all $n \ge 1$ and all $\Xi \in \mathsf{A}_n^{\zeta_0}$.

Let us begin with the definition in the case $n = 1$. Put

$$
(\mathcal{E}_{\mathrm{old}}, \mu_{\mathcal{E}_{\mathrm{old}}}) = (\mathcal{E}_{\zeta_0}, \mu_{\mathcal{E}_{\zeta_0}}).
$$

In view of (11.12) and (11.10), $(\mathcal{E}_{\mathrm{old}}, \mu_{\mathcal{E}_{\mathrm{old}}})$ satisfies the conditions in Lemma 9.1 with $\Upsilon = e^{Dt}$, $\mathsf{R} = 0$, and c depending only on M. Recall also that $0 < \kappa \le \varepsilon/10^6$. By Lemma 9.1, thus, there exists $L_{\mu_{\mathcal{E}_{\mathrm{old}}}} \subset [0,1]$ with

$$
\big|[0,1]\setminus L_{\mu_{\mathcal{E}_{\mathrm{old}}}}\big| \ll e^{-\kappa^2 t/4},
$$

and for every $r \in L_{\mu_{\mathcal{E}_{\rm old}}}$, there exists a subset

$$\mathcal{E}_{{\rm old},r} \subset \hat{\mathcal{E}}_{\rm old} = \bigcup \hat{\mathsf{E}}.\{\exp(w)y_0 : w \in F_{\rm old}\}, \qquad (\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}})$$

satisfying $\mu_{\mathcal{E}_{\rm old}}(\mathcal{E}_{\rm old} \setminus \mathcal{E}_{{\rm old},r}) \ll e^{-\kappa^2 t/64}$ and the following: for all $z \in \mathcal{E}_{{\rm old},r}$,

$$f_{\hat{\mathcal{E}}_{\rm old},b,\mathsf{R}_1}(a_\ell u_r, z) \le 200 L_1 e^{-\alpha\ell}\Upsilon^{1+8\kappa} + 200 e^{2\alpha\ell}\psi_{\hat{\mathcal{E}}_{\rm old},b}(a_\ell u_r, z); \tag{11.15}$$

where $L_1 = L\kappa^{-L}$ and $\mathsf{R}_1 = 1 + L_1\Upsilon^\kappa$. We assumed Υ is large (depending on κ) and the fact that $\mathsf{R} = 1$ in the above bound, see also Theorem 6.2.

Recall that $d_1 = 100\lceil\frac{4D-3}{2\varepsilon}\rceil$, and fix a maximal $e^{-6d_1\ell}$-separated subset

$$\mathcal{L}_{\mathcal{E}_{\rm old}} = \{r_{{\rm old},q}\} \subset L_{\mu_{\mathcal{E}_{\rm old}}}.$$

For every $r_0 \in \mathcal{L}_{\mathcal{E}_{\rm old}}$, let

$$\{(\mathcal{E}_\zeta, \mu_{\mathcal{E}_\zeta}) : \zeta \in \mathcal{Z}''_{\zeta_0,r_0}\}$$

be the set of offsprings of $a_\ell u_{r_0}\mathcal{E}_{\rm old}$, see (8.21) and (8.22). In particular, $\mathcal{E}_\zeta = \mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta\}$ where

$$F_\zeta \subset \big\{w \in B_{\mathfrak{r}}(0,\beta) : \mathsf{Q}^H_\ell.\exp(w)y_\zeta \subset a_\ell u_{r_0}\mu_{\mathcal{E}_{\rm old}}\big\},$$

and $y_\zeta \in X_{3\eta/2}$. Moreover, (8.18) implies that for every $\zeta \in \mathcal{Z}''_{\zeta_0,r_0}$,

$$\beta^9 \cdot (\#F_{\rm old}) \le \#F_\zeta \le \beta^8 \cdot (\#F_{\rm old}). \tag{11.16}$$

Let us put $\hat{\hat{\mathsf{E}}} = \overline{\mathsf{E} \setminus \partial_{100\beta^2}\mathsf{E}}$, and define

$$\hat{\hat{\mathcal{E}}}_{\rm old} = \hat{\hat{\mathsf{E}}}.\{\exp(v)y_0 : v \in F_{\rm old}\}.$$

Then, we have

$$\mu_{\mathcal{E}_{\rm old}}\big(\mathcal{E}_{\rm old} \setminus (\mathcal{E}_{{\rm old},r_0} \cap \hat{\hat{\mathcal{E}}}_{\rm old})\big) \ll \beta + e^{-\kappa^2 t/64}. \tag{11.17}$$

Let $F_{\zeta,r_0} = \big\{w \in F_\zeta : \mathsf{Q}^H_\ell.\exp(w)y_\zeta \cap a_\ell u_{r_0}\big(\mathcal{E}_{{\rm old},r_0} \cap \hat{\hat{\mathcal{E}}}_{\rm old}\big) = \emptyset\big\}$. If $\#F_{\zeta,r_0} \le 10^{-6}\cdot(\#F_\zeta)$, replace $\mathcal{E}_\zeta$ with

$$\mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta \setminus F_{\zeta,r_0}\}$$

otherwise, discard the set $\mathcal{E}_\zeta$ entirely. Such replacements will increase the set $a_\ell u_{r_0}\mathcal{E}_{\rm old}\setminus\bigcup_\zeta \mathcal{E}_\zeta$. But thanks to (11.17), this doesn't affect the properties that we will need later, or more precisely the inequality (11.26) in Lemma 11.6 below.

Let $\mathcal{Z}'_{\zeta_0,r_0} \subset \mathcal{Z}''_{\zeta_0,r_0}$ be the set of indices which survive the above process. Abusing the notation, for every $\zeta \in \mathcal{Z}'_{\zeta_0,r_0}$, we denote $F_\zeta \setminus F_{\zeta,r_0}$ by F_ζ and denote $\mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta \setminus F_{\zeta,r_0}\}$ by $\mathcal{E}_\zeta$.

Thus, we obtain a collection $\{(\mathcal{E}_\zeta, \mu_{\mathcal{E}_\zeta}) : \zeta \in \mathcal{Z}'_{\zeta_0,r_0}\}$ satisfying the following: If $\zeta \in \mathcal{Z}'_{0,r_0}$ and $w \in F_\zeta$, then

$$\mathsf{Q}^H_\ell.\exp(w)y_\zeta \cap a_\ell u_{r_0}\big(\mathcal{E}_{{\rm old},r_0} \cap \hat{\hat{\mathcal{E}}}_{\rm old}\big) \ne \emptyset;$$

moreover, the following analogue of (11.16) holds

$$0.5\beta^9 \cdot (\#F_{\rm old}) \le \#F_\zeta \le 2\beta^8 \cdot (\#F_{\rm old}). \tag{11.18}$$

With this notation, define

$$\mathsf{B}_1^{\zeta_0} = \big\{(\zeta_0, r_0, \zeta) : r_0 \in \mathcal{L}_{\mathcal{E}_{\zeta_0}}, \zeta \in \mathcal{Z}'_{\zeta_0, r_0}\big\}, \tag{11.19}$$

and for every $\Xi = (\zeta_0, r_0, \zeta) \in \mathsf{B}_1^{\zeta_0}$, put

$$\mathcal{E}_\Xi = \mathsf{E}.\{\exp(w)y_\Xi : w \in F_\Xi\},$$

where $y_\Xi = y_\zeta$ and $F_\Xi = F_\zeta$.

11.5. **Lemma.** *Let $\Xi = (\zeta_0, r_0, \zeta) \in \mathsf{B}_1^{\zeta_0}$, and write $F = F_\Xi$, $y = y_\Xi$, and $\mathcal{E} = \mathcal{E}_\Xi$. Let $w_0 \in F_{\zeta_0}$ be so that*

$$\psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w_0)y_0) = \sup_{w'} \psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w')y_0).$$

Then one of the following properties holds:

(1) If $e^{Dt} \ge e^{\varepsilon t/2}\psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w_0)y_0)$, then

$$f_{\mathcal{E}, b, \mathsf{R}_1}(e, z) \le e^{-0.6\ell} e^{Dt} + 10\psi_{\mathcal{E}, b}(e, z) \quad \textit{for all } z \in \mathcal{E}, \tag{11.20}$$

where $\mathsf{R}_1 = 1 + L\kappa^{-L}e^{\kappa D t}$.

(2) If $e^{Dt} < e^{\varepsilon t/2}\psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w_0)y_0)$, then both of the following hold

(a) Let $z = h\exp(w)y_0 \in \mathcal{E}_{\zeta_0}$ where $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$, then

$$f_{\mathcal{E}_{\zeta_0}, b, \mathsf{R}}(e, z) \le e^{\varepsilon t/2}\psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w_0)y_0) \le C_6 e^{\varepsilon t/2}\psi_{\mathcal{E}_{\zeta_0}, b}(e, z), \tag{11.21}$$

(indeed the first inequality above holds for every *$z \in \mathcal{E}_{\zeta_0}$).*

(b) For all $z \in \mathcal{E}$, we have

$$f_{\mathcal{E}, b, \mathsf{R}_1}(e, z) \le e^{-0.6\ell}\big(e^{\varepsilon t/2} \cdot \psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w_0)y_0)\big) + 10\psi_{\mathcal{E}, b}(e, z). \tag{11.22}$$

Indeed case (2) does not hold and we are always in case (1).

Proof. Note that $e^{\kappa D t} \le e^{\ell t/100}$. Moreover, in view of (11.10) and the fact that for every $w \in F_{\zeta_1}$, we have

$$\mathsf{Q}_\ell^H.\exp(w)y \cap a_\ell u_{r_0}\big(\mathcal{E}_{{\rm old}, r} \cap \hat{\hat{\mathcal{E}}}_{\rm old}\big) \ne \emptyset,$$

Lemma 10.7 is applicable with $\mathcal{E}_{\zeta_0}$ and $\mathcal{E}$. Applying loc. cit. with $\mathcal{E}_{\zeta_0}$ and $\mathcal{E}$ thus implies all but the final claim in this lemma.

To see the final claim, note that by (11.4), we have

$$e^{\varepsilon t/2}\psi_{\mathcal{E}_{\zeta_0}, b}(e, \exp(w)y_0) \le e^{\varepsilon t/2} \cdot (2\eta b^{-\alpha}) \cdot (\beta^{-3}e^t) \le e^{2t}.$$

Moreover, $D \ge 10$, see Proposition 4.8, hence, case (2) cannot hold. □

Let $\Upsilon_0 = e^{Dt}$. For every $\Xi = (\zeta_0, r_0, \zeta) \in \mathsf{B}_1^{\zeta_0}$, define $\Upsilon_{\Xi,1}$ as follows: if

$$e^{-0.6\ell} e^{Dt} \geq 10 \sup_{z \in \mathcal{E}_\Xi} \psi_{\mathcal{E}_\Xi, b}(e, z),$$

then we put

$$\Upsilon_{\Xi,1} = e^{-\ell/2} e^{Dt}. \tag{11.23}$$

Otherwise, i.e., if $e^{-0.6\ell} e^{Dt} < 10 \sup_{z \in \mathcal{E}_\Xi} \psi_{\mathcal{E}_\Xi, b}(e, z)$, then we put

$$\Upsilon_{\Xi,1} = 20 \sup_{z \in \mathcal{E}_\Xi} \psi_{\mathcal{E}_\Xi, b}(e, z). \tag{11.24}$$

11.6. **Lemma.** *The following three statements hold:*

(1) For every $\Xi = (\zeta_0, r_0, \zeta) \in \mathsf{B}_1^{\zeta_0}$, we have $\Upsilon_{\Xi,1} \leq e^{Dt}$.

(2) Let $\Xi = (\zeta_0, r_0, \zeta) \in \mathsf{B}_1^{\zeta_0}$, then

$$f_{\mathcal{E}_\Xi, b, \mathsf{R}_1}(e, z) \leq \Upsilon_{\Xi,1}, \tag{11.25}$$

where $\mathsf{R}_1 = 1 + L\kappa^{-L} e^{\kappa D t}$.

(3) Let $r_0 \in \mathcal{L}_{\mathcal{E}_{\zeta_0}}$. Then

$$\begin{aligned} \Big| \int \varphi(a_\tau u_s.z)\, \mathrm{d}(a_\ell u_{r_0} \mu_{\mathcal{E}_{\zeta_0}})(z) - \sum_{\mathsf{B}_1^{\zeta_0}} c_{\Xi'} \int \varphi(a_\tau u_s z)\, \mathrm{d}\mu_{\mathcal{E}_{\Xi'}}(z) \Big| & \\ \ll \max\{\eta^{1/2}, e^{-\kappa^2 t/64}\} \operatorname{Lip}(\varphi), & \end{aligned} \tag{11.26}$$

for every $\varphi \in C_c^\infty(X)$, every $0 < \tau \leq 2d_1\ell$, and all $|s| \leq 2$,

Proof. The claim in part (1) is clear if $\Upsilon_{\Xi,1} = e^{-\ell/2} e^{Dt}$. Assume thus that

$$\Upsilon_{\Xi,1} = 20 \sup_{z \in \mathcal{E}_\Xi} \psi_{\mathcal{E}_\Xi, b}(e, z).$$

Then by the definition of ψ, (11.4) and (11.18), we have

$$\Upsilon_{\Xi,1} \ll b^{-\alpha} \eta^{-\alpha} \cdot (\# F_\Xi) \leq e^{2t},$$

where we also used $b = e^{-\sqrt{\varepsilon} t}$ and $\eta \geq e^{-0.01\varepsilon t}$. The claim follows as $D \geq 10$.

Part (2) follows from the definition of $\Upsilon_{\Xi,1}$ and Lemma 11.5.

To see part (3), apply Lemma 8.9, with $\mathsf{d}_0 = 3d_1\ell$ (note that $\tau + \ell \leq \mathsf{d}_0$) and r_0. By that lemma thus

$$\begin{aligned} \Big| \int \varphi(a_d u_s.z)\, \mathrm{d}(a_\ell u_{r_0} \mu_{\mathcal{E}_{\zeta_0}})(z) - \sum c_\zeta \int \varphi(a_d u_s z)\, \mathrm{d}\mu_{\mathcal{E}_\zeta}(z) \Big| & \\ \ll \max\{\eta^{1/2}, e^{-\kappa^2 t/64}\} \operatorname{Lip}(\varphi), & \end{aligned}$$

where the sum is over $\zeta \in \mathcal{Z}''_{\zeta_0, r_0}$.

We can replace the summation over $\mathcal{Z}''_{\zeta_0, r_0}$ by summation over $\mathcal{Z}'_{\zeta_0, r_0}$ (hence over $\mathsf{B}_1^{\zeta_0}$) in view of (11.17) and the definition of $\mathcal{Z}'_{\zeta_0, r_0}$. □

11.7. **Regularizing F_Ξ.** In preparation for the next step of the inductive construction, we will refine the set $\mathsf{B}_1^{\zeta_0}$ by decomposing F_Ξ (for $\Xi \in \mathsf{B}_1^{\zeta_0}$) into sets satisfying estimates similar to those in (6.7).

To that end, let $\Xi = (\zeta_0, r_0, \zeta_1) \in \mathsf{B}_1^{\zeta_0}$, and let $F = F_\Xi$, $y = y_\Xi$, and $\mathcal{E} = \mathcal{E}_\Xi$. In view of Lemma 11.6(2) and Lemma 9.4,

$$\mathcal{G}_{F_w,\mathsf{R}_1}(w') \le 10^6 \Upsilon_{\Xi,1} \qquad \text{for every } w' \in F_w,$$

where $F_w = F \cap B_{\mathfrak{r}}(w, 4b\mathrm{inj}(y))$.

Let $k_1 > k_0$ be positive integers defined as follows:

$$2^{k_0} < (b\,\mathrm{inj}(y))^{-1} \le 2^{k_0+1} \quad \text{and} \quad 2^{k_1} < 10^6 \Upsilon_{\Xi,1} \le 2^{k_1+1}. \tag{11.27}$$

Let M be as above, see (11.2). Applying Lemma 6.4, we can write

$$F = F' \bigcup \left(\textstyle\bigcup_l F_l\right) \tag{11.28}$$

where $\#F' \le \beta^{1/4} \cdot (\#F)$ and $\#F_l \ge \beta^2 \cdot (\#F)$. In view of (11.18), we have

$$\begin{aligned} 0.5\beta^{11} \cdot (\#F_{\zeta_0}) \le \beta^2 \cdot (\#F) \le \#F_l& \\ \le \#F \le 2\beta^8 \cdot (\#F_{\zeta_0})&, \end{aligned} \tag{11.29}$$

and for every $k_0 - 10 \le k \le k_1$, there exists some $\tau_k = \tau_k^l$ so that

$$\text{either} \quad 2^{\mathsf{M}(\tau_k - 2)} \le \#F_l \cap Q \le 2^{\mathsf{M}\tau_k} \quad \text{or} \quad F_l \cap Q = \emptyset, \tag{11.30}$$

for all $Q \in \mathcal{Q}_{\mathsf{M}k}$.

Let us also note that combining (11.29) and (11.9), we conclude

$$\frac{1}{2}\beta^{22} e^t \le \#F_l \le 2\beta^5 e^t. \tag{11.31}$$

Let $\mathcal{Z}_{\zeta_0,r_0}$ be an enumeration of $\{(\zeta', l) : \zeta' \in \mathcal{Z}'_{0,r_0}, l \in \mathcal{K}_{(\zeta_0,r_0,\zeta')}\}$ where for every $\Xi = (\zeta_0, r_0, \zeta') \in \mathsf{B}_1^{\zeta_0}$, we let

$$\mathcal{K}_\Xi = \{l : F_l \text{ as in (11.28)}\}.$$

If $\zeta \in \mathcal{Z}_{\zeta_0,r_0}$ corresponds to (ζ', l), put $y_\zeta = y_{\zeta'}$ and $F_\zeta = (F_{\zeta'})_l$, see (11.28).

Define

$$\mathsf{A}_1^{\zeta_0} = \left\{(\zeta_0, r_0, \zeta_1) : r_0 \in \mathcal{L}_{\mathcal{E}_{\zeta_0}}, \zeta_1 \in \mathcal{Z}_{\zeta_0,r_0}\right\}, \tag{11.32}$$

and for every $\Xi = (\zeta_0, r_0, \zeta_1) \in \mathsf{A}_1^{\zeta_0}$, put

$$\mathcal{E}_\Xi = \mathsf{E}.\{\exp(w) y_\Xi : w \in F_\Xi\},$$

where $y_\Xi = y_{\zeta_1}$ and $F_\Xi = F_{\zeta_1}$.

11.8. **Lemma.** *Let $\Xi = (\zeta_0, r_0, \zeta_1) \in \mathsf{A}_1^{\zeta_0}$, and suppose ζ_1 correspond to (ζ', l) as above. Put $\Upsilon_{\Xi,1} = \Upsilon_{\Xi',1}$ where $\Xi' = (\zeta_0, r_0, \zeta') \in \mathsf{B}_1^{\zeta_0}$. Then both of the following hold:*

(1) We have

$$f_{\mathcal{E}_\Xi, b, \mathsf{R}_1}(e, z) \le \Upsilon_{\Xi,1}, \tag{11.33}$$

where $\mathsf{R}_1 = 1 + L\kappa^{-L} e^{\kappa D t}$.

(2) Let $r_0 \in \mathcal{L}_{\mathcal{E}_{\zeta_0}}$. Then

$$\Big|\int \varphi(a_\tau u_s.z)\,\mathrm{d}(a_\ell u_{r_0}\mu_{\mathcal{E}_{\zeta_0}})(z) - \sum_{\mathsf{A}_1^{\zeta_0}} c_\Xi \int \varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_\Xi}(z)\Big| \ll \max\{\eta^{1/2}, e^{-\kappa^2 t/64}\}\,\mathrm{Lip}(\varphi), \tag{11.34}$$

for every $\varphi \in C_c^\infty(X)$, every $0 < \tau \le 2d_1\ell$, and all $|s| \le 2$,

Proof. Part (1) follows from Lemma 11.6(2) and the fact that $\mathcal{E}_\Xi \subset \mathcal{E}_{\Xi'}$. Part (2) follows from Lemma 11.6(3) in view of (11.28) if we put

$$c_\Xi = c_{\Xi'}\mu_{\mathcal{E}_{\Xi'}}(\mathcal{E}_\Xi)$$

and use the fact that $\mu_{\mathcal{E}_{\Xi'}}$ is admissible, see Lemma 8.8. □

11.9. **Random walk trajectories: n-steps.** We now assume that $\mathsf{A}_n^{\zeta_0}$ is defined for some $n \ge 1$, and will define $\mathsf{A}_{n+1}^{\zeta_0}$. The construction is similar to the case $n = 1$. Indeed, as it was done in that case, we will define $\mathsf{A}_{n+1}^{\zeta_0}$ using the collection of $2n+3$ tuples

$$(\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta_{n+1})$$

satisfying the following properties

- $\hat{\Xi} := (\zeta_0, r_0, \ldots, \zeta_n) \in \mathsf{A}_n^{\zeta_0}$,
- $r_n \in \mathcal{L}_{\mathcal{E}(\hat{\Xi})}$, and
- $\zeta_{n+1} \in \mathcal{Z}'_{n,r_n}$,

where $\mathcal{L}_{\mathcal{E}(\hat{\Xi})} \subset L_{\mu_{\mathcal{E}(\hat{\Xi})}}$ is a maximal e^{-6d_1}-separated subset, see Lemma 9.1 for $L_{\mathcal{E}(\hat{\Xi})}$, and

$$\mathcal{Z}'_{n,r_n} \subset \mathcal{Z}''_{n,r_n}$$

where $\mathcal{Z}''_{n,r_n}$ is the index set enumerating the offsprings of $a_\ell u_{r_n}\mathcal{E}(\hat{\Xi})$, see (8.21) and (8.22) for offsprings.

We now turn to the details: Recall that $0 < \kappa \le \varepsilon/10^6$, for all $m \in \mathbb{N}$ put

$$\mathsf{R}_m = 1 + mL\kappa^{-L}e^{\kappa D t}, \tag{11.35}$$

see Lemma 11.8 for R_1.

Let $\hat{\Xi} = (\zeta_0, r_0, \ldots, \zeta_n) \in \mathsf{A}_n^{\zeta_0}$, and put

$$(\mathcal{E}_{\mathrm{old}}, \mu_{\mathcal{E}_{\mathrm{old}}}) = (\mathcal{E}_{\hat{\Xi}}, \mu_{\mathcal{E}_{\hat{\Xi}}});$$

note that $\mathcal{E}_{\hat{\Xi}} = \mathsf{E}.\{\exp(w)y_{\hat{\Xi}} : w \in F_{\hat{\Xi}}\}$, where

$$\frac{1}{2^n}\beta^{11(n+1)}e^t \le \#F_{\hat{\Xi}} \le 2^n\beta^{8n-3}e^t, \tag{11.36}$$

see (11.4) and (11.31).

Then, by inductive hypothesis, we have

$$f_{\mathcal{E}_{\hat{\Xi}},b,\mathsf{R}_n}(e,z) \le \Upsilon_{\hat{\Xi},n} \quad \text{for all } z \in \mathcal{E}_{\hat{\Xi}}, \tag{11.37}$$

where $\Upsilon_{\hat{\Xi},n}$ is defined inductively. Recall that $\Upsilon_0 = e^{Dt}$ also see (11.23) and (11.24) for the definition of $\Upsilon_{\hat{\Xi},1}$. In particular, we have

$$\Upsilon_{\hat{\Xi},n} \le e^{Dt}, \tag{11.38}$$

see Lemma 11.6(1).

Recall that $d_1 = 100\lceil\frac{4D-3}{2\varepsilon}\rceil$. Fix a maximal $e^{-6d_1\ell}$-separated subset

$$\mathcal{L}_{\mathcal{E}_{\rm old}} \subset L_{\mu_{\mathcal{E}_{\rm old}}}.$$

For every $r_n \in \mathcal{L}_{\mathcal{E}_{\rm old}}$, let

$$\{(\mathcal{E}_\zeta, \mu_{\mathcal{E}_\zeta}) : \zeta \in \mathcal{Z}''_{\hat{\Xi},r_n}\}$$

be the set of all offsprings of $a_\ell u_{r_n}\mathcal{E}_{\rm old} = a_\ell u_{r_n}\mathcal{E}_{\hat{\Xi}}$, see (8.21) and (8.22). In particular, $\mathcal{E}_\zeta = \mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta\}$ where

$$F_\zeta \subset \big\{w \in B_{\mathfrak{r}}(0,\beta) : \mathsf{Q}_\ell^H.\exp(w)y_\zeta \subset a_\ell u_{r_n}\mu_{\mathcal{E}_{\rm old}}\big\}$$

for some $y_\zeta \in X_{3\eta/2}$.

Moreover, (8.18) implies that for every $\zeta \in \mathcal{Z}''_{\hat{\Xi},r_n}$, we have

$$\beta^9 \cdot (\#F_{\rm old}) \le \#F_\zeta \le \beta^8 \cdot (\#F_{\rm old}). \tag{11.39}$$

Let us put $\hat{\hat{\mathsf{E}}} = \overline{\mathsf{E} \setminus \partial_{100\beta^2}\mathsf{E}}$, and define

$$\hat{\hat{\mathcal{E}}}_{\rm old} = \hat{\hat{\mathsf{E}}}.\{\exp(v)y_{\rm old} : v \in F_{\rm old}\}.$$

Then, we have

$$\mu_{\mathcal{E}_{\rm old}}\big(\mathcal{E}_{\rm old} \setminus (\mathcal{E}_{{\rm old},r_n} \cap \hat{\hat{\mathcal{E}}}_{\rm old})\big) \ll \beta + e^{-\kappa^2 t/64}. \tag{11.40}$$

Let $F_{\zeta,r_n} = \big\{w \in F_\zeta : \mathsf{Q}_\ell^H.\exp(w)y_\zeta \cap a_\ell u_{r_n}\big(\mathcal{E}_{{\rm old},r_n} \cap \hat{\hat{\mathcal{E}}}_{\rm old}\big) = \emptyset\big\}$. If $\#F_{\zeta,r_n} \le 10^{-6}\cdot(\#F_\zeta)$, replace $\mathcal{E}_\zeta$ with

$$\mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta \setminus F_{\zeta,r_n}\}$$

otherwise, discard the set $\mathcal{E}_\zeta$ entirely. As in how (11.17) was used, the inequality (11.40) assures that such replacements causes no damage later.

Let $\mathcal{Z}'_{\hat{\Xi},r_n} \subset \mathcal{Z}''_{\hat{\Xi},r_n}$ be the set of indices which survive the above process. Abusing the notation, for every $\zeta \in \mathcal{Z}'_{\hat{\Xi},r_n}$, we denote $F_\zeta \setminus F_{\zeta,r_n}$ by F_ζ and denote $\mathsf{E}.\{\exp(w)y_\zeta : w \in F_\zeta \setminus F_{\zeta,r_n}\}$ by $\mathcal{E}_\zeta$.

Thus, we obtain a collection $\big\{(\mathcal{E}_\zeta, \mu_{\mathcal{E}_\zeta}) : \zeta \in \mathcal{Z}'_{\hat{\Xi},r_n}\big\}$ satisfying the following: If $\zeta \in \mathcal{Z}'_{\hat{\Xi},r_n}$ and $w \in F_\zeta$, then

$$\mathsf{Q}_\ell^H.\exp(w)y_\zeta \cap a_\ell u_{r_n}\big(\mathcal{E}_{{\rm old},r_n} \cap \hat{\hat{\mathcal{E}}}_{\rm old}\big) \ne \emptyset;$$

moreover, the following analogue of (11.39) holds

$$0.5\beta^9 \cdot (\#F_{\rm old}) \le \#F_\zeta \le 2\beta^8 \cdot (\#F_{\rm old}). \tag{11.41}$$

With this notation, define

$$\mathsf{B}_{n+1}^{\zeta_0} = \left\{ (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta) : \begin{array}{c} \hat{\Xi} = (\zeta_0, r_0, \ldots, \zeta_n) \in \mathsf{A}_n^{\zeta_0}, \\ r_n \in \mathcal{L}_{\mathcal{E}_{\hat{\Xi}}}, \zeta \in \mathcal{Z}'_{\hat{\Xi}, r_n} \end{array} \right\}. \tag{11.42}$$

For every $\Xi = (\zeta_0, \ldots, \zeta_n, r_n, \zeta) \in \mathsf{B}_{n+1}^{\zeta_0}$, put

$$\mathcal{E}_\Xi = \mathsf{E}.\{\exp(w) y_\Xi : w \in F_\Xi\},$$

where $y_\Xi = y_\zeta$ and $F_\Xi = F_\zeta$.

11.10. **Lemma.** *Let $\Xi = (\zeta_0, \ldots, \zeta_n, r_n, \zeta) \in \mathsf{B}_{n+1}^{\zeta_0}$, and write*

$$\hat{\Xi} = (\zeta_0, \ldots, \zeta_n), \;\; F = F_\Xi, \;\; y = y_\Xi, \text{ and } \mathcal{E} = \mathcal{E}_\Xi.$$

Let $w_0 \in F_{\hat{\Xi}}$ be so that

$$\psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w_0) y_{\hat{\Xi}}) = \sup_{w'} \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w') y_{\hat{\Xi}}).$$

Then one of the following properties holds:

(A-1) If $\Upsilon_{\hat{\Xi}, n} \geq e^{\varepsilon t/2} \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w_0) y_{\hat{\Xi}})$, then

$$f_{\mathcal{E}, b, \mathsf{R}_{n+1}}(e, z) \leq e^{-0.6\ell} \Upsilon_{\hat{\Xi}, n} + 10 \psi_{\mathcal{E}, b}(e, z) \quad \text{for all } z \in \mathcal{E}, \tag{11.43}$$

where $\mathsf{R}_{n+1} = 1 + (n+1) L \kappa^{-L} e^{\kappa D t}$, see (11.35).

(A-2) If $\Upsilon_{\hat{\Xi}, n} < e^{\varepsilon t/2} \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w_0) y_{\hat{\Xi}})$, then both of the following hold

(a) Let $z = h \exp(w) y_{\hat{\Xi}} \in \mathcal{E}_{\hat{\Xi}}$ where $h \in \overline{\mathsf{E} \setminus \partial_{10b} \mathsf{E}}$, then

$$f_{\mathcal{E}_{\hat{\Xi}}, b, \mathsf{R}_n}(e, z) \leq e^{\varepsilon t/2} \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w_0) y_{\hat{\Xi}}) \leq C_6 e^{\varepsilon t/2} \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, z), \tag{11.44}$$

(indeed the first inequality above holds for every *$z \in \mathcal{E}_{\hat{\Xi}}$).*

(b) For all $z \in \mathcal{E}$, we have

$$f_{\mathcal{E}, b, \mathsf{R}_{n+1}}(e, z) \leq e^{-0.6\ell} \big(e^{\varepsilon t/2} \cdot \psi_{\mathcal{E}_{\hat{\Xi}}, b}(e, \exp(w_0) y_{\hat{\Xi}}) \big) + 10 \psi_{\mathcal{E}, b}(e, z). \tag{11.45}$$

Proof. Recall that $\Upsilon_{\hat{\Xi}, n} \leq e^{Dt}$, see (11.38); we have $e^{\kappa D t} \leq e^{\ell t/100}$. Moreover, note that for every $w \in F_{\hat{\Xi}}$, we have

$$\mathsf{Q}_\ell^H . \exp(w) y \cap a_\ell u_{r_n} \big(\mathcal{E}_{\text{old}, r_n} \cap \hat{\hat{\mathcal{E}}}_{\text{old}} \big) \neq \emptyset.$$

Moreover, using $\Upsilon_{\hat{\Xi}, n} \leq e^{Dt}$ again, we have

$$\mathsf{R}_n + L \kappa^{-L} \Upsilon_{\hat{\Xi}, n}^\kappa \leq \mathsf{R}_{n+1}.$$

The claims in the lemma thus follow from Lemma 10.7 applied with $\mathcal{E}_{\hat{\Xi}}$, $\mathcal{E}$ and $\mathsf{R} = \mathsf{R}_n$. □

Let $\Xi = (\zeta_0, \ldots, \zeta_n, r_n, \zeta) \in \mathsf{B}_{n+1}^{\zeta_0}$ and put $\hat{\Xi} = (\zeta_0, \ldots, \zeta_n)$. We define $\Upsilon_{\Xi, n+1}$ as follows: If case (A-1) holds and

$$e^{-0.6\ell} \Upsilon_{\hat{\Xi}, n} \geq 10 \sup_{z \in \mathcal{E}_\Xi} \psi_{\mathcal{E}_\Xi, b}(e, z),$$

then we put

$$\Upsilon_{\Xi,n+1} = e^{-\ell/2}\Upsilon_{\hat{\Xi},n}. \tag{11.46}$$

If case (A-1) holds and $e^{-0.6\ell}\Upsilon_{\hat{\Xi},n} < 10\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z)$, then we put

$$\Upsilon_{\Xi,n+1} = 20\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z). \tag{11.47}$$

If case (A-2) holds and

$$e^{-0.6\ell}\big(e^{\varepsilon t/2}\cdot\psi_{\mathcal{E}_{\hat{\Xi}},b}(e,\exp(w_0)y_{\hat{\Xi}})\big) \geq 10\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z),$$

we put

$$\Upsilon_{\Xi,n+1} = e^{-\ell/2}\big(e^{\varepsilon t/2}\cdot\psi_{\mathcal{E}_{\hat{\Xi}},b}(e,\exp(w_0)y_{\hat{\Xi}})\big). \tag{11.48}$$

If case (A-2) holds and

$$e^{-0.6\ell}\big(e^{\varepsilon t/2}\cdot\psi_{\mathcal{E}_{\hat{\Xi}},b}(e,\exp(w_0)y_{\hat{\Xi}})\big) < 10\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z),$$

then we put

$$\Upsilon_{\Xi,n+1} = 20\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z). \tag{11.49}$$

11.11. **Lemma.** *The following three statements hold:*

(1) For every $\Xi = (\zeta_0,\ldots,\zeta_n,r_n,\zeta)\in\mathsf{B}^{\zeta_0}_{n+1}$, we have $\Upsilon_{\Xi,n+1}\leq e^{Dt}$.

(2) Let $\Xi = (\zeta_0,\ldots,\zeta_n,r_n,\zeta)\in\mathsf{B}^{\zeta_0}_{n+1}$, then

$$f_{\mathcal{E}_\Xi,b,\mathsf{R}_{n+1}}(e,z)\leq\Upsilon_{\Xi,n+1}, \tag{11.50}$$

where $\mathsf{R}_{n+1} = 1+(n+1)L\kappa^{-L}e^{\kappa Dt}$.

(3) Let $\hat{\Xi}\in\mathsf{A}^{\zeta_0}_n$ and let $r_n\in\mathcal{L}_{\mathcal{E}_{\hat{\Xi}}}$. Then for every $\varphi\in C^\infty_c(X)$, every $0<\tau\leq 2d_1\ell$, and all $|s|\leq 2$, we have

$$\Big|\int\varphi(a_\tau u_s.z)\,\mathrm{d}(a_\ell u_{r_n}\mu_{\mathcal{E}_{\hat{\Xi}}})(z) - \sum c_\Xi\int\varphi(a_\tau u_s z)\,\mathrm{d}\mu_{\mathcal{E}_\Xi}(z)\Big| \ll \max\big\{\eta^{1/2}, e^{-\kappa^2 t/64}\big\}\,\mathrm{Lip}(\varphi), \tag{11.51}$$

where the sum is over $\mathcal{Z}'_{\hat{\Xi},r_n}$, and for every $\zeta\in\mathcal{Z}'_{\hat{\Xi},r_n}$, we let

$$\Xi = (\zeta_0,r_0,\ldots,\zeta_n,r_n,\zeta).$$

Proof. Let $\Xi = (\zeta_0,\ldots,\zeta_n,r_n,\zeta)$ and put $\hat{\Xi} = (\zeta_0,\ldots,\zeta_n)$. The claim in part (1) follows from (11.38) if $\Upsilon_{\Xi,n+1} = e^{-\ell/2}\Upsilon_{\hat{\Xi},n}$.

We now consider the other two possibilities. First suppose that

$$\Upsilon_{\Xi,n+1} = 20\sup_{z\in\mathcal{E}_\Xi}\psi_{\mathcal{E}_\Xi,b}(e,z).$$

Then by the definition of ψ, (11.36) and (11.41), we have

$$\Upsilon_{\Xi,n+1}\ll b^{-\alpha}\eta^{-\alpha}\cdot(\#F_\Xi)\leq e^{2t},$$

where we also used $b = e^{-\sqrt{\varepsilon}t}$ and $\eta \geq e^{0.01\varepsilon t}$. The claim in this case also follows as $D \geq 10$.

Finally, let us assume

$$\Upsilon_{\Xi,n+1} = e^{-\ell/2}\big(e^{\varepsilon t/2}\cdot \sup_{w'} \psi_{\mathcal{E}_{\hat\Xi},b}(e, \exp(w')y_{\hat\Xi})\big).$$

Then again using the definition of ψ, and (11.36), we have

$$\Upsilon_{\Xi,n+1} \ll e^{\varepsilon t/2} b^{-\alpha}\eta^{-\alpha}\cdot(\#F_{\hat\Xi}) \leq e^{2t},$$

which completes the proof of part (1).

Part (2) follows from the definition of $\Upsilon_{\Xi,n+1}$ and Lemma 11.10.

To see part (3), apply Lemma 8.9, with $\mathsf{d}_0 = 3d_1\ell$ (note that $\tau + \ell \leq \mathsf{d}_0$) and r_n. The claim then follows from Lemma 8.9 and (11.40). □

11.12. **Regularizing F_Ξ.** Similar to what was done in §11.7, we will define the set $\mathsf{A}^{\zeta_0}_{n+1}$ by decomposing F_Ξ (for $\Xi \in \mathsf{B}^{\zeta_0}_{n+1}$) into sets satisfying estimates similar to those in (6.7).

To that end, let $\Xi = (\zeta_0, \ldots, \zeta_n, r_n, \zeta_{n+1}) \in \mathsf{B}^{\zeta_0}_{n+1}$, and let $F = F_\Xi$, $y = y_\Xi$, $\mathcal{E} = \mathcal{E}_\Xi$. In view of Lemma 11.11(2) and Lemma 9.4,

$$\mathcal{G}_{F_w,\mathsf{R}_1}(w') \leq 10^6\Upsilon_{\Xi,n+1} \qquad \text{for every } w' \in F_w,$$

where $F_w = F \cap B_{\mathfrak{r}}(w, 4b\mathrm{inj}(y))$.

Let $k_1 > k_0$ be positive integers defined as follows:

$$2^{k_0} < (b\,\mathrm{inj}(y))^{-1} \leq 2^{k_0+1} \quad \text{and} \quad 2^{k_1} < 10^6\Upsilon_{\Xi,n+1} \leq 2^{k_1+1} \tag{11.52}$$

Let M be as above, see (11.2). Applying Lemma 6.4, we can write

$$F = F' \bigcup \big(\textstyle\bigcup_l F_l\big) \tag{11.53}$$

where $\#F' \leq \beta^{1/4}\cdot(\#F)$ and $\#F_l \geq \beta^2\cdot(\#F)$. In view of (11.41), we have

$$\begin{aligned} 0.5\beta^{11}\cdot(\#F_{\hat\Xi}) \leq \beta^2\cdot(\#F) &\leq \#F_l \\ &\leq \#F \leq 2\beta^8\cdot(\#F_{\hat\Xi}), \end{aligned} \tag{11.54}$$

and for every $k_0 - 10 \leq k \leq k_1$, there exists some $\tau_k = \tau_k^l$ so that

$$\text{either} \quad 2^{\mathsf{M}(\tau_k-2)} \leq \#F_l \cap Q \leq 2^{\mathsf{M}\tau_k} \quad \text{or} \quad F_l \cap Q = \emptyset, \tag{11.55}$$

for all $Q \in \mathcal{Q}_{\mathsf{M}k}$.

Let us also note that combining (11.54) and (11.36), we conclude

$$\frac{1}{2^{n+1}}\beta^{11(n+2)}e^t \leq \#F_l \leq 2^{n+1}\beta^{8(n+1)-3}e^t. \tag{11.56}$$

Let $\hat\Xi = (\zeta_0, \ldots, \zeta_n) \in \mathsf{A}^{\zeta_0}_n$, and let $r_n \in \mathcal{L}_{\mathcal{E}_{\hat\Xi}}$. We let $\mathcal{Z}_{\hat\Xi,r_n}$ denote an enumeration of

$$\{(\zeta', l) : \zeta' \in \mathcal{Z}'_{\hat\Xi,r_n}, l \in \mathcal{K}_\Xi\}$$

where for $\Xi = (\zeta_0, \ldots, \zeta_n, r_n, \zeta') \in \mathsf{B}^{\zeta_0}_{n+1}$, we let $\mathcal{K}_\Xi = \{l : F_l \text{ as in } (11.53)\}$. If $\zeta \in \mathcal{Z}_{\hat\Xi,r_n}$ corresponds to (ζ', l), then we put $y_\zeta = y_{\zeta'}$ and $F_\zeta = (F_{\zeta'})_l$, see (11.53) and the discussion leading to Lemma 11.10.

Define

$$\mathsf{A}_{n+1}^{\zeta_0} = \left\{ (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta_{n+1}) : \begin{array}{c} \hat{\Xi} = (\zeta_0, r_0, \ldots, \zeta_n) \in \mathsf{A}_n^{\zeta_0}, \\ r_n \in \mathcal{L}_{\mathcal{E}_{\hat{\Xi}}}, \zeta_{n+1} \in \mathcal{Z}_{\hat{\Xi}, r_n} \end{array} \right\}, \tag{11.57}$$

and for every $\Xi = (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta_{n+1}) \in \mathsf{A}_{n+1}^{\zeta_0}$, put

$$\mathcal{E}_\Xi = \mathsf{E}.\{\exp(w) y_\Xi : w \in F_\Xi\},$$

where $y_\Xi = y_{\zeta_{n+1}}$ and $F_\Xi = F_{\zeta_{n+1}}$.

11.13. **Lemma.** *Let $\Xi = (\zeta_0, r_0, , \ldots, \zeta_n, r_n, \zeta_{n+1}) \in \mathsf{A}_{n+1}^{\zeta_0}$. Suppose ζ_{n+1} corresponds to (ζ', l) as above, i.e., $\Xi' = (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta') \in \mathsf{B}_{n+1}^{\zeta_0}$ and $l \in \mathcal{K}_{\Xi'}$. Put $\Upsilon_{\Xi, n+1} = \Upsilon_{\Xi', n+1}$. Both of the following hold:*

(1) Let $\Xi = (\zeta_0, r_0, , \ldots, \zeta_n, r_n, \zeta_{n+1}) \in \mathsf{A}_{n+1}^{\zeta_0}$, then

$$f_{\mathcal{E}_\Xi, b, \mathsf{R}_{n+1}}(e, z) \leq \Upsilon_{\Xi, n+1}, \tag{11.58}$$

where $\mathsf{R}_{n+1} = 1 + (n+1) L \kappa^{-L} e^{\kappa D t}$.

(2) Let $\hat{\Xi} = (\zeta_0, r_0, \ldots, \zeta_n) \in \mathsf{A}_n^{\zeta_0}$ and let $r_n \in \mathcal{L}_{\mathcal{E}_{\hat{\Xi}}}$. Then for every $\varphi \in C_c^\infty(X)$, every $0 < \tau \leq 2 d_1 \ell$, and all $|s| \leq 2$, we have

$$\begin{aligned} \Big| \int \varphi(a_\tau u_s.z) \, \mathrm{d}(a_\ell u_{r_0} \mu_{\mathcal{E}_{\hat{\Xi}}})(z) - \sum c_\Xi \int \varphi(a_\tau u_s z) \, \mathrm{d}\mu_{\mathcal{E}_\Xi}(z) \Big| \\ \ll \max\{\eta^{1/2}, e^{-\kappa^2 t/64}\} \operatorname{Lip}(\varphi), \end{aligned} \tag{11.59}$$

where the sum is over $\mathcal{Z}_{\hat{\Xi}, r_n}$, and for every $\zeta \in \mathcal{Z}_{\hat{\Xi}, r_n}$, we let

$$\Xi = (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta).$$

Proof. Part (1) follows from Lemma 11.11(2) and the fact that $\mathcal{E}_\Xi \subset \mathcal{E}_{\Xi'}$.

As for part (2), we again use the above notation, i.e.,

$$\Xi = (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta).$$

where $\hat{\Xi} = (\zeta_0, r_0, \ldots, \zeta_n)$. Suppose ζ corresponds to (ζ', l) as above, that is, $\Xi' = (\zeta_0, r_0, \ldots, \zeta_n, r_n, \zeta') \in \mathsf{B}_{n+1}^{\zeta_0}$ and $l \in \mathcal{K}_{\Xi'}$. Then part (2) in the lemma follows from Lemma 11.11(3) in view of (11.53) if we put

$$c_\Xi = c_{\Xi'} \mu_{\mathcal{E}_{\Xi'}}(\mathcal{E}_\Xi)$$

and use the fact that $\mu_{\mathcal{E}_{\Xi'}}$ is admissible, see Lemma 8.8. □

12. Final sets and the proof of Proposition 10.1

We will complete the proof of Proposition 10.1 in this section. Let $\zeta_0 \in \mathcal{Z}$, see §11.1 in particular (11.14), and let $\mathsf{A}_n^{\zeta_0}$ be defined as in (11.57).

Recall that $0 < \varepsilon < 1$ is a small parameter (in our application, ε will depend on κ_7, see (13.1)) and $t > 1$ is a large parameter (which will be chosen to be $\asymp \log R$ where R is as in Theorem 1.1); let $b = e^{-\sqrt{\varepsilon} t}$. Recall also from Proposition 10.1 that we fixed

$$\kappa = 10^{-6} d_1^{-1} \leq 10^{-6} \varepsilon; \tag{12.1}$$

where $d_1 = 100\lceil (4D-3)/(2\varepsilon)\rceil$, see Proposition 10.1.

Set $\beta = e^{-\kappa t}$ and $\eta^2 = \beta$. Recall from (11.35) that

$$\mathsf{R}_n = 1 + nL\kappa^{-L}e^{\kappa D t}.$$

In particular, so long as t is large enough, we have

$$\mathsf{R}_{d_1} = 1 + d_1 L\kappa^{-L}e^{\kappa D t} \le e^{0.01\varepsilon t}. \tag{12.2}$$

Recall also our assumption that Proposition 4.8(1) holds, and that

$$x_2 = a_{8t}u_{r_1}x_1$$

where $r_1 \in I(x_1)$. Then $x_2 \in X_\eta$, and the map $h \mapsto hx_2$ is injective over $\mathsf{B}^{s,H}_\beta \cdot a_t \cdot U_1$, see Proposition 4.8(1).

Motivated by the conditions in (A-1) and (A-2) of Lemma 11.10, we make the following definition.

Definition 12.1. Let $d_2 := d_1 - \lceil \frac{10^4}{\sqrt{\varepsilon}}\rceil$ where $d_1 = 100\lceil\frac{4D-3}{2\varepsilon}\rceil$, and let

$$d_2 \le d \le d_1.$$

Let $\zeta_0 \in \mathcal{Z}$. An element $\Xi \in \mathsf{A}_d^{\zeta_0}$ is said to be *final* if

$$\Upsilon_{\Xi,d} < e^{\varepsilon t/2} \sup_{w\in F_\Xi} \psi_{\mathcal{E}_\Xi,b}(e, \exp(w)y_\Xi), \tag{12.3}$$

where $\mathcal{E}_\Xi = \mathsf{E}.\{\exp(w)y_\Xi : w \in F_\Xi\}$.

It will be more convenient to distinguish elements of $\mathsf{A}_d^{\zeta_0}$ satisfying (12.3) for $d < d_2$ as well. Thus, for every $0 \le d \le d_1$, let

$$\hat{\mathsf{A}}_d^{\zeta_0} = \big\{\Xi \in \mathsf{A}_d^{\zeta_0} : \Xi \text{ satisfies (12.3)}\big\}.$$

Note that if $d_2 \le d \le d_1$, then $\Xi \in \hat{\mathsf{A}}_d^{\zeta_0}$ if and only if it is final.

12.2. **Lemma.** *If $\Xi \in \hat{\mathsf{A}}_d^{\zeta_0}$, then*

$$f_{\mathcal{E}_\Xi,b,\mathsf{R}_d}(e,z) \le C_6 e^{\varepsilon t/2}\psi_{\mathcal{E}_\Xi,b}(e,z)$$

for all $z = h\exp(w)y_\Xi \in \mathcal{E}_\Xi$ with $h \in \overline{\mathsf{E}\setminus\partial_{10b}\mathsf{E}}$.

Proof. Let z be as in the statement. Then by Lemma 10.3, we have

$$\sup_w \psi_{\mathcal{E}_\Xi,b}(e,\exp(w)y) \le C_6\psi_{\mathcal{E}_\Xi,b}(e,z).$$

Moreover, by (11.25), we have

$$f_{\mathcal{E}_\Xi,b,\mathsf{R}_d}(e,z') \le \Upsilon_{\Xi,d}, \quad \text{for all } z' \in \mathcal{E}_\Xi.$$

The claim in the lemma follows from these, in view of (12.3). ☐

We fix the following notation: Let $0 \le d \le d_1$, for any

$$\Xi = (\zeta_0, r_0, \dots, \zeta_{d-1}, r_{d-1}, \zeta_d) \in \mathsf{A}_d^{\zeta_0},$$

and $0 \le n \le d$, put $\Xi_n := (\zeta_0, r_0, \dots, \zeta_n)$.

12.3. **Lemma.** *Let $\Xi \in \mathsf{A}^{\zeta_0}_{d_2}$, and let $d_2 \le d \le d_1$. Let $\Xi' \in \mathsf{A}^{\zeta_0}_{d}$ be so that $\Xi'_{d_2} = \Xi$. Then at least one of the following holds.*

(1) There exists $d_2 \le n \le d$ so that $\Xi'_n \in \hat{\mathsf{A}}^{\zeta_0}_n$.

(2) There exists $d < d' \le d_1$ and $\Xi'' \in \hat{\mathsf{A}}^{\zeta_0}_{d'}$ so that $\Xi''_d = \Xi'$.

In particular,

(3) For every $\Xi \in \mathsf{A}^{\zeta_0}_{d_2}$ and every $\Xi' \in \mathsf{A}^{\zeta_0}_{d_1}$ with $\Xi'_{d_2} = \Xi$, there exists $d_2 \le d \le d_1$ so that $\Xi'_d \in \hat{\mathsf{A}}^{\zeta_0}_d$.

Proof. First note that (3) is a direct consequence of (1)–(2). Thus it is enough to prove the latter.

For every $\Xi \in \mathsf{A}^{\zeta_0}_{d_2}$, put

$$\mathrm{past}(\Xi) = \left\{ n_i \le d_2 : \Xi_{n_i} \in \hat{\mathsf{A}}^{\zeta_0}_{n_i} \right\} \tag{12.4}$$

if such n_i exists, otherwise put $\mathrm{past}(\Xi) = \emptyset$; in the former case, we will write $\mathrm{past}(\Xi) = \{n_1 < \cdots < n_{m_\Xi}\}$. It follows from the definition (see (12.3)) that if $n \in \mathrm{past}(\Xi)$, then

$$\Upsilon_{\Xi_n, n} < e^{\varepsilon t/2} \sup_w \psi_{\mathcal{E}_{\Xi_n}, b}(e, \exp(w) y_{\Xi_n}).$$

Let d and $\Xi' \in \mathsf{A}^{\zeta_0}_d$ be as in the statement; note that for every $d \le d' \le d_1$, we have

$$\left\{ \Xi'' \in \mathsf{A}^{\xi_0}_{d'} : \Xi''_d = \Xi' \right\} \ne \emptyset;$$

see the discussion leading to (11.57).

We will consider two cases, $\mathrm{past}(\Xi) = \emptyset$ and $\mathrm{past}(\Xi) \ne \emptyset$, separately (though the argument in both cases is similar).

Case 1. Assume that $\mathrm{past}(\Xi) = \emptyset$.

Suppose that the claim in the lemma fails. Then for every $\Xi'' \in \mathsf{A}^{\xi_0}_{d_1}$ with $\Xi''_d = \Xi'$ and all $0 \le n \le d_1$ we have

$$\Upsilon_{\Xi''_n, n} \ge e^{\varepsilon t/2} \sup_w \psi_{\mathcal{E}_{\Xi''_n}, b}(e, \exp(w) y_{\Xi''_n}). \tag{12.5}$$

For $0 \le n \le d_2$, (12.5) follows from $\mathrm{past}(\Xi) = \emptyset$ and $\Xi''_{d_2} = \Xi$; for $d_2 \le n \le d_1$, it follows from the fact that $\Xi''_n \notin \hat{\mathsf{A}}^{\zeta_0}_n$, see (12.3).

We will show that (12.5) leads to a contradiction. To that end, put

$$\mathcal{E}'' = \mathcal{E}_{\Xi''} = \mathsf{E}.\{\exp(w) y : w \in F''\}.$$

Recall that $\ell = 0.01\varepsilon t$ and $d_1 = 100\lceil \frac{4D-3}{2\varepsilon} \rceil$. Thus $\frac{\ell d_1}{2} \ge \frac{(4D-3)t}{4}$ and

$$e^{-\ell d_1/2} e^{Dt} \le e^{-(4D-3)t/4} e^{Dt} \le e^{3t/4}. \tag{12.6}$$

In view of (12.5), we have (A-1) and (11.46) hold for all $0 \le n \le d_1$. That is $\Upsilon_{\Xi''_n, n} = e^{-\ell/2} \Upsilon_{\Xi''_{n-1}, n-1}$ for all $0 < n \le d_1$. Since $\Upsilon_0 = e^{Dt}$, we conclude from (12.6) that

$$\Upsilon_{\Xi'', d_1} = e^{-d_1 \ell/2} e^{Dt} \le e^{3t/4}. \tag{12.7}$$

We will compare (12.7) with a lower bound for $\psi_{\mathcal{E}'',b}$ which we now obtain. In view of (11.36), we have

$$\#F'' \geq (0.5)^{d_1}\beta^{11(d_1+1)}e^t.$$

This and (6.8) imply that for all $w \in F''$,

$$\psi_{\mathcal{E}'',b}(e,\exp(w)y) \geq e^{-4\sqrt{\varepsilon}t}(\#F'') \geq e^{-4\sqrt{\varepsilon}t}\beta^{11d_1+12}e^t \geq e^{0.9t} \tag{12.8}$$

where in the last inequality we used $\beta = e^{-\kappa t}$ and $100d_1\kappa \leq 0.01$, see (12.1).

We conclude from (12.7) and (12.8) that

$$\Upsilon_{\Xi'',d_1} \leq \sup_w \psi_{\mathcal{E}'',b}(e,\exp(w)y).$$

This contradicts $\Xi'' \notin \hat{\mathsf{A}}^{\zeta_0}_{d_1}$, and completes the proof in this case.

Case 2. Assume that $\mathrm{past}(\Xi) \neq \emptyset$.

Let us write $\mathrm{past}(\Xi) = \{n_1 < \cdots < n_{m_\Xi}\}$, and let Ξ' be as in the statement. We will write $n_m = n_{m_\Xi}$ for simplicity in the notation. Assume again that the claim in the lemma fails. First note that $n_m < d_2$ otherwise part (1) would hold with $n = d_2$, which contradicts our assumption. Similar to (12.5), for every $\Xi'' \in \mathsf{A}^{\xi_0}_{d_1}$ with $\Xi''_d = \Xi'$ and all $n_m < n \leq d_1$ we have

$$\Upsilon_{\Xi''_n,n} \geq e^{\varepsilon t/2}\sup_w \psi_{\mathcal{E}_{\Xi''_n},b}(e,\exp(w)y_{\Xi''_n}). \tag{12.9}$$

For $n_m < n \leq d_2$, this follows from $\mathrm{past}(\Xi) = \{n_1,\ldots,n_m\}$ and $\Xi''_{d_2} = \Xi$; for $d_2 \leq n \leq d_1$, it follows from our assumption that $\Xi''_n \notin \hat{\mathsf{A}}^{\zeta_0}_n$.

As in Case 1, we will show that (12.9) leads to a contradiction. Put

$$\mathcal{E}'' = \mathcal{E}_{\Xi''} = \mathsf{E}.\{\exp(w)y : w \in F''\}.$$

We will now inductively estimate $\Upsilon_{\Xi''_n,n}$ for $n_m < n \leq d_1$. Since $\Xi_{n_m} = \Xi''_{n_m} \in \hat{\mathsf{A}}^{\zeta_0}_{n_m}$ and $\Xi''_{n_m+1} \notin \hat{\mathsf{A}}^{\zeta_0}_{n_m+1}$ (see (12.4)), we conclude that (A-2) and (11.48) are used to define $\Upsilon_{\Xi''_{n_m+1},n_m+1}$. Thus there exists some $w_0 \in F_{\Xi_{n_m}}$ so that

$$\begin{aligned}\Upsilon_{\Xi''_{n_m+1},n_m+1} &= e^{-\ell/2}e^{\varepsilon t/2}\psi_{\mathcal{E}_{\Xi_{n_m}},b}\big(e,\exp(w_0)y_{\Xi_{n_m}}\big)\\ &\leq 2e^{-\ell/2}e^{\varepsilon t/2}\eta^{-\alpha}b^{-\alpha}\cdot\big(\#F_{\Xi_{n_m}}\big)\end{aligned} \tag{12.10}$$

where we used the definition of ψ in the last inequality.

We now turn to $\Upsilon_{\Xi''_n,n}$ for $n > n_m + 1$. In view of (12.9) applied for n and $n-1$, we have (A-1) and (11.46) hold. Thus

$$\Upsilon_{\Xi''_n,n} = e^{-\ell/2}\Upsilon_{\Xi''_{n-1},n-1} \quad \text{for all } n_m + 1 < n \leq d_1.$$

This and (12.10), imply that

$$\Upsilon_{\Xi'',d_1} \leq e^{-\ell(d_1-n_m)/2}\cdot\big(2e^{\varepsilon t/2}\eta^{-\alpha}b^{-\alpha}\big)\cdot\big(\#F_{\Xi_{n_m}}\big). \tag{12.11}$$

We will compare (12.11) with a lower bound for $\psi_{\mathcal{E}'',b}$ which we now obtain. In view of (11.54), we have

$$\#F'' \geq (0.5)^{d_1}\beta^{11(d_1-n_m)}\cdot(\#F_{\Xi_{n_m}}).$$

This and (6.8) imply that for all $w \in F''$,

$$\begin{aligned}\psi_{\mathcal{E}'',b}(e,\exp(w)y) &\geq e^{-4\sqrt{\varepsilon}t}(\#F'') \\ &\geq e^{-4\sqrt{\varepsilon}t}(0.5)^{d_1}\beta^{11(d_1-n_m)}\cdot(\#F_{\Xi_{n_m}}).\end{aligned} \tag{12.12}$$

Since $\Xi'' \notin \hat{\mathsf{A}}^{\zeta_0}_{d_1}$, we have

$$\Upsilon_{\Xi'',d_1} \geq e^{\varepsilon t/2}\sup_w \psi_{\mathcal{E}'',b}(e,\exp(w)y).$$

Combining this with (12.11) and (12.12), we conclude that

$$\begin{aligned}e^{-\frac{\ell(d_1-n_m)}{2}}\cdot\left(2e^{\frac{\varepsilon t}{2}}\eta^{-\alpha}b^{-\alpha}\right)\cdot\left(\#F_{\Xi_{n_m}}\right) &\geq \Upsilon_{\Xi'',d_1} \\ &\geq e^{\frac{\varepsilon t}{2}}\sup_w \psi_{\mathcal{E}'',b}(e,\exp(w)y) \\ &\geq e^{\frac{\varepsilon t}{2}}e^{-4\sqrt{\varepsilon}t}(0.5)^{d_1}\beta^{11(d_1-n_m)}\cdot(\#F_{\Xi_{n_m}}).\end{aligned}$$

Comparing the first and last terms, cancelling $\#F_{\Xi_{n_m}}$ and $e^{\varepsilon t/2}$ from both sides, and multiplying by $\beta^{-11(d_1-n_m)}$ and replacing 2^{d_1+1} by β^{-1},

$$e^{-\ell(d_1-n_m)/2}\beta^{-11(d_1-n_m)-1}\cdot\left(\eta^{-\alpha}b^{-\alpha}\right) \geq e^{-4\sqrt{\varepsilon}t}.$$

Recall now that $\beta = e^{-\kappa t}$, $0 < \kappa \leq \varepsilon/10^6$, see (12.1), and that $\ell = 0.01\varepsilon t$. Therefore,

$$e^{-\ell(d_1-n_m)/2}\beta^{-11(d_1-n_m)-1} \leq e^{-\ell(d_1-n_m)/3}$$

This and the above thus imply that

$$e^{-\ell(d_1-n_m)/3}\cdot\left(\eta^{-\alpha}b^{-\alpha}\right) \geq e^{-4\sqrt{\varepsilon}t}. \tag{12.13}$$

However, $\ell = 0.01\varepsilon t$ and $d_1 - n_m \geq d_1 - d_2 \geq 10^4/\sqrt{\varepsilon}$. Therefore, we have $\ell(d_1-n_m) \geq 100\sqrt{\varepsilon}t$. This, together with $\eta \geq e^{-\varepsilon t}$ and $b = e^{-\sqrt{\varepsilon}t}$, implies

$$e^{-\ell(d_1-n_m)/3}\cdot\left(\eta^{-\alpha}b^{-\alpha}\right) \leq e^{-30\sqrt{\varepsilon}t}$$

which contradicts (12.13) and finishes the proof in Case 2 as well. □

In view of this lemma, let $\hat{\mathsf{A}}^{\zeta_0}_{d_2,d_2} = \hat{\mathsf{A}}^{\zeta_0}_{d_2}$, and for every $d_2 < d \leq d_1$, let

$$\hat{\mathsf{A}}^{\zeta_0}_{d_2,d} = \left\{\Xi \in \hat{\mathsf{A}}^{\zeta_0}_{d} : \Xi_n \notin \hat{\mathsf{A}}^{\zeta_0}_{n} \text{ for any } d_2 \leq n < d\right\}.$$

Let $N^{\zeta_0}_d = \#\hat{\mathsf{A}}^{\zeta_0}_{d_2,d}$. For all d as above and all $1 \leq i \leq N^{\zeta_0}_d$, let $\mathcal{E}^i_d$ and $\mu_{\mathcal{E}^i_d}$ denote $\mathcal{E}_{\Xi^i_d}$ and $\mu_{\mathcal{E}_{\Xi^i_d}}$, respectively — we note that $\mathcal{E}^i_d$ and $\mu_{\mathcal{E}^i_d}$ also depend on ζ_0, however, this abuse of notation will not cause confusion in what follows.

12.4. **Lemma.** *For every $\varphi \in C^\infty_c(X)$, all $0 < \tau \leq d_1\ell$ and $|s| \leq 2$ we have*

$$\Big|\int \varphi(a_\tau u_s h x_2)\,\mathrm{d}\mu_{t,\ell,d_1}(h) - \sum_{\mathcal{Z}}\sum_{d,i} c_{d,i}\int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d)}_\ell * \mu_{\mathcal{E}^i_d}(z)\Big| \ll \operatorname{Lip}(\varphi)\beta^\star$$

where for every $\zeta_0 \in \mathcal{Z}$, the inner sum is over $d_2 \le d \le d_1$ and $1 \le i \le N_d^{\zeta_0}$, $c_{d,i} \ge 0$ with $\sum_{d,i} c_{d,i} = 1 - O(\beta^\star)$, $\mathrm{Lip}(\varphi)$ is the Lipschitz norm of φ, and the implied constants depend on X.

Proof. We will use the above notation also the notation from §11. Let

$$\{(\mathcal{E}_{\zeta_0}, \mu_{\zeta_0}) : \zeta_0 \in \mathcal{Z}\}$$

be as in (11.14). For every $\zeta_0 \in \mathcal{Z}$, let $\mathsf{A}_{d_2}^{\zeta_0}$ be as in (11.57). Then by part (2) in Lemma 11.13, for $0 < \tau' \le 2d_1\ell$, we have

$$\Big|\int \varphi(a_{\tau'} u_s h x_1)\,\mathrm{d}\mu_{t,\ell,d_2}(h) - \sum_{\zeta_0\in\mathcal{Z}} \sum_{\Xi\in\mathsf{A}_{d_2}^{\zeta_0}} c_\Xi \int \varphi(a_{\tau'} u_s z)\,\mathrm{d}\mu_{\mathcal{E}_\Xi}(z)\Big| \ll \max\big\{\eta^{1/2}, e^{-\kappa^2 t/64}\big\}\,\mathrm{Lip}(\varphi). \tag{12.14}$$

Recall that $a_{\ell_1} u_r a_{\ell_2} = a_{\ell_1+\ell_2} u_{e^{-\ell_2} r}$ for all $\ell_1, \ell_2, r \in \mathbb{R}$. Arguing as in Lemma 7.4, (12.14) (applied with $\tau' = \tau + (d_1 - d_2)\ell \le 2d_1\ell$) implies that

$$\Big|\int \varphi(a_\tau u_s h x_1)\,\mathrm{d}\mu_{t,\ell,d_1}(h) - \sum c_\Xi \int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d_2)} * \mu_{\mathcal{E}_\Xi}(z)\Big| \ll \max\big\{\eta^{1/2}, e^{-\kappa^2 t/64}\big\}\,\mathrm{Lip}(\varphi) \tag{12.15}$$

where $\sum = \sum_{\zeta_0\in\mathcal{Z}} \sum_{\Xi\in\mathsf{A}_{d_2}^{\zeta_0}}$.

Let $\zeta_0 \in \mathcal{Z}$ and let $\Xi \in \mathsf{A}_{d_2}^{\zeta_0}$. For every $d_2 \le d \le d_1$, put

$$\hat{\mathsf{A}}_{d_2,d}^{\zeta_0}(\Xi) = \{\Xi' \in \hat{\mathsf{A}}_{d_2,d}^{\zeta_0} : \Xi'_{d_2} = \Xi\};$$

note in particular that if $\Xi \in \hat{\mathsf{A}}_{d_2}^{\zeta_0}$, then $\hat{\mathsf{A}}_{d_2,d}^{\zeta_0}(\Xi) = \emptyset$ for all $d > d_2$.

We claim that

$$\Big|\int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d_2)} * \mu_{\mathcal{E}_\Xi} - \sum c_{\Xi'} \int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d)} * \mu_{\mathcal{E}_{\Xi'}}\Big| \ll \max\big\{\eta^{1/2}, e^{-\kappa^2 t/64}\big\}\,\mathrm{Lip}(\varphi) \tag{12.16}$$

where now $\sum = \sum_{d_2\le d\le d_1} \sum_{\hat{\mathsf{A}}_{d_2,d}^{\zeta_0}(\Xi)}$ and again $\sum c_{\Xi'} > 1 - O(\beta^\star)$.

Note that (12.16) and (12.15) finish the proof of the lemma. Thus, we need to prove (12.16).

As it was mentioned, if $\Xi \in \hat{\mathsf{A}}_{d_2}^{\zeta_0}$, then $\hat{\mathsf{A}}_{d_2,d}^{\zeta_0}(\Xi) = \emptyset$ for all $d > d_2$, and there is nothing to prove. Let now $\Xi \in \mathsf{A}_{d_2}^{\zeta_0} \setminus \hat{\mathsf{A}}_{d_2}^{\zeta_0}$. Then we have

$$\int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d_2)} * \mu_{\mathcal{E}_\Xi} = \int_0^1 \int \varphi(a_\tau u_s z)\,\mathrm{d}(\nu^{d_1-d_2-1} * (a_\ell u_r \mu_{\mathcal{E}_\Xi}))\,\mathrm{d}r.$$

Thus by Lemma 8.9 applied to the right side of the above, see also Lemma 11.13, we have

$$\Big|\int \varphi(a_\tau u_s z)\,\mathrm{d}\nu^{(d_1-d_2)} * \mu_{\mathcal{E}_\Xi} - \sum c_{\Xi'} \int \varphi((a_\tau u_s z))\,\mathrm{d}(\nu^{d_1-d_2-1} * \mu_{\mathcal{E}_{\Xi'}})\Big| \ll \eta^{1/2}\,\mathrm{Lip}(\varphi),$$

where the sum is over $\Xi' \in \mathsf{A}^{\zeta_0}_{d_2+1}$ with $\Xi'_{d_2} = \Xi$.

We now continue inductively, i.e., write

$$\{\Xi' \in \mathsf{A}^{\zeta_0}_{d_2+1} : \Xi'_{d_2} = \Xi\} = \hat{\mathsf{A}}^{\zeta_0}_{d_2,d_2+1}(\Xi) \cup \{\Xi' \in \mathsf{A}^{\zeta_0}_{d_2+1} : \Xi'_{d_2} = \Xi, \Xi' \notin \hat{\mathsf{A}}^{\zeta_0}_{d_2,d_2+1}(\Xi)\}$$

and decompose the sum $\sum_{\Xi'}$ accordingly. Repeat the above for all $\Xi' \in \mathsf{A}^{\zeta_0}_{d_2+1}$ with $\Xi'_{d_2} = \Xi$ but $\Xi' \notin \hat{\mathsf{A}}^{\zeta_0}_{d_2,d_2+1}(\Xi)$. In view of Lemma 12.3, this process terminates at some $d \le d_1$, and the claim in (12.16) follows. ☐

Proof of Proposition 10.1. Proposition 10.1 follows from Lemma 12.4, as we now explicate. The decomposition in Lemma 12.4 is of the form claimed in (10.3).

Moreover, the sets provided by Lemma 12.4 satisfy (10.1) in view of (11.55); they also satisfy (10.2) thanks to Lemma 12.2. In view of Lemma 8.3 and Lemma 8.8, the measures are $(\lambda_\bullet, M_\bullet)$-admissible with $M_\bullet$ depending only on X and the number of steps, which is $\le d_1$. Finally, in view of (12.2),

$$\mathsf{R}_d \le \mathsf{R}_{d_1} \le e^{0.01\varepsilon t}.$$

The proof is complete. ☐

13. From large dimension to equidistribution

Let $0 < \kappa_7 \le 1$ be the constant given by Proposition 5.2; recall that this constant is closely related to the spectral gap (or mixing rate) in G/Γ, c.f. (5.1). Throughout this section, we fix ε as follows

$$0 < \sqrt{\varepsilon} \le 10^{-8}\kappa_7. \tag{13.1}$$

We also recall that $\beta = e^{-\kappa t}$ and $\eta^2 = \beta$ where $0 < \kappa \le \varepsilon/10^6$.

The following is the main result of this section.

13.1. **Proposition.** *The following holds for all large enough t. Let $F \subset B_{\mathfrak{r}}(0,\beta)$ be a finite set with $\#F \ge e^{0.9t}$. Let*

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\} \subset X_\eta$$

be equipped with an admissible measure $\mu_{\mathcal{E}}$ (the definition is recalled below). Assume further that the following two properties are satisfied:

(1) For all $w \in F$, we have

$$\#\big(B_{\mathfrak{r}}(w, 4b\,\mathrm{inj}(y)) \cap F\big) \ge e^{-\varepsilon t} \sup_{w' \in F} \#\big(B_{\mathfrak{r}}(w', 4b\,\mathrm{inj}(y)) \cap F\big). \tag{13.2}$$

(2) For all $z = h\exp(w)y$ *with* $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$*, we have*

$$f_{\mathcal{E},b,\mathsf{R}}(e,z) \le e^{\varepsilon t}\psi_{\mathcal{E},b}(e,z) \tag{13.3}$$

where $\mathsf{R} \le e^{0.01\varepsilon t}$*,* $e^{-\sqrt{\varepsilon}t} \le b \le e^{-\sqrt{\varepsilon}t/2}$*, and* $\alpha = 1-\sqrt{\varepsilon}$*, see* §9*.*

Let $2\sqrt{\varepsilon}t \le \tau \le 0.01\kappa_7 t$*. Then*

$$\Big| \int_0^1 \int \varphi(a_\tau u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X \Big| \ll \mathcal{S}(\varphi)e^{-\varepsilon^2 t}$$

for all $\varphi \in C_c^\infty(X)$*.*

The proof, which is based on Proposition 5.2 and Theorem 6.2, or more precisely Theorem C.3, will be completed in several steps.

Let us first recall from §7.6 that a probability measure $\mu_{\mathcal{E}}$ on $\mathcal{E}$ is said to be (λ, M)-admissible if

$$\mu_{\mathcal{E}} = \frac{1}{\sum_{w\in F}\mu_w(X)}\sum_{w\in F}\mu_w$$

where for every $w \in F$, μ_w is a measure on $\mathsf{E}.\exp(w)y$ satisfying that

$$\mathrm{d}\mu_w(\mathsf{h}\exp(w)y) = \lambda\varrho_w(\mathsf{h})\,\mathrm{d}m_H(\mathsf{h}) \quad \text{where } 1/M \le \varrho_w(\cdot) \le M; \tag{13.4}$$

moreover, there is a subset $\mathsf{E}_w = \bigcup_{p=1}^M \mathsf{E}_{w,p} \subset \mathsf{E}$ so that

(1) $\mu_w\big((\mathsf{E}\setminus\mathsf{E}_w).\exp(w)y\big) \le M\beta\mu_w(\mathsf{E}.\exp(w)y)$,
(2) The complexity of $\mathsf{E}_{w,p}$ is bounded by M for all p, and
(3) $\mathrm{Lip}(\varrho_w|_{\mathsf{E}_{w,p}}) \le M$ for all p.

13.2. **Localizing the set** F**.** Recall that $F \subset B_{\mathfrak{r}}(0,\beta)$, and the set

$$\mathcal{E} = \mathsf{E}.\{\exp(w)y : w \in F\}$$

is equipped with a (λ, M)-admissible measure $\mu_{\mathcal{E}}$. In order to use Proposition 5.2, we need to *move* F to the direction of $\mathrm{Lie}(V) \subset \mathfrak{r}$, while controlling the errors in other directions. To facilitate this, we cover F with subsets contained in cubes of size $\asymp b\,\mathrm{inj}(y)$ — *localized* Margulis functions were considered in the improving the dimension phase, precisely for this reason.

Let $\bar\eta > 0$ be so that $\bar\eta/2 \le \mathrm{inj}(z) \le 2\bar\eta$ for all $z \in \mathcal{E}$, and that $\bar\eta b$ is a dyadic number. For every $v \in B_{\mathfrak{r}}(0,\beta)$, let $Q(v)$ be a cube with center v and size $4\bar\eta b$. Fix a covering $\{Q(v_i) : v_i \in F\}$ of F with multiplicity bounded by K (absolute).

Since $\#\{Q(v_i) : v_i \in F\} \ll (\bar\eta b)^{-3}$, (13.2) implies that for all i and j,

$$e^{-\varepsilon t}\cdot(\#(Q(v_j)\cap F)) \le \#(Q(v_i)\cap F) \le e^{\varepsilon t}\cdot(\#(Q(v_j)\cap F)) \quad \text{and} \tag{13.5a}$$

$$\#(Q(v_i)\cap F) \ge (\bar\eta b)^4\cdot(\#F) \tag{13.5b}$$

where we used $e^{-\sqrt{\varepsilon}t} \le b \le e^{-\sqrt{\varepsilon}t/2}$ and $\bar\eta \ge e^{-0.001\varepsilon t}$, and assumed t is large to account for implied multiplicative constants.

For every i, define $\rho_i : Q(v_i) \to \{1/j : j = 1, \ldots, K\}$ by

$$\rho_i(w) = (\#\{Q(v_j) : w \in Q(v_j)\})^{-1};$$

we extend ρ_i to $\mathfrak{r}$ by defining it to be zero outside $Q(v_i)$.

For every i, let $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y : w \in Q(v_i)\}$. Let

$$\mathrm{d}\mu_{\mathcal{E}_i}(\mathsf{h}\exp(w)y) = \rho_i(w)\,\mathrm{d}\mu_{\mathcal{E}}(\mathsf{h}\exp(w)y).$$

Then $\mu_{\mathcal{E}} = \sum_i \mu_{\mathcal{E}_i}$.

13.3. **A decomposition of the integral.** Recall that $\tau \geq 2\sqrt{\varepsilon}t$. Let $\ell_2 = |\log 128\bar{\eta}b|$ (then $\sqrt{\varepsilon}t/2 \leq \ell_2 \leq \sqrt{\varepsilon}t + \varepsilon t$) and let $\ell_1 = \tau - \ell_2$. Let $0 < \delta \leq 1$, and let $\varphi \in C_c^\infty(X)$. Then

$$\begin{aligned}(13.6)\quad \int_0^1 \int \varphi(a_\tau u_r z)\,\mathrm{d}\mu_{\mathcal{E}}(z)\,\mathrm{d}r = \\ \delta^{-1}\int_0^\delta \int_0^1 \int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\mu_{\mathcal{E}}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 + O(e^{-\ell_2}\,\mathrm{Lip}(\varphi))\end{aligned}$$

where the implied constant depends on X. Note that in the integral above r_1 runs over $[0, \delta]$ and r_2 over $[0, 1]$.

Thus we will investigate the first term on the right side of (13.6). Using the decomposition $\mu_{\mathcal{E}} = \sum \mu_{\mathcal{E}_i}$ and Fubini's theorem we have

$$\begin{aligned}(13.7)\quad \delta^{-1}\int_0^\delta \int_0^1 \int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\mu_{\mathcal{E}}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \delta^{-1}\int_0^\delta \int_0^1 \sum_i \int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 = \\ \sum_i \delta^{-1}\int_0^\delta \int_0^1 \int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\mu_{\mathcal{E}_i}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1.\end{aligned}$$

The following lemma will complete the proof of Proposition 13.1.

13.4. **Lemma.** *Fix some i, and let $\bar{\mu}_{\mathcal{E}_i} = \frac{1}{\mu_{\mathcal{E}_i}(\mathcal{E}_i)}\mu_{\mathcal{E}_i}$, i.e., the probability measure proportional to $\mu_{\mathcal{E}_i}$. Then*

$$\left|\delta^{-1}\int_0^\delta \int_0^1 \int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\bar{\mu}_{\mathcal{E}_i}(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 - \int \varphi\,\mathrm{d}m_X\right| \ll e^{-\varepsilon^2 t}\mathcal{S}(\varphi).$$

Proof. Recall that $\mathcal{E}_i = \mathsf{E}.\{\exp(w)y : w \in Q(v_i)\}$. Let $z_i = \exp(v_i)y$. It will be more convenient to replace y in the definition of $\mathcal{E}_i$ by z_i: Note that

$$\begin{aligned}(13.8)\quad \mathsf{h}\exp(w)y &= \mathsf{h}\exp(w)\exp(-v_i)\exp(v_i)y \\ &= \mathsf{h}\mathsf{h}_w\exp(v_w)z_i\end{aligned}$$

where $\|\mathsf{h}_w - I\| \ll b\beta$ and $\frac{1}{2}\|w - v_i\| \leq \|v_w\| \leq 2\|w - v_i\|$, see Lemma 3.2.

Note also that the map $w \mapsto v_w$ is one-to-one. Let $F_i = \{v_w : w \in Q(v_i)\}$ and let $\hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{20b}\mathsf{E}}$. Put

$$\hat{\mathcal{E}}_i := \hat{\mathsf{E}}.\{\exp(v)z_i : v \in F_i\}.$$

Then by (13.8) and since $\|\mathsf{h}_w - I\| \ll b\beta$, we have $\hat{\mathcal{E}}_i \subset \mathcal{E}_i$; moreover, $\bar{\mu}_{\mathcal{E}_i}(\mathcal{E}_i \setminus \hat{\mathcal{E}}_i) \ll b$. Thus it suffices to show the claim in the lemma with $\bar{\mu}_{\mathcal{E}_i}$ replaced by $\hat{\mu}_i := \frac{1}{\bar{\mu}_{\mathcal{E}_i}(\hat{\mathcal{E}}_i)}\bar{\mu}_{\mathcal{E}_i}|_{\hat{\mathcal{E}}_i}$.

For later reference, let us also record that (13.8) and $\|\mathsf{h}_w - I\| \ll b\beta$ implies also that in fact

$$\hat{\mathcal{E}}_i \subset \mathcal{E}' := \mathsf{E}'.\{\exp(w)y : w \in F\} \tag{13.9}$$

where $\mathsf{E}' = \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$. In particular, (13.3) holds true for all $z \in \hat{\mathcal{E}}_i$.

Recall that $\hat{\mu}_i$ is the probability measure proportional to $\sum_w \hat{\mu}_{i,w}$ where $\mathrm{d}\hat{\mu}_{i,w} = \hat{\rho}_{i,w}\,\mathrm{d}m_H$ and $(KM)^{-1} \le \hat{\rho}_{i,w} \le M$. We will use Fubini's theorem to change the order of disintegration of $\hat{\mu}_i$ as follows. Let $z \in \hat{\mathcal{E}}_i$, then

$$z = \mathsf{h}\exp(v)z_i = \exp(\mathrm{Ad}(\mathsf{h})v)\mathsf{h}z_i \in \hat{\mathcal{E}}_i.$$

Moreover, $\mathrm{Ad}(\mathsf{h})v \in B_{\mathfrak{r}}(0, 8\bar{\eta}b)$. Since $\bar{\eta}/2 \le \mathrm{inj}(z') \le 2\bar{\eta}$ for every $z' \in \mathcal{E}_i$, we conclude that

$$\mathrm{Ad}(\mathsf{h})v \in I_{\mathcal{E}_i,32b}(e, \mathsf{h}z_i).$$

Let $\pi : \hat{\mathcal{E}}_i \to \mathsf{E}.z_i$ denote the projection $z = \mathsf{h}\exp(v)z_i \mapsto \mathsf{h}z_i$. Using Fubini's theorem, we have

$$\hat{\mu}_i = \int \hat{\mu}_i^{\mathsf{h}}\,\mathrm{d}\pi_*\hat{\mu}_i(\mathsf{h}.z_i),$$

where $\hat{\mu}_i^{\mathsf{h}}$ denotes the conditional measure of $\hat{\mu}_i$ for the factor map π. Note that $\hat{\mu}_i^{\mathsf{h}}$ is supported on $\{\exp(w)\mathsf{h}z_i : w \in I_{\mathcal{E}_i,32b}(e, \mathsf{h}z_i)\}$. In view of the above discussion, $\mathrm{d}\pi_*\hat{\mu}_i$ is proportional to $\hat{\rho}\,\mathrm{d}m_H$ restricted to the support of $\pi_*\hat{\mu}_i$ where $1 \ll \hat{\rho} \ll 1$, moreover, for every i, and every $w \in \mathrm{supp}(\hat{\mu}_i^{\mathsf{h}})$,

$$\hat{\mu}_i^{\mathsf{h}}(w) \asymp (\#F_i)^{-1} \tag{13.10}$$

where the implied constant depends on K and M.

Now, using Fubini's theorem we have

$$\delta^{-1}\int_0^\delta\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}z)\,\mathrm{d}\hat{\mu}_i(z)\,\mathrm{d}r_2\,\mathrm{d}r_1 =$$
$$\delta^{-1}\int_{\hat{\mathsf{E}}.z_i}\int_0^\delta\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z_i)\,\mathrm{d}\hat{\mu}_i^{\mathsf{h}}(w)\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}\pi_*\hat{\mu}_i(\mathsf{h}.z_i).$$

Fix some i and $\mathsf{h} \in \hat{\mathsf{E}} = \overline{\mathsf{E} \setminus \partial_{20b}\mathsf{E}}$. We will investigate

$$\delta^{-1}\int_0^\delta\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z_i)\,\mathrm{d}\hat{\mu}_i^{\mathsf{h}}(w)\,\mathrm{d}r_2\,\mathrm{d}r_1. \tag{13.11}$$

Discretized dimension of $\hat{\mu}_i^{\mathsf{h}}$. Let us put

$$F_i^{\mathsf{h}} := \mathrm{supp}(\hat{\mu}_i^{\mathsf{h}}) = \{\mathrm{Ad}(\mathsf{h})v : v \in F_i\}.$$

Moreover, recall from (13.9) that $\exp(\mathrm{Ad}(\mathsf{h})v)\mathsf{h}z_i = \mathsf{h}\exp(v)z_i \in \hat{\mathcal{E}}_i \subset \mathcal{E}'$. Since $\|v\| \le 4\bar{\eta}b$, for every $v \in F_i$, we conclude that

$$F_i^{\mathsf{h}} \subset I_{\mathcal{E}',32b}(e, \mathsf{h}z_i). \tag{13.12}$$

Furthermore, by (13.5b) and since $\#F \geq e^{0.9t}$, we have

$$F_i^{\mathsf{h}} = \#F_i = \#(Q(v_i) \cap F) \geq (\bar{\eta}b)^4 \cdot (\#F) \geq e^{0.8t}. \tag{13.13}$$

Recall now that

$$f_{\mathcal{E},b,\mathsf{R}}(e, z') \leq e^{\varepsilon t}\psi_{\mathcal{E},b}(e, z') \leq e^{\varepsilon t} \sup_{z''\in\mathcal{E}} \psi_{\mathcal{E},b}(e, z'') \tag{13.14}$$

for all $z' \in \mathcal{E}'$, where we used (13.3) to get the first bound.

Apply Lemma 9.2 with $\Upsilon = e^{\varepsilon t}\sup_{z'\in\mathcal{E}} \psi_{\mathcal{E},b}(e, z')$, $z = \mathsf{h}z_i$, and $I_{\mathsf{h}z_i} = I_{\mathcal{E}',32b}(e, \mathsf{h}z_i)$. We thus conclude that

$$\mathcal{G}_{I_{\mathsf{h}z_i},\mathsf{R}}(w) \ll \Upsilon \qquad \text{for every } w \in I_{\mathsf{h}z_i}. \tag{13.15}$$

Moreover, by (13.5a) and Lemma 10.2, we have

$$\begin{aligned} F_i^{\mathsf{h}} = \#F_i = \#(Q(v_i) \cap F) &\gg e^{-\varepsilon t} \sup_{z'} \#I_{\mathcal{E},b}(e, z') \\ &= e^{-\varepsilon t} \sup_{z'} \big((\mathrm{inj}(z')b)^\alpha \psi_{\mathcal{E},b}(e, z')\big) \\ &\gg e^{-2\varepsilon t}(\bar{\eta}b)^\alpha \Upsilon, \end{aligned}$$

where we also used the definition of Υ in the last inequality.

Recall that $\mathsf{R} \leq e^{0.01\varepsilon t}$. Therefore, (13.12), (13.13), and (13.15), in view of the above, imply that

$$\mathcal{G}_{F_i^{\mathsf{h}},\mathsf{R}}(w) \ll \Upsilon \ll e^{2\varepsilon t}(\bar{\eta}b)^{-\alpha} \cdot (\#F_i^{\mathsf{h}}) \qquad \text{for every } w \in F_i^{\mathsf{h}}.$$

Using $\mathsf{R} \leq e^{0.01\varepsilon t}$ and (13.13) again, we conclude that

$$\sigma_i^{\mathsf{h}}(B(w, b')) \ll e^{2\varepsilon t}(b'/\bar{\eta}b)^\alpha \quad \text{for all } b' \geq (\#F_i^{\mathsf{h}})^{-1},$$

where σ_i^{h} is the uniform measure on F_i^{h}. This and (13.10) imply that

$$\hat{\mu}_i^{\mathsf{h}}(B(w, b')) \ll e^{2\varepsilon t}(b'/\bar{\eta}b)^\alpha \quad \text{for all } b' \geq (\#F_i^{\mathsf{h}})^{-1}, \tag{13.16}$$

where the implied constant depends only on M and K.

Projecting the dimension. Recall that $0 < \kappa_7 \leq 1$, we have

$$2\sqrt{\varepsilon}t \leq \tau \leq 0.01\kappa_7 t \leq 0.01t.$$

For every $r \in [0, 1]$ and $w \in B_{\mathfrak{r}}(0, 128\bar{\eta}b)$, write

$$\exp(\mathrm{Ad}(u_r)w) = \begin{pmatrix} d_{r,w} & 0 \\ c_{r,w} & 1/d_{r,w} \end{pmatrix} \begin{pmatrix} 1 & \xi_r(w) \\ 0 & 1 \end{pmatrix} \tag{13.17}$$

where $|d_{r,w} - 1|, |c_{r,w}| \ll e^{-\ell_2}$.

In view of (13.16), we may apply Theorem C.3 with F_i^{h}, $b_1 = e^{-\ell_2} = 128\bar{\eta}b$, $b_0 = (\#F_i^{\mathsf{h}})^{-1}$, $\hat{\mu}_i^{\mathsf{h}}$, ε, and

$$b' = e^{-3\ell_1-\ell_2} \geq e^{-4\tau} \geq e^{-0.04t} \geq (\#F_i^{\mathsf{h}})^{-1},$$

where we used $\tau \leq 0.01t$ and (13.13).

Let $J_{b'} \subset [0,1]$ and $\Theta_{b',r_2} \subset F_i^{\mathsf{h}}$ (for every $r_2 \in J_{b'}$) be as in Theorem C.3. Set $J^{\mathsf{h}} := J_{b'}$. Let $\bar\mu^{\mathsf{h}}_{i,r_2}$ denote the projection of $\hat\mu_i^{\mathsf{h}}|_{\Theta_{b',r_2}}$ under the map $w \mapsto \xi_{r_2}(w)$. Then, by Theorem C.3, we have

$$\bar\mu^{h}_{i,r_2}(I) \le L\varepsilon^{-L} e^{2\varepsilon n}(b'/\bar\eta b)^{\alpha-7\varepsilon} \tag{13.18}$$

for every interval I of length b' where L is absolute.

Moreover, $\big|[0,1]\setminus J^{\mathsf{h}}\big| \le L\varepsilon^{-L} b'^{\varepsilon}$ which is $\le L\varepsilon^{-L} e^{-\varepsilon^{3/2}t}$ since $b' < e^{-2\sqrt{\varepsilon}t}$. Thus for any $r_1 \in [0,\delta]$

$$\begin{aligned}\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z_i)\,\mathrm{d}\hat\mu_i^{\mathsf{h}}(w)\,\mathrm{d}r_2 &= \\ \int_{J^h}\int \varphi\big(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z_i\big)\,\mathrm{d}\hat\mu_i^{\mathsf{h}}(w)\,\mathrm{d}r_2 &+ O\big(\mathcal{S}(\varphi)L\varepsilon^{-L}e^{-\varepsilon^{3/2}t}\big).\end{aligned} \tag{13.19}$$

Approximating orbits using the projection ξ_{r_2}. In view of (13.19), we need to investigate the contribution of the first term on the right side of (13.19) to (13.11). We begin by fixing the size of δ and some algebraic considerations.

Recall that $\sqrt{\varepsilon}t/2 \le \ell_2 \le \sqrt{\varepsilon}t + \varepsilon t$ and $\ell_1 = \tau - \ell_2 \ge \sqrt{\varepsilon}t - \varepsilon t$. Define $0 < \delta \le 1$ by the following equation

$$e^{\ell_1}\delta = e^{\sqrt{\varepsilon}t/4} \le e^{\ell_2/2}. \tag{13.20}$$

For any $r_2 \in [0,1]$, put $z^{\mathsf{h}}_{i,r_2} = a_{\ell_2}u_{r_2}\mathsf{h}z_i$. Using (13.17) and (13.12), for any $w \in F_i^{\mathsf{h}}$ and all $r_1 \in [0,\delta]$, we have

$$\begin{aligned}a_{\ell_1}u_{r_1}\exp(\mathrm{Ad}(a_{\ell_2}u_{r_2})w)z^{\mathsf{h}}_{i,r_2} &= \\ &a_{\ell_1}u_{r_1}\begin{pmatrix} d_{r_2,w} & 0 \\ e^{-\ell_2}c_{r_2,w} & 1/d_{r_2,w}\end{pmatrix}\begin{pmatrix} 1 & e^{\ell_2}\xi_{r_2}(w) \\ 0 & 1\end{pmatrix} z^{\mathsf{h}}_{i,r_2}\end{aligned}$$

where $|c_{r_2,w}|, |d_{r_2,w}-1| \ll e^{-\ell_2}$. From this, we conclude that

$$a_{\ell_1}u_{r_1}\exp(\mathrm{Ad}(a_{\ell_2}u_{r_2})w)z^{\mathsf{h}}_{i,r_2} = g a_{\ell_1}u_{r_1}\begin{pmatrix} 1 & e^{\ell_2}\xi_{r_2}(w) \\ 0 & 1\end{pmatrix} z^{\mathsf{h}}_{i,r_2} \tag{13.21}$$

where $\|g - I\| \ll e^{\ell_1}\delta e^{-\ell_2} \ll e^{-\ell_2/2} \le e^{-\sqrt{\varepsilon}t/4}$, see (13.20).

Applying Proposition 5.2. Fix $r_2 \in J^{\mathsf{h}}$. Let $\hat\mu^{\mathsf{h}}_{i,r_2}$ denote the image of $\bar\mu^{\mathsf{h}}_{i,r_2}$ under the map $s \mapsto e^{\ell_2}s$. In view of (13.21) and the fact that $\hat\mu_i^{\mathsf{h}}(F_i^{\mathsf{h}} \setminus \Theta_{b',r_2}) \le L\varepsilon^{-L}e^{-\varepsilon^{3/2}t}$ we have

$$\begin{aligned}\delta^{-1}\int_0^\delta\int \varphi\big(a_{\ell_1}u_{r_1}\exp(\mathrm{Ad}(a_{\ell_2}u_{r_2})w)z^{\mathsf{h}}_{i,r_2}\big)\,\mathrm{d}\hat\mu_i^{\mathsf{h}}(w)\,\mathrm{d}r_1 &= \\ \delta^{-1}\int_0^\delta\int \varphi\big(a_{\ell_1}u_{r_1}v_s z^{\mathsf{h}}_{i,r_2}\big)\,\mathrm{d}\hat\mu^{\mathsf{h}}_{i,r_2}(s)\,\mathrm{d}r_1 &+ O\big(\mathcal{S}(\varphi)L\varepsilon^{-L}e^{-\varepsilon^{3/2}t}\big).\end{aligned}$$

Recall that $\alpha = 1-\sqrt{\varepsilon}$. By (13.18), the measure $\hat\mu^{\mathsf{h}}_{i,r_2}$ satisfies the condition (5.2) in Proposition 5.2 for

$$\theta = \sqrt{\varepsilon}+7\varepsilon, \qquad \mathfrak{b} = e^{-3\ell_1}, \quad \text{and} \quad C = L\varepsilon^{-L}e^{2\varepsilon t}.$$

Apply Proposition 5.2 for $t=\ell_1$, and the above chosen δ; note that $|\log \mathfrak{b}|/4 \le t = \ell_1 \le |\log \mathfrak{b}|/2$ so that in particular (5.3) holds. Then as $\mathfrak{b}^{1/2} \le e^{-\ell_1}$ the first term in the right hand side of (5.4) dominates and

$$\text{(13.22)} \quad \Big|\delta^{-1}\int_0^\delta\int \varphi\big(a_{\ell_1}u_{r_1}v_s z^{\mathsf{h}}_{i,r_2}\big)\,\mathrm{d}\hat\mu^{\mathsf{h}}_{i,r_2}(s)\,\mathrm{d}r_1 - \int\varphi\,\mathrm{d}m_X\Big| \\ \ll \mathcal{S}(\varphi)\big(L\varepsilon^{-L}e^{2\varepsilon t}e^{3(\sqrt{\varepsilon}+7\varepsilon)\ell_1}\big)^{1/2}(e^{\ell_1}\delta)^{-\kappa_7}.$$

Recall that $\ell_1 \le \tau \le 0.01\kappa_7 t$. Therefore,

$$e^{3\sqrt{\varepsilon}\ell_1} \le e^{0.03\kappa_7\sqrt{\varepsilon}t}.$$

Moreover, $\ell_1 \le \tau \le 0.01t$, hence $21\varepsilon\ell_1 \le \varepsilon t$, and using (13.1) we get

$$3\varepsilon = 3(\sqrt{\varepsilon})^2 \le 0.01\kappa_7\sqrt{\varepsilon}.$$

Thus, $e^{2\varepsilon t}\cdot e^{21\varepsilon\ell_1} \le e^{3\varepsilon t} \le e^{0.01\kappa_7\sqrt{\varepsilon}t}$. Altogether, we conclude that

$$e^{2\varepsilon t}e^{3(\sqrt{\varepsilon}+7\varepsilon)\ell_1} \le e^{0.04\kappa_7\sqrt{\varepsilon}t}.$$

Since $e^{\ell_1}\delta = e^{\sqrt{\varepsilon}t/4}$. The above implies that the right side of (13.22) is

$$\ll \mathcal{S}(\varphi)L\varepsilon^{-L}e^{-\kappa_7\sqrt{\varepsilon}t/5} \ll \mathcal{S}(\varphi)L\varepsilon^{-L}e^{-\varepsilon t}$$

where in the second inequality is a consequence of (13.1).

Choosing t large enough so that $L\varepsilon^{-L}e^{-\varepsilon^{3/2}t} \le e^{-\varepsilon^2 t}$, we conclude that

$$\Big|\delta^{-1}\int_0^\delta\int_0^1\int \varphi(a_{\ell_1}u_{r_1}a_{\ell_2}u_{r_2}\exp(w)\mathsf{h}z_i)\,\mathrm{d}\hat\mu^{\mathsf{h}}_i(w)\,\mathrm{d}r_2\,\mathrm{d}r_1 - \int\varphi\,\mathrm{d}m_X\Big| \\ \ll \mathcal{S}(\varphi)e^{-\varepsilon^2 t}.$$

The proof is complete. □

Proof of Proposition 13.1. In view of (13.6) and (13.7), the proposition follows from Lemma 13.4. □

14. Proof of Theorem 1.1

The proof will be completed in some steps and it is based on various propositions which were discussed so far.

Fixing the parameters. Fix ε as follows

$$0 < \sqrt{\varepsilon} < 10^{-8}\kappa_7 \tag{14.1}$$

where κ_7 is as in Proposition 5.2.

Let $D = D_0D_1 + 2D_1$ where D_0 is as in Proposition 4.6 and D_1 is as in Proposition 4.8; we will always assume $D_1, D_0 \geq 10$. We will show the claim holds with

$$A = 15 + 2D_0.$$

Let us assume (as we may) that

$$R \geq \max\{(10C_4)^3 \mathrm{inj}(x_0)^{-2}, e^{C_4}, e^{s_0}, C_1\}, \tag{14.2}$$

see Proposition 4.3 and Proposition 4.6. Let $T \geq R^A$, and suppose that Theorem 1.1(2) does not hold with this A. That is, for every $x \in X$ so that Hx is periodic with $\mathrm{vol}(Hx) \leq R$,

$$d_X(x, x_0) > R^A(\log T)^A T^{-1} \geq (\log S)^{D_0} S^{-1} \tag{14.3}$$

where $S := R^{-A}T$.

Since $D_0, D_1 \geq 10$, we have

$$A = 15 + 2D_0 \geq 10 + (10 + 2D)D_1^{-1} \geq 10 + (\tfrac{5}{2}\sqrt{\varepsilon} + 9 + \tfrac{4D-3}{2})D_1^{-1}.$$

Therefore,

$$\begin{aligned} \log T - \big((\tfrac{5}{2}\sqrt{\varepsilon} + 9 + \tfrac{4D-3}{2})D_1^{-1}\big)\log R &\geq \log T - A\log R + 10\log R \\ &\geq \log S + 2|\log \mathrm{inj}(x_0)| + 8\log R \end{aligned} \tag{14.4}$$

we used $R \geq \mathrm{inj}(x_0)^{-2}$ and $\log S = \log T - A\log R$ in the last inequality.

Let $t = \frac{1}{D_1}\log R$, $\ell = \varepsilon t/100$, and $d_1 = 100\lceil \frac{4D-3}{2\varepsilon}\rceil$. Then

$$\tfrac{4D-3}{2}t \leq d_1\ell \leq \tfrac{4D-3}{2}t + \varepsilon t. \tag{14.5}$$

As it was done in (12.1), fix

$$0 < \kappa < \min\{10^{-6}d_1^{-1}, 10^{-6}\varepsilon\}.$$

Let $\beta = e^{-\kappa t}$ and let $\eta = \beta^{1/2}$; note that $\eta \geq e^{-0.1\ell}$.

Let us write $\log T = t_3 + t_2 + t_1 + t_0$ where

$$\begin{aligned} &t_0 = \log T - ((\tfrac{5}{2}\sqrt{\varepsilon} + 9 + \tfrac{4D-3}{2})D_1^{-1})\log R \\ &t_1 = 8t, \quad \text{and} \quad t_2 = t + d_1\ell. \end{aligned} \tag{14.6}$$

Note that $t_0, t_1, t_2 \geq t$ (see (14.4) for $t_0 > t$). We now estimate t_3; indeed

$$\begin{aligned} t_3 &= \log T - (t_0 + t_1 + t_2) \\ &= (\tfrac{5}{2}\sqrt{\varepsilon} + 9 + \tfrac{4D-3}{2})D_1^{-1}\log R - 9t - d_1\ell \\ &= (\tfrac{5}{2}\sqrt{\varepsilon} + 9 + \tfrac{4D-3}{2})t - 9t - d_1\ell \end{aligned}$$

where we used $t = \frac{1}{D_1}\log R$ in the last equation. This and (14.5) imply

$$2\sqrt{\varepsilon}t \leq t_3 \leq 3\sqrt{\varepsilon}t. \tag{14.7}$$

Recall that $a_{\ell_1} u_r a_{\ell_2} = a_{\ell_1+\ell_2} u_{e^{-\ell_2} r}$. Thus, for any $\varphi \in C_c^\infty(X)$, we have

$$\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty e^{-t}) + \int_0^1\int_0^1\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}a_{t_1}u_{r_1}a_{t_0}u_{r_0}x_0)\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0 \tag{14.8}$$

where the implied constant is absolute and we used $t_0, t_1, t_2 \geq t$.

Improving the Diophantine condition. Apply Proposition 4.6 with $S = R^{-A}T$, then for all

$$\tau \geq \max\{\log S, 2|\log \mathrm{inj}(x_0)|\} + s_0,$$

we have the following

$$\left|\left\{r \in [0,1] : \begin{array}{c} a_\tau u_r x_0 \notin X_\eta \text{ or } \exists\, x \text{ with } \mathrm{vol}(Hx) \leq R \\ \text{so that } d_X(x, a_\tau u_r x_0) \leq R^{-D_0-1} \end{array}\right\}\right| \ll \eta^{1/2}, \tag{14.9}$$

where we also used $\eta^{1/2} \geq R^{-1}$ and $R \geq C_1$.

Let $J_0 \subset [0,1]$ be the set of those $r_0 \in [0,1]$ so that $a_{t_0}u_{r_0}x_0 \in X_\eta$ and

$$d_X(x, a_{t_0}u_{r_0}x_0) > R^{-D_0-1} = e^{-D_1(D_0+1)t}$$

for all x with $\mathrm{vol}(Hx) \leq R = e^{D_1 t}$. Then since by (14.4) and (14.2) we have

$$t_0 \geq \log S + 2|\log \mathrm{inj}(x_0)| + 8\log R \geq \max\{\log S, 2|\log \mathrm{inj}(x_0)|\} + s_0,$$

the assertion in (14.9) implies that $|[0,1] \setminus J_0| \ll \eta^{1/2}$. In consequence,

$$\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty \eta^{1/2}) + \int_{J_0}\int_0^1\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}a_{t_1}u_{r_1}x(r_0))\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0 \tag{14.10}$$

where $x(r_0) = a_{t_0}u_{r_0}x_0$ and the implied constant depends on X.

Applying the closing lemma. For every $r_0 \in J_0$, we now apply Proposition 4.8 with $x(r_0)$, $D = D_0D_1 + 2D_1$ and the parameter t. For any such r_0, we have

$$d_X(x, x(r_0)) > e^{-D_1(D_0+1)t} = e^{(-D+D_1)t}$$

for all x with $\mathrm{vol}(Hx) \leq e^{D_1 t}$. Thus Proposition 4.8(1) holds. Let

$$J_1(r_0) = I(x(r_0)) = I(a_{t_0}u_{r_0}x_0)$$

Then

$$\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r = O(\|\varphi\|_\infty \eta^{1/2}) + \int_{J_0}\int_{J_1(r_0)}\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}x(r_0,r_1))\,\mathrm{d}r_3\,\mathrm{d}r_2\,\mathrm{d}r_1\,\mathrm{d}r_0 \tag{14.11}$$

where $x(r_0, r_1) = a_{t_1}u_{r_1}a_{t_0}u_{r_0}x_0$ and the implied constant is absolute.

Improving the dimension phase. Fix some $r_0 \in J_0$, and let $r_1 \in J(r_0)$. Put $x_1 = x(r_0, r_1)$. Recall from (8.10) that

$$\mu_{t,\ell,d_1} = \nu_\ell * \cdots * \nu_\ell * \sigma * \nu_t$$

where ν_ℓ appears d_1 times in the above expression. In view of Lemma 7.4,

$$\Big|\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}x_1)\,\mathrm{d}r_3\,\mathrm{d}r_2 - \iint_0^1 \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}r_3\mu_{t,\ell,d_1}(h)\Big| \ll \mathrm{Lip}(\varphi)e^{-\ell} \ll \mathrm{Lip}(\varphi)\eta^{1/2}. \tag{14.12}$$

We now apply Proposition 10.1 with x_1, t_3 and $r_3 \in [0,1]$. Then

$$\int_0^1\int \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}\mu_{t,\ell,d_1}(h)\,\mathrm{d}r_3 = \sum_{d,i} c_{d,i}\int_0^1\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r_3 + O(\mathrm{Lip}(\varphi)\beta^{\kappa_4}) \tag{14.13}$$

where the sum is over

$$d_1 - \lceil 10^4\varepsilon^{-1/2}\rceil = d_2 \le d \le d_1,$$

$c_{d,i} \ge 0$ and $\sum_{d,i} c_{d,i} = 1 - O(\beta^{\kappa_4})$ and the implied constants depend on X. Moreover, for all d, i both of the following hold

$$\#\big(B_{\mathfrak{r}}(w, 4b\,\mathrm{inj}(y)) \cap F_{d,i}\big) \ge e^{-\varepsilon t}\sup_{w'\in F_{d,i}} \#\big(B_{\mathfrak{r}}(w', 4b\,\mathrm{inj}(y)) \cap F_{d,i}\big) \tag{14.14a}$$

$$f_{\mathcal{E}_{d,i},b,\mathsf{R}}(e,z) \le e^{\varepsilon t}\psi_{\mathcal{E}_{d,i},b}(e,z) \quad \text{where } \mathsf{R} \le e^{0.01\varepsilon t} \tag{14.14b}$$

for all $w \in F_{d,i}$ and all $z = h\exp(w)y_{d,i} \in \mathcal{E}_{d,i}$ with $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$.

From large dimension to equidistribution. For every $d_2 \le d \le d_1$, set

$$\tau_d := t_3 + (d_1 - d)\ell.$$

Since $0 \le d_1 - d \le \lceil 10^4\varepsilon^{-1/2}\rceil$, $\ell = 0.01\varepsilon t$, and $2\sqrt{\varepsilon}t \le t_3 \le 3\sqrt{\varepsilon}t$, see (14.7),

$$2\sqrt{\varepsilon}t \le \tau_d \le (4 + 10^2)\sqrt{\varepsilon}t \le 0.01\kappa_7 t \tag{14.15}$$

where in the last inequality we used $0 < \sqrt{\varepsilon} < 10^{-8}\kappa_7$, see (14.1).

In view of Lemma 7.4, for all d, i as above, we have

$$\int_0^1\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r_3 = \int_0^1\int \varphi(a_{\tau_d}u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r + O(\mathrm{Lip}(\varphi)e^{-\ell}) \tag{14.16}$$

where the implied constant depends on X.

We now apply Proposition 13.1 with $\mathcal{E}_{d,i}$ (in view of (14.14a) and (14.14b) the conditions in that proposition are satisfied) and τ_d which is in the admissible range thanks to (14.15). Hence, for all d, i as above, we have

$$\Big| \int_0^1 \int \varphi(a_{\tau_d} u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X \Big| \ll \mathcal{S}(\varphi) e^{-\varepsilon^2 t} \tag{14.17}$$

where the implied constant depends on X.

Let $\kappa_1 = \min\{\varepsilon^2, \kappa_4\kappa, \kappa/4\}$. Then (14.17), (14.16), (14.13), (14.12), (14.11), (14.10), and (14.8), imply that

$$\Big| \int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X \Big| \ll \mathcal{S}(\varphi) e^{-\kappa_1 t} \ll \mathcal{S}(\varphi) R^{-\kappa_1/D_1}$$

where the implied constant depends on X. The proof is complete. □

15. Proof of Theorem 1.3

The argument is similar to the proof of Theorem 1.1, the main difference here is that even though Proposition 4.6 holds without the arithmeticity assumption on Γ, its output, i.e., points which are not near periodic H-orbits, is too weak for our closing lemma, in the absence of arithmeticity. Indeed the assertion (2') in §4.7 only guarantees that if Proposition 4.8(1) fails, then we can find a nearby point x whose stabilizer contains a non-elementary Fuchsian subgroup which is generated by *small* elements; without the arithmeticity assumption on Γ, however, the orbit Hx need not be periodic, see e.g., [BO18, §12], in contrast to what happens in the arithmetic case (cf. Lemma B.1). Therefore, the proof of Theorem 1.3 will not include the improving Diophantine condition step which was present in the proof of Theorem 1.1 (see p. 93). To remedy this issue, we will choose the parameter D in the proof to be $O(1/\delta)$; this is responsible for the error rate $T^{-\delta^2\kappa_1}$ in Theorem 1.3(1). Let us now turn to the details.

Fixing the parameters. Fix ε as follows

$$0 < \sqrt{\varepsilon} < 10^{-8}\kappa_7 \tag{15.1}$$

where κ_7 is as in Proposition 5.2.

Let $0 < \delta < 1/4$ be as in the statement of Theorem 1.3, and let D_1 be as in Proposition 4.8. Put $t = \frac{\delta}{D_1}\log T$, and define D by

$$\tfrac{4D-3}{2} + 9 + \tfrac{5}{2}\sqrt{\varepsilon} = D_1/\delta \tag{15.2}$$

Since $\delta < 1/4$, we have $D \geq 2D_1$. Let

$$A' = \big(\tfrac{4D-3}{2} + 9 + \tfrac{5}{2}\sqrt{\varepsilon}\big)/(D - D_1); \tag{15.3}$$

note that $A' \ll 1$ where the implied constant is absolute.

We assume T is large enough so that

$$e^t > (10C_4)^3 \mathrm{inj}(x_0)^{-2}.$$

Suppose that Theorem 1.3(2) does not hold with this A'. That if $x \in X$ satisfies the following: there are elements γ_1 and γ_2 in $\mathrm{Stab}_H(x)$ with $\|\gamma_i\| \leq T^\delta$ for $i = 1, 2$ so that $\langle \gamma_1, \gamma_2 \rangle$ is Zariski dense in H, then

$$d_X(x, x_0) > T^{-1/A'} = e^{(-D+D_1)t}. \tag{15.4}$$

We will show that Theorem 1.3(1) holds.

Put $\ell = \varepsilon t/100$, and $d_1 = 100\lceil \frac{4D-3}{2\varepsilon} \rceil$. Then

$$\tfrac{4D-3}{2}t \leq d_1 \ell \leq \tfrac{4D-3}{2}t + \varepsilon t. \tag{15.5}$$

We define the parameter κ as follows:

$$\kappa = \tfrac{1}{2} \min\{10^{-6} d_1^{-1}, 10^{-6}\varepsilon\}, \tag{15.6}$$

and let $\beta = e^{-\kappa t}$ and let $\eta = \beta^{1/2}$; note that $\eta \geq e^{-0.1\ell}$ and that $\kappa \asymp \delta$.

Let us write $\log T = t_3 + t_2 + t_1$ where

$$t_1 = 8t \quad \text{and} \quad t_2 = t + d_1 \ell. \tag{15.7}$$

Note that $t_1, t_2 \geq t$. We now estimate t_3; indeed

$$\begin{aligned} t_3 &= \log T - (t_1 + t_2) \\ &= tD_1/\delta - 9t - d_1\ell \\ &= \left(\tfrac{4D-3}{2} + 9 + \tfrac{5}{2}\sqrt{\varepsilon}\right)t - 9t - d_1\ell \end{aligned}$$

where we used $tD_1/\delta = \log T$ in the second equation and (15.2) in the last equation. This and (15.5) imply

$$2\sqrt{\varepsilon} t \leq t_3 \leq 3\sqrt{\varepsilon} t. \tag{15.8}$$

Recall that $a_{\ell_1} u_r a_{\ell_2} = a_{\ell_1+\ell_2} u_{e^{-\ell_2} r}$. Thus, for any $\varphi \in C_c^\infty(X)$, we have

$$\int_0^1 \varphi(a_{\log T} u_r x_0)\, \mathrm{d}r = O(\|\varphi\|_\infty e^{-t}) + \int_0^1 \int_0^1 \int_0^1 \varphi(a_{t_3} u_{r_3} a_{t_2} u_{r_2} a_{t_1} u_{r_1} x_0)\, \mathrm{d}r_3\, \mathrm{d}r_2\, \mathrm{d}r_1\, \mathrm{d}r_0 \tag{15.9}$$

where the implied constant is absolute and we used $t_1, t_2 \geq t$.

The rest of the argument follows, mutatis mutandis, the same steps as in the proof of Theorem 1.1, as we now explicate.

Applying the closing lemma. We now apply Proposition 4.8 with x_0, D as in (15.2) and the parameter t (which is assumed to be large). In view of (15.4), Proposition 4.8(1) holds. Let $J_1 = I(x_0)$. Then

$$\int_0^1 \varphi(a_{\log T} u_r x_0)\, \mathrm{d}r = O(\|\varphi\|_\infty \eta^{1/2}) + \int_{J_1} \int_0^1 \int_0^1 \varphi(a_{t_3} u_{r_3} a_{t_2} u_{r_2} x(r_1))\, \mathrm{d}r_3\, \mathrm{d}r_2\, \mathrm{d}r_1 \tag{15.10}$$

where $x(r_1) = a_{t_1} u_{r_1} x_0$ and the implied constant is absolute.

Improving the dimension phase. Fix some $r_1 \in J_1$, and put $x_1 = x(r_1)$. Recall from (8.10) that

$$\mu_{t,\ell,d_1} = \nu_\ell * \cdots * \nu_\ell * \sigma * \nu_t$$

where ν_ℓ appears d_1 times in the above expression. In view of Lemma 7.4,

$$\Big|\int_0^1\int_0^1 \varphi(a_{t_3}u_{r_3}a_{t_2}u_{r_2}x_1)\,\mathrm{d}r_3\,\mathrm{d}r_2 - \int\int_0^1 \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}r_3\mu_{t,\ell,d_1}(h)\Big| \\ \ll \mathrm{Lip}(\varphi)e^{-\ell} \ll \mathrm{Lip}(\varphi)\eta^{1/2}. \tag{15.11}$$

We now apply Proposition 10.1 with x_1, t_3 and $r_3 \in [0,1]$. Then

$$\int_0^1\int \varphi(a_{t_3}u_{r_3}hx_1)\,\mathrm{d}\mu_{t,\ell,d_1}(h)\,\mathrm{d}r_3 = \\ \sum_{d,i} c_{d,i}\int_0^1\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r_3 + O(\mathrm{Lip}(\varphi)\beta^{\kappa_4}) \tag{15.12}$$

where the sum is over

$$d_1 - \lceil 10^4\varepsilon^{-1/2}\rceil = d_2 \le d \le d_1,$$

$c_{d,i} \ge 0$ and $\sum_{d,i} c_{d,i} = 1 - O(\beta^{\kappa_4})$ and the implied constants depend on X. Moreover, for all d, i both of the following hold

$$\#\big(B_{\mathfrak{r}}(w, 4b\,\mathrm{inj}(y)) \cap F_{d,i}\big) \ge e^{-\varepsilon t}\sup_{w'\in F_{d,i}} \#\big(B_{\mathfrak{r}}(w', 4b\,\mathrm{inj}(y)) \cap F_{d,i}\big) \tag{15.13a}$$

$$f_{\mathcal{E}_{d,i},b,\mathsf{R}}(e,z) \le e^{\varepsilon t}\psi_{\mathcal{E}_{d,i},b}(e,z) \quad \text{where } \mathsf{R} \le e^{0.01\varepsilon t} \tag{15.13b}$$

for all $w \in F_{d,i}$ and all $z = h\exp(w)y_{d,i} \in \mathcal{E}_{d,i}$ with $h \in \overline{\mathsf{E} \setminus \partial_{10b}\mathsf{E}}$.

From large dimension to equidistribution. For every $d_2 \le d \le d_1$, set

$$\tau_d := t_3 + d_1 - d.$$

Since $0 \le d_1 - d \le \lceil 10^4\varepsilon^{-1/2}\rceil$, $\ell = 0.01\varepsilon t$, and $2\sqrt{\varepsilon}t \le t_3 \le 3\sqrt{\varepsilon}t$, see (15.8),

$$2\sqrt{\varepsilon}t \le \tau_d \le (4 + 10^2)\sqrt{\varepsilon}t \le 0.01\kappa_7 t \tag{15.14}$$

where in the last inequality we used $0 < \sqrt{\varepsilon} < 10^{-8}\kappa_7$, see (15.1).

In view of Lemma 7.4, for all d, i as above, we have

$$\int_0^1\int \varphi(a_{t_3}u_{r_3}z)\,\mathrm{d}\nu_\ell^{(d_1-d)} * \mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r_3 = \\ \int_0^1\int \varphi(a_{\tau_d}u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r + O(\mathrm{Lip}(\varphi)e^{-\ell}) \tag{15.15}$$

where the implied constant depends on X.

We now apply Proposition 13.1 with $\mathcal{E}_{d,i}$, in view of (14.14a) and (14.14b) the conditions in that proposition are satisfied, and τ_d which is in the admissible range thanks to (14.15). Hence, for all d, i as above, we have

$$\Big|\int_0^1 \int \varphi(a_{\tau_d} u_r z)\,\mathrm{d}\mu_{\mathcal{E}_{d,i}}(z)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)e^{-\varepsilon^2 t} \tag{15.16}$$

where the implied constant depends on X.

Let $\hat{\kappa} = \min\{\varepsilon^2, \kappa_4\kappa, \kappa/4\}$. Then (15.16), (15.15), (15.12), (15.11), (15.10), and (15.9), imply that

$$\Big|\int_0^1 \varphi(a_{\log T} u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \ll \mathcal{S}(\varphi)e^{-\hat{\kappa}t} = \mathcal{S}(\varphi)T^{-\delta\hat{\kappa}/D_1}$$

where the implied constant depends on X.

In view of the definition of $\hat{\kappa}$ and (15.6), we have $\hat{\kappa} \gg \delta$ where the implied constant depends only on X. The proof is complete. □

16. Proof of Theorem 1.2

The proof is based on Theorem 1.1 and the following lemma, which is a special case of [LMMS19, Thm. 1.4] tailored to our application here.

16.1. **Lemma.** *There exist A_3, D_3, and C_8 (depending on X) so that the following holds. Let $S, M > 0$, and $0 < \eta < 1/2$ satisfy*

$$S \geq M^{A_3} \quad \textit{and} \quad M \geq C_8\eta^{-A_3}.$$

Let $x_1 \in X_\eta$, and suppose there exists $\mathrm{Exc} \subset \{r \in [-S, S] : u_r x_1 \in X_\eta\}$ with

$$|\mathrm{Exc}| > C_8\eta^{1/D_3}S$$

so that for every $r \in \mathrm{Exc}$, there exists $y_r \in X$ with

$$\mathrm{vol}(H.y_r) \leq M \quad \textit{and} \quad d(u_r x_1, y_r) \leq M^{-A_3}.$$

Then one of the following holds

(1) There exists $x \in G/\Gamma$ with $\mathrm{vol}(H.x) \leq M^{A_3}$, and for every $r \in [-S, S]$ there exists $g \in G$ with $\|g\| \leq M^{A_3}$ so that

$$d_X(u_s x_1, gH.x) \leq M^{A_3}\left(\frac{|s-r|}{S}\right)^{1/D_3} \quad \textit{for all } s \in [-S, S].$$

(2) For every $r \in [-S, S]$ and $t \in [\log M, \log S]$, the injectivity radius at $a_{-t}u_r x_1$ is at most $M^{A_3}e^{-t}$.

The lemma will be proved using [LMMS19, Thm. 1.4] or more precisely [LMMS19, Cor. 7.2]. The statements in [LMMS19] use a slightly different language than the one we used in this paper, thus we begin by recalling some terminology to relate Lemma 16.1 to [LMMS19, Thm. 1.4].

Arithmetic groups. Let $\mathbf{G} = \mathrm{SL}_2 \times \mathrm{SL}_2$ if $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, and $\mathbf{G} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2)$ if $G = \mathrm{SL}_2(\mathbb{C})$. Then $\mathbf{G}$ is defined over $\mathbb{R}$ and $G = \mathbf{G}(\mathbb{R})$; moreover, $H = \mathbf{H}(\mathbb{R})$ where $\mathbf{H} \subset \mathbf{G}$ is an algebraic subgroup.

Recall that Γ is assumed to be arithmetic. Therefore, there exists a semisimple simply connected $\mathbb{Q}$-group $\tilde{\mathbf{G}} \subset \mathrm{SL}_N$, for some N, and an epimorphism

$$\rho : \tilde{\mathbf{G}}(\mathbb{R}) \to \mathbf{G}(\mathbb{R}) = G$$

of $\mathbb{R}$-groups with compact kernel so that Γ is commensurable with $\rho(\tilde{\mathbf{G}}(\mathbb{Z}))$. Note that $\tilde{\mathbf{G}}$ can be chosen to be $\mathbb{Q}$-almost simple unless $\Gamma \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a reducible lattice, in which case $\tilde{\mathbf{G}}$ can be chosen to have two $\mathbb{Q}$-almost simple factors. We assume $\tilde{\mathbf{G}}$ is thus chosen.

Moreover, since $\tilde{\mathbf{G}}$ is simply connected, we can identify $\tilde{G}(\mathbb{R})$ with $G \times G'$ where $G' = \ker(\rho)$ is compact.

We are allowed to choose the parameter M in the lemma to be large depending on Γ, therefore, by passing to a finite index subgroup, we will assume that both of the following hold:

- $\Gamma \subset \tilde{\Gamma} := \rho(\tilde{\mathbf{G}}(\mathbb{Z}))$, where $\tilde{\mathbf{G}}(\mathbb{Z}) = \tilde{\mathbf{G}}(\mathbb{R}) \cap \mathrm{SL}_N(\mathbb{Z})$, and
- if $\Gamma \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is reducible, then $\Gamma = \Gamma_1 \times \Gamma_2$.

With this notation, every $\gamma \in \Gamma$ lifts uniquely to $(\gamma, \sigma(\gamma)) \in \tilde{\Gamma}$, where σ is (a collection of) Galois automorphisms. For every $g \in G$, we put

$$\hat{g} = (g, 1) \in G \times G'.$$

Suppose now that $g \in G$ is so that $Hg\Gamma$ is periodic. Let $\Delta_g = \Gamma \cap g^{-1}Hg$, and let $\tilde{\Delta}_g = \rho^{-1}(\Delta_g) \cap \tilde{\Gamma}$. Let $\tilde{\mathbf{H}}_g$ be the Zariski closure of $\tilde{\Delta}_g$. Then $\tilde{\mathbf{H}}_g$ is a semisimple $\mathbb{Q}$-subgroup, and the restriction of ρ to $\tilde{\mathbf{H}}_g$ surjects onto $g^{-1}\mathbf{H}g$. Let $\tilde{H}_g = \tilde{\mathbf{H}}_g(\mathbb{R})$, then

$$\overline{\hat{g}^{-1}\hat{H}\hat{g}\tilde{\Gamma}} = \tilde{H}_g\tilde{\Gamma}$$

Lie algebras and the adjoint representation. We continue to write $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$; these are considered as 6-dimensional (resp. 3-dimensional) $\mathbb{R}$-vector spaces.

Let v_H be a unit vector on the line $\wedge^3\mathfrak{h}$. Note that

$$N_G(H) = \{g \in G : gv_H = v_H\}$$

which contains H as a subgroup of index two.

Let $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{\mathbf{G}}(\mathbb{R}))$, this Lie algebra has a natural $\mathbb{Q}$-structure. Moreover, $\tilde{\mathfrak{g}}_{\mathbb{Z}} := \tilde{\mathfrak{g}} \cap \mathfrak{sl}_N(\mathbb{Z})$ is a $\tilde{\mathbf{G}}(\mathbb{Z})$-stable lattice in $\tilde{\mathfrak{g}}$.

If there exists $g \in G$ so that $Hg\Gamma$ is periodic, fix $\mathsf{g}_1, \ldots, \mathsf{g}_m$ so that $\mathrm{vol}(H\mathsf{g}_i\Gamma) \ll 1$ (the implied constant and m depend on Γ) and that every $\tilde{H}_g$ is conjugate to some $\tilde{H}_i = \tilde{H}_{\mathsf{g}_i}$ in $\tilde{G}$. Let $\mathbf{v}_i$ be a primitive integral vector on the line

$$\wedge^{\dim \tilde{H}_i}(\mathrm{Lie}(\tilde{H}_i)) \subset \wedge^{\dim \tilde{H}_i}\tilde{\mathfrak{g}}.$$

Then $N_{\tilde G}(\tilde H_i) = \{g \in \tilde G : gv_i = v_i\}$, and $\tilde H_i \subset N_{\tilde G}(\tilde H_i)$ has finite index. For all i, $\mathbf{v}_i = c_i \cdot \big((g_i^{-1}v_H) \wedge v_i'\big)$ where $v_i' \in \wedge\mathrm{Lie}(G')$ and $|c_i| \asymp 1$.

More generally, if $\mathbf{L} \subset \tilde{\mathbf{G}}$ is a $\mathbb{Q}$-algebraic group, we let $\mathbf{v}_L$ be a primitive integral vector on the line $\wedge^{\dim L}\mathrm{Lie}(L) \subset \wedge^{\dim L}\tilde{\mathfrak{g}}$ where $L = \mathbf{L}(\mathbb{R})$.

Volume and height of periodic orbits. Let $\mathbf{L} \subset \tilde{\mathbf{G}}$ be a $\mathbb{Q}$-algebraic group. Recall the definition of the height of $\mathbf{L}$ from [LMMS19]

$$\mathrm{ht}(\mathbf{L}) = \|\mathbf{v}_L\|.$$

Recall that $\tilde G = G \times G'$. We fix a right invariant metric on $\tilde G$ defined using the killing form and the maximal compact subgroup $\tilde K = K \times G'$ where $K = \mathrm{SO}(2)\times\mathrm{SO}(2)$ if $G = \mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SU}(2)$ if $G = \mathrm{SL}_2(\mathbb{C})$; this metric induces the right invariant metric on G which we fixed on p. 3.

16.2. **Lemma.** *Let $Hg\Gamma$ be a periodic orbit, and let $\tilde{\mathbf{H}}_g$ be as above. Both of the following properties hold:*

$$\mathrm{ht}(\tilde{\mathbf{H}}_g)^\star \ll \mathrm{vol}(\tilde H_g\tilde\Gamma/\tilde\Gamma) \ll \mathrm{ht}(\tilde{\mathbf{H}}_g)^\star$$

$$\|g\|^{-\star}\mathrm{vol}(Hg\Gamma) \ll \mathrm{vol}(\tilde H_g\tilde\Gamma/\tilde\Gamma) \ll \|g\|^\star\mathrm{vol}(Hg\Gamma)$$

Proof. For the first claim see [EMV09, §17] or [EMMV20, App. B] (for the upper bound, see also [ELMV09, §2], which treats the case of tori but the proof there works for the semisimple case as well).

To see the second claim, note that $\tilde H_g\tilde\Gamma$ projects onto $g^{-1}Hg\Gamma$ and the fiber is compact which volume $\asymp 1$. Therefore,

$$\mathrm{vol}(\tilde H_g\tilde\Gamma) \asymp \mathrm{vol}(g^{-1}Hg\Gamma).$$

Moreover, left multiplication by g changes the volume by $\|g\|^\star$.

The claim follows. □

Proof of Lemma 16.1. In view of our assumption in the lemma, periodic H orbits exists. Let $\tilde H_1, \ldots, \tilde H_m$ be as above. Let A_3 and D_3 be large constants which will be explicated later, in particular, we will let $A_3 > \max(A, D_2)$, $D_3 > D$ and $C_8 > \max\{mE_1, C_5\}$ where A, D, and E_1 are as in [LMMS19, Thm. 1.4] applied with $\{\hat u_r\} \subset \tilde G$, and D_2 and C_5 are as in Lemma 4.4.

We first interpret the condition in the lemma as a condition about the action of $\{\hat u_r\}$ on $\tilde G/\tilde\Gamma$. Let us write $x_1 = g_1\Gamma$, where $\|g_1\| \le C_5\eta^{-D_2} \le M$, see Lemma 4.4 and our assumption in this lemma. Similarly, for every $r \in \mathrm{Exc}$, let us write $y_r = g(r)\Gamma$ where $\|g(r)\| \le M$ and for every such r, there exists $\gamma_r \in \Gamma$ so that

$$\|u_rg_1\gamma_r\| \le M+1 \quad\text{and}\quad u_rg_1\gamma_r = \epsilon(r)g(r), \tag{16.1}$$

where $\|\epsilon(r)\| \ll M^{-A_3}$.

For every $1 \le i \le m$, let

$$\mathrm{Exc}_i = \{r \in \mathrm{Exc} : \tilde H^r := \tilde H_{g(r)} \text{ is a conjugate of } \tilde H_i\}.$$

Then, there exists some i so that $|\mathrm{Exc}_i| \geq |\mathrm{Exc}|/m$. Replacing Exc by Exc_i, we assume that $\tilde{H}^r$ is a conjugate of $\tilde{H}_i$ for all $r \in \mathrm{Exc}$. Let us write $\tilde{H}^r = \tilde{g}(r)^{-1}\tilde{H}_i\tilde{g}(r)$. Then

$$\tilde{g}(r) = (\mathsf{g}_i^{-1}g(r), \tilde{g}'(r)) \in G \times G',$$

and $\mathbf{v}^r := \frac{\|\mathbf{v}_{\tilde{H}^r}\|}{\|\tilde{g}(r)^{-1}\mathbf{v}_i\|}\tilde{g}(r)^{-1}\mathbf{v}_i = \pm\mathbf{v}_{\tilde{H}^r}$. Moreover, we have

$$\mathbf{v}^r = c_r \cdot \big((g(r)^{-1}v_H) \wedge (\tilde{g}'(r)^{-1}v_i')\big) \quad \text{where } |c_r| \ll M\mathrm{ht}(\tilde{\mathbf{H}}_g) \ll M^\star \tag{16.2}$$

where we used Lemma 16.2 to conclude $M\mathrm{ht}(\tilde{\mathbf{H}}_g) \ll M^\star$.

Recall that $\hat{g} = (g, 1)$ for all $g \in G$. In view of (16.1), we have

$$\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r = c_r \cdot \Big((\epsilon(r)v_H) \wedge \big((\sigma(\gamma_r)\tilde{g}'(r)^{-1})v_i'\big)\Big). \tag{16.3}$$

Since G' is compact, we conclude from (16.3) that

$$\|\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r\| \leq M^{A_3'}, \tag{16.4}$$

for some A_3'.

Let $z \in \mathfrak{g}$ be a vector so that $u_r = \exp(rz)$. Using (16.3) and associativity of the exterior algebra, we have

$$\begin{aligned}\|z \wedge \big(\hat{u}_r\hat{g}_1(\gamma_r, \sigma(\gamma_r)).\mathbf{v}^r\big)\| &= |c_r|\big\|\big(z \wedge \epsilon(r)v_H\big) \wedge \big((\sigma(\gamma_r)\tilde{g}'(r)^{-1})v_i'\big)\big\| \\ &\ll M^\star M^{-A_3} < \eta^A M^{-AA_3'}/E_1. \end{aligned}\tag{16.5}$$

where we used $\|\epsilon(r)\| \ll M^{-A_3}$ in the second to last inequality, A and E_1 are as in [LMMS19, Thm. 1.4], and we choose A_3 large enough so that the last estimate holds.

In view of (16.4) and (16.5), conditions in [LMMS19, Cor. 7.2] are satisfied. Hence, there exist $\tilde{\gamma} = (\gamma, \sigma(\gamma)) \in \tilde{\Gamma}$, $r \in \mathrm{Exc}$, and a subgroup

$$\tilde{\mathbf{H}}' \subset \tilde{\gamma}^{-1}\tilde{\mathbf{H}}^r\tilde{\gamma} \cap \tilde{\mathbf{H}}^r$$

satisfying that $\tilde{\mathbf{H}}'(\mathbb{C})$ is generated by unipotent subgroups (see [LMMS19, p. 3]) so that both of the following hold for all $r \in [-S, S]$

$$\|u_r g_1 \mathbf{v}_{\tilde{H}'}\| \ll M^\star \tag{16.6a}$$

$$\|z \wedge (u_r g_1 \mathbf{v}_{\tilde{H}'})\| \ll S^{-1/D}M^\star. \tag{16.6b}$$

Let $\tilde{H}' = \tilde{\mathbf{H}}'(\mathbb{R})$. Since $\|g_1\| \leq M$, we conclude from (16.6a), applied with $r = 0$, that

$$\|\mathbf{v}_{\tilde{H}'}\| \ll M^\star. \tag{16.7}$$

Let us consider two possibilities:

Case 1. $\rho(\tilde{H}')$ is a conjugate of H.

First note that this implies

$$\rho(\tilde{H}') = g(r_0)^{-1}Hg(r_0) \quad \text{where } r_0 \in \mathrm{Exc} \text{ is as above.}$$

Let us write $g' = g(r_0)$. Then $\|g'\| \leq M$, and we have

$$\begin{aligned} \mathrm{vol}(Hg'\Gamma/\Gamma) &\ll \|g'\|^\star \mathrm{vol}(g'^{-1}Hg'\Gamma/\Gamma) \\ &\ll M^\star \mathrm{ht}(\tilde{\mathbf{H}}') \ll M^\star \end{aligned} \tag{16.8}$$

where we used Lemma 16.2 in the second and (16.7) in the last inequality.

Recall that H is a symmetric subgroup of G, i.e., there exists an involution $\tau : G \to G$ so that H is the connected component of the identity in $\mathrm{Fix}(\tau)$. In particular, $G = KA'H$ for an $\mathbb{R}$-diagonalizable subgroup A'. For every $r \in [-S, S]$, let us write

$$u_r g_1 = g'^{-1}k_r b_r g' g'^{-1} h_r g' \in g'^{-1}KA'g'g'^{-1}Hg',$$

and put $g'_r = g'^{-1}k_r b_r g'$. Then (16.6a) and (16.7) imply that

$$\|g'_r\| \ll \|g'_r \mathbf{v}_{\tilde{H}'}\|^\star \|\mathbf{v}_{\tilde{H}'}\|^\star \|g'\|^\star \ll \|u_r g_1 \mathbf{v}_{\tilde{H}'}\|^\star M^\star \ll M^\star.$$

Since the map $r \mapsto u_r g_1 \mathbf{v}_{\tilde{H}'}$ is a polynomial map whose coefficients are $\ll M^\star$, we conclude that

$$g'_s = \epsilon'(s, r) g'_r \quad \text{where } \|\epsilon'(s, r)\| \ll M^\star \big(|s - r|/S\big)^\star.$$

Since $u_s g_1 = g'_s g'^{-1} h_s g'$ and d is right invariant, the above implies

$$d(u_s g_1, g'_r g'^{-1} H g') \ll M^\star \big(|s - r|/S\big)^\star;$$

hence part (1) in the lemma holds if for every $r \in [-S, S]$ we let $g = g'_r g'^{-1}$.

Case 2. $\rho(\tilde{H}') = g'^{-1}Ug'$ where $U = \{u_r\}$.

First note that if this holds, then $\tilde{\mathbf{G}} = \mathbf{G}$ (as $\mathbb{R}$-groups). Indeed in this case Γ is a non-uniform arithmetic lattice, thus $\tilde{\mathbf{G}} = R_{k/\mathbb{Q}}(\mathrm{SL}_2)$ for a quadratic extension $k/\mathbb{Q}$ if $G = \mathrm{SL}_2(\mathbb{C})$ or $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and Γ is irreducible. If $\Gamma = \Gamma_1 \times \Gamma_2$ in $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, then since the projection of $g'^{-1}Ug'$ to both factors is a nontrivial unipotent subgroup, Γ_1 and Γ_2 are both non-uniform arithmetic lattices; hence, $\tilde{\mathbf{G}} = \mathrm{SL}_2 \times \mathrm{SL}_2$.

Moreover, note that in this case $\mathbf{v}_{\tilde{H}'} \in \mathrm{Lie}(G)$, and we have

$$\exp(\mathbf{v}_{\tilde{H}'}) \in \tilde{H}' \cap \Gamma.$$

Let us consider the case of $G = \mathrm{SL}_2(\mathbb{C})$, the computations in the other case is similar by considering each component. Put

$$g_1.\mathbf{v}_{\tilde{H}'} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Then (16.6a) implies that for every $r \in [-S, S]$ we have

$$\left\| u_r \begin{pmatrix} a & b \\ c & -a \end{pmatrix} u_{-r} \right\| = \left\| \begin{pmatrix} a + cr & -cr^2 - 2ar + b \\ c & -a - cr \end{pmatrix} \right\| \ll M^\star$$

Hence $|c|S^2 \ll M^\star$ and $|a|S \ll M^\star$, which implies $|a + cr| \ll M^\star S^{-1}$.

Let now $t \in [\log M, \log S]$, then

$$\left\|a_{-t}u_r \begin{pmatrix} a & b \\ c & -a \end{pmatrix} u_{-r}a_t\right\| = \left\|\begin{pmatrix} a+cr & e^{-t}(-cr^2-2ar+b) \\ e^t c & -a-cr \end{pmatrix}\right\| \ll M^\star e^{-t},$$

where we used $e^t|c|, |a+cr| \ll M^\star S^{-1} \leq M^\star e^{-t}$.

Since $\exp(\mathbf{v}_{\tilde H'}) \in \tilde H' \cap \Gamma$, the above implies the claim in part (2). $\square$

16.3. **Proof of Theorem 1.2.** Let A be as Theorem 1.1, and let A_3, D_3 and C_8 be as in Lemma 16.1. Increasing A_3 and D_3 if necessary, we may assume $A_3, D_3 \geq 10A$. We will show the theorem holds with

$$A_1 = \star A_3 \geq 4A_3 \quad \text{and} \quad A_2 = D_3$$

Let $C = \max\{(10C_4)^3, e^{C_4}, e^{s_0}, C_1, C_8\}$, see (14.2). Let $R \geq C^2$, and put

$$d = 3A_3 \log R \quad \text{and} \quad \eta = (C/R)^{1/A_3}.$$

Let $T > R^{A_1}$, and put $T_1 = e^{-d}T \geq R^{A_3}$. Then

$$\begin{aligned}(16.9)\quad \frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r &= \frac{1}{T_1}\int_0^{T_1} \varphi(a_d u_{r_1} a_{-d} x_0)\,\mathrm{d}r_1 \\ &= \frac{1}{T_1}\int_0^{T_1}\int_0^1 \varphi(a_d u_r u_{r_1} a_{-d} x_0)\,\mathrm{d}r\,\mathrm{d}r_1 + O(\|\varphi\|_\infty T_1^{-1})\end{aligned}$$

where the implied constant is ≤ 2.

Put $x_1 = a_{-d}x_0$, and define

$$(16.10a)\qquad \mathrm{Exc}_1 = \{r_1 \in [0, T_1] : u_{r_1}x_1 \notin X_\eta\}$$

$$(16.10b)\qquad \mathrm{Exc}_2 = \left\{r_1 \in [0, T_1] : \begin{array}{c}\text{there exists } x \text{ with } \mathrm{vol}(Hx) \leq R \\ \text{and } d(u_{r_1}x_1, x) \leq R^A d^A e^{-d}\end{array}\right\}.$$

Let us first assume that

$$(16.11)\qquad |\mathrm{Exc}_1| \leq C\eta^{1/2}T_1 \quad \text{and} \quad |\mathrm{Exc}_2| \leq 2C^2R^{-\kappa}T_1,$$

where $\kappa = \min\{1/(2A_3), 1/(2D_3)\}$.

For every

$$r_1 \in [0, T_1] \setminus \big(\mathrm{Exc}_1 \cup \mathrm{Exc}_2\big),$$

put $x(r_1) = u_{r_1}x_1$. Then

$$R \geq C\eta^{-A_3} \geq C\mathrm{inj}(x(r_1))^{-2},$$

see (14.2); moreover, $e^d > R^A$. Thus conditions of Theorem 1.1 hold true with e^d, R, and $x(r_1)$. Moreover, in view of the definition of Exc_2, part (2) in Theorem 1.1 does not hold with these choices. Altogether, we conclude that for every r_1 as above,

$$\Big|\int_0^1 \varphi(a_d u_r x(r_1))\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq \mathcal{S}(\varphi)R^{-\kappa_1}$$

This, (16.11) and (16.9) imply that

$$\Big|\frac{1}{T}\int_0^T \varphi(u_r x_0)\,\mathrm{d}r - \int \varphi\,\mathrm{d}m_X\Big| \leq (R^{-\kappa_1} + 3C^2R^{-\kappa} + 2T_1^{-1})\mathcal{S}(\varphi),$$

where we used $C\eta^{1/2} \le C^2 R^{-\kappa}$.

Hence, part (1) in Theorem 1.2 holds with $\kappa_2 = \min(\kappa_1, \kappa)/2$ if we assume R is large enough.

We now assume to the contrary that (16.11) fails:

Assume that $|\mathrm{Exc}_1| > C\eta^{1/2}T_1$**.** We will show that part (3) in the theorem holds under this condition; the argument is similar to Case 2 in Lemma 16.1.

Let us write $x_0 = g_0\Gamma$. Then

$$\begin{aligned}\{u_{r_1}x_1 : r_1 \in [0, T_1]\} &= \{a_{\log T_1} u_r a_{-d-\log T_1} x_0 : r \in [0,1]\}\\ &= \{a_{\log T_1} u_r a_{-\log T} g_0\Gamma : r \in [0,1]\}.\end{aligned}$$

Our assumption $|\mathrm{Exc}_1| > C\eta^{1/2}T_1$ and the change variable thus imply

$$|\{r \in [0,1] : a_{\log T_1} u_r a_{-\log T} g_0\Gamma \notin X_\eta\}| > C_4\eta^{1/2},$$

where we used $C \ge C_4$, see Proposition 4.2 for C_4.

This and Proposition 4.2, applied with $a_{-\log T}g_0\Gamma$, the interval $[0,1]$, $\log T_1$, and $\varepsilon = \eta$, implies that

$$\mathrm{inj}(a_{-\log T} g_0\Gamma) \ll T_1^{-1};$$

the implied constant depends on X. Hence, there is some $\gamma \in \Gamma$ so that

$$a_{-\log T} g_0 \gamma g_0^{-1} a_{\log T} \in \mathsf{B}^G_{C' T_1^{-1}}$$

where C' depends on X. Assuming R and hence T_1 is large enough, the above implies that γ is a unipotent element. In particular, we have

$$a_{-\log T} g_0 \gamma g_0^{-1} a_{\log T} = \exp\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right)$$

where $|a|, |b|, |c| \ll T_1^{-1} = e^d T^{-1} = R^{3A_3}T^{-1}$. Hence,

$$g_0\gamma g_0^{-1} = \exp\left(\begin{pmatrix} a & Tb \\ T^{-1}c & -a \end{pmatrix}\right).$$

Let $b' = Tb$ and $c' = c/T$. Then

$$|b'| \ll R^{3A_3} \quad \text{and} \quad |c'| \ll R^{3A_3}T^{-2},$$

which implies that $|a + c'r| \ll R^{3A_3}T^{-1}$ for every $r \in [0, T]$. Therefore, for every $r \in [0, T]$ and every $t \in [\log R, \log T]$ we have

$$a_{-t} u_r g_0 \gamma g_0^{-1} u_{-r} a_t = \begin{pmatrix} a + c'r & e^{-t}(-c'r^2 - 2ar + b') \\ e^t c' & -a - c'r \end{pmatrix}.$$

Note that $|a + c'r| \ll R^{3A_3}T^{-1}$, $e^t|c'| \ll R^{3A_3}T^{-1}$, and

$$e^{-t}| - c'r^2 - 2ar + b'| \ll R^{3A_3}e^{-t}.$$

In consequence, part (3) in the theorem holds with $A_1 = 3A_3 + 1$ if we assume R is large enough.

Assume that $|\mathrm{Exc}_2| > 2C^2R^{-\kappa}T_1$**.** If $|\mathrm{Exc}_1| > C\eta^{1/2}T_1$, then part (3) in the theorem holds as we just discussed. Thus, we may assume that

$$|\mathrm{Exc}_2| > 2C^2R^{-\kappa}T_1 \quad \text{and} \quad |\mathrm{Exc}_1| \le C\eta^{1/2}T_1.$$

Put $\mathrm{Exc}' := \mathrm{Exc}_2 \setminus \mathrm{Exc}_1$. Then

$$\mathrm{Exc}' = \left\{ r_1 \in [0, T_1] : \begin{array}{c} u_{r_1}x_1 \in X_\eta \text{ and there exists } x \text{ with} \\ \mathrm{vol}(Hx) \le R \text{ and } d(u_{r_1}x_1, x) \le R^A d^A e^{-d} \end{array} \right\},$$

and $|\mathrm{Exc}'| \ge C^2R^{-\kappa}T_1 \ge C_8 R^{-1/D_3}T_1$. Moreover, assuming R is large enough, we have

$$R^A d^A e^{-d} = R^A (3A_3 \log R)^A R^{-3A_3} \le R^{-A_3}.$$

Fix some $r_1 \in \mathrm{Exc}'$ for the rest of the argument. Put

$$x_2 = u_{r_1}x_1 = u_{r_0}a_{-d}x_0 \quad \text{and} \quad \mathrm{Exc} = \mathrm{Exc}' - r_1 \subset [-T_1, T_1].$$

Then the conditions in Lemma 16.1 are satisfied with x_2, Exc, η, $M = R$, and $S = T_1 = R^{3A_3}T^{-1}$.

Assume first that part (1) in Lemma 16.1 holds. Then there exists $x \in G/\Gamma$ with $\mathrm{vol}(H.x) \le R^{A_3}$, and for every $r \in [-T_1, T_1]$ there exists $g \in G$ with $\|g\| \le R^{A_3}$ so that

$$d_X(u_s x_2, gHx) \le R^{A_3} \left(\frac{|s - r|}{T_1} \right)^{1/D_3} \qquad \text{for all } s \in [-T_1, T_1].$$

Since $s - r_1, r - r_1 \in [-T_1, T_1]$ for all $s, r \in [0, T_1]$, the above implies

$$\begin{aligned} d_X(u_{e^d s}x_0, a_d gHx) &= d_X(a_d u_s a_{-d}x_0, a_d gHx) \\ &= d_X(a_d u_{s-r_1} u_{r_1} a_{-d}x_0, a_d gHx) \\ &= d_X(a_d u_{s-r_1} x_2, a_d gHx) \\ &\ll e^{\star d} d_X(u_{s-r_1}x_2, gHx) \le R^{\star A_3} \Big(\frac{|e^d s - e^d r|}{T} \Big)^{1/D_3}. \end{aligned}$$

That is part (1) holds with $A_1 = \star A_3$ and $A_2 = D_3$ for all large enough R.

Assume now that part (2) in Lemma 16.1 holds. Therefore, for every $r \in [-T_1, T_1]$ and every $t_1 \in [\log R, \log T_1]$, the injectivity radius of $a_{-t_1}u_r x_2$ is at most $R^{A_3}e^{-t_1}$.

Let $t_1 \in [\log R, \log T_1]$ and $r \in [0, T_1]$, then

$$\begin{aligned} \mathrm{inj}(a_{-t_1}u_{e^d r}x_0) &= \mathrm{inj}(a_{-t_1}a_d u_{r-r_1}u_{r_1}a_{-d}x_0) \\ &\ll e^{\star d}\mathrm{inj}(a_{-t_1}u_{r-r_1}x_2) \le R^{\star A_3}e^{-t_1}. \end{aligned}$$

This implies part (3) of the theorem for all $t \in [\log R, \log T_1]$ and large enough R.

Let now $t \in [\log T_1, \log T]$. Then $t = s + \log T_1$ where $0 \le s \le 3A_3 \log R$, and we have

$$\mathrm{inj}(a_{-t}u_{e^d r}x_0) = \mathrm{inj}(a_{-s}a_{-\log T_1}u_{e^d r}x_0) \le R^{\star A_3}T_1^{-1} \le R^{\star A_3}e^{-t}.$$

Altogether, part (3) in the theorem holds, again with $A_1 = \star A_3$ and assuming R is large enough depending on X. $\square$

Appendix A. Proof of Proposition 4.6

In this section we prove Proposition 4.6. The proof is based on the study of a certain Margulis function whose definition will be recalled in (A.4).

For every $d > 0$, define the probability measure σ_d on H by

$$\int \varphi(h)\,\mathrm{d}\sigma_d(h) = \frac{1}{3}\int_{-1}^{2} \varphi(a_d u_r)\,\mathrm{d}r.$$

Let us first remark our choice of the interval $[-1, 2]$: We will define a function f_Y in (A.4) below. In Lemmas A.1–A.4, certain estimates for

$$\int f_Y(h\,\bullet)\,\mathrm{d}(\sigma_{d_1} * \cdots * \sigma_{d_n})(h)$$

will be obtained, then in Lemma A.5, we will convert these estimates to similar estimates for

$$\int_0^1 f_Y(a_{d_1+\cdots+d_n} u_r\,\bullet)\,\mathrm{d}r.$$

The argument in Lemma A.5 is based on commutation relations between a_d and u_r. Similar arguments have been used several times throughout the paper, however, since the function f_Y can have a rather large Lipschitz constant, we will not appeal to continuity properties of f_Y in Lemma A.5. Instead, we will use the fact that $[0, 1] \subset [-1, 2] + r$ for any $|r| \leq 1/2$.

We begin with the following linear algebra lemma.

A.1. **Lemma** (cf. Lemma 5.2, [EMM98]). *For all $0 \neq w \in \mathfrak{r}$, we have*

$$\int \|\operatorname{Ad}(h)w\|^{-1/3}\,\mathrm{d}\sigma_d(h) \leq C' e^{-d/3}\|w\|^{-1/3}$$

where C' is an absolute constant.

Proof. We may assume $\|w\| = 1$. Let us write $w = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$. Then

$$\operatorname{Ad}(a_t u_r)w = \begin{pmatrix} x + zr & e^t(-zr^2 - 2xr + y) \\ e^{-t}z & -x - zr \end{pmatrix}$$

For every $\varepsilon > 0$, let

$$I(\varepsilon) = \{r \in [-1, 2] : \varepsilon/2 \leq |-zr^2 - 2xr + y| \leq \varepsilon\},$$

then $|I(\varepsilon)| \leq C''\varepsilon^{1/2}$ where C'' is absolute, see e.g. [KM98, Prop. 3.2]. (This estimate is responsible for our choice of exponent 1/3 which is $< 1/2$.)

Moreover, for every $r\in I(\varepsilon)$, we have $\|\operatorname{Ad}(a_tu_r)w\|\geq e^t\varepsilon/2$. Note also that $\sup_{[-1,2]}|-zr^2-2xr+y|\leq 10$. Altogether, we have

$$\int \|\operatorname{Ad}(h)w\|^{-1/3}\,\mathrm{d}\sigma_d \leq \sum_{-4}^{\infty}\int_{I(2^k)}\|\operatorname{Ad}(a_tu_r)w\|^{-1/3}\,\mathrm{d}r$$
$$\leq C''\sum_{k=-4}^{\infty}2^{-k/2}\big(e^{-t/3}2^{(k+1)/3}\big)\leq 2C''e^{-t/3}\sum_{k=-4}^{\infty}2^{-k/6}.$$

The claim follows. □

We also need the following

A.2. **Proposition.** *There exists $C\geq C'$ (absolute) so that*

$$\int \operatorname{inj}(hx)^{-1/3}\,\mathrm{d}\sigma_d^{(\ell)}(h)\leq C^\ell e^{-\ell d/3}\operatorname{inj}(x)^{-1/3}+\bar{B}e^{2d/3}$$

where $\sigma_d^{(\ell)}$ denotes the ℓ-fold convolution and $\bar{B}\geq 1$ depends only of X.

Proof. This follows from [LM21, Prop. A.3] if one replaces the use of Equation (2.12) in that proof by Lemma A.1, see also [LM21, Lemma 2.4]. □

Let $Y=Hy$ be a periodic orbit. For every $x\in X\setminus Y$, define

$$I_Y(x)=\{w\in\mathfrak{r}:0<\|w\|<\operatorname{inj}(x),\exp(w)x\in Y\}.$$

Recall from [LM21, §9], that

$$\#I_Y(x)\leq E\operatorname{vol}(Y) \tag{A.1}$$

for a constant E depending only on X.

For every $h=a_du_r$ with $d\geq 0$ and $r\in[-1,2]$, and all $w\in\mathfrak{g}$, we have

$$\|\operatorname{Ad}(h^{\pm1})w\|\leq 10e^d\|w\|. \tag{A.2}$$

Replacing 10 by a bigger constant c, if necessary, we also assume that

$$c^{-1}e^{-d}\operatorname{inj}(x)\leq\operatorname{inj}(h^{\pm1}x)\leq ce^d\operatorname{inj}(x) \tag{A.3}$$

for all such h and all $x\in X$.

Define

$$f_Y(x)=\begin{cases}\sum_{w\in I_Y(x)}\|w\|^{-1/3} & I_Y(x)\neq\emptyset\\ \operatorname{inj}(x)^{-1/3} & \text{otherwise}\end{cases}. \tag{A.4}$$

A.3. **Lemma.** *Let C be as in Proposition A.2, and let $d\geq 3\log(4C)$. Then*

$$\int f(hx)\,\mathrm{d}\sigma_d(h)\leq$$
$$Ce^{-d/3}f_Y(x)+ce^dE\operatorname{vol}(Y)\cdot(Ce^{-d/3}\operatorname{inj}(x)^{-1/3}+\bar{B}e^d)$$

where $\bar{B}$ is as in Proposition A.2.

Proof. Since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

Let $d \geq 0$ and let $h = a_d u_r$ for some $r \in [-1, 2]$. Let $x \in X$. First, let us assume that there exists some $w \in I(hx)$ with

$$\|w\| < c^{-2} e^{-2d} \mathrm{inj}(hx) =: \Upsilon.$$

This in particular implies that both $I(hx)$ and $I(z)$ are non-empty. Hence, we have

$$\begin{aligned} f(hx) &= \sum_{w \in I(hx)} \|w\|^{-1/3} \\ &= \sum_{\|w\| < \Upsilon} \|w\|^{-1/3} + \sum_{\|w\| \geq \Upsilon} \|w\|^{-1/3} \\ &\leq \sum_{w \in I(x)} \|\operatorname{Ad}(h) w\|^{-1/3} + c^{2/3} e^{2d/3} \big(\# I(hx)\big) \cdot \mathrm{inj}(hx)^{-1/3}. \end{aligned} \tag{A.5}$$

Note also that if $\|w\| \geq \Upsilon = c^{-2} e^{-2d} \mathrm{inj}(hx)$ for all $w \in I(hx)$ (which in view of the choice of c includes the case $I(x) = \emptyset$) or if $I(hx) = \emptyset$, then

$$f(hx) \leq c^{2/3} e^{2d/3} \big(\# I(hx)\big) \cdot \mathrm{inj}(hx)^{-1/3}. \tag{A.6}$$

Averaging (A.5) and (A.6) over $[-1, 2]$ and using (A.1), we conclude that

$$\int f(hx) \, \mathrm{d}\sigma_d(h) \leq \sum_{w \in I(x)} \int \|hw\|^{-1/3} \, \mathrm{d}\sigma_d(h) \quad + \\ c^{2/3} e^{2d/3} E \mathrm{vol}(Y) \cdot \int \mathrm{inj}(hx)^{-1/3} \, \mathrm{d}\sigma_d(h);$$

we replace the summation on the right by 0 if $I(x) = \emptyset$.

Thus by Lemma A.1 and Proposition A.2, we conclude that

$$\int f(hx) \, \mathrm{d}\sigma_d(h) \leq C e^{-d/3} \cdot \sum_{w \in I(x)} \|w\|^{-1/3} \quad + \\ c e^d E \mathrm{vol}(Y) \cdot (C e^{-d/3} \mathrm{inj}(x)^{-1/3} + \bar{B} e^d)$$

where we replaced $2d/3$ by d. This may be rewritten as

$$\int f(hx) \, \mathrm{d}\sigma_d(h) \leq C e^{-d/3} f(x) + c e^d E \mathrm{vol}(Y) \cdot (C e^{-d/3} \mathrm{inj}(x)^{-1/3} + \bar{B} e^d).$$

The proof is complete. □

A.4. **Lemma.** *There is an absolute constant T_0 so that the following holds. Let $T \geq T_0$ and define*

$$d_i = 10^{-2} \cdot (2^{-i} \log T)$$

for all $i = 1, \ldots, k$ where k is the largest integer so that $d_k \geq 3 \log(4C)$ and C is as in Proposition A.2 — note that $\frac{1}{2} \log\log T \leq k \leq 2 \log\log T$.

Then

$$\int f_Y(hx)\,\mathrm{d}\sigma_{d_1}^{(100)} * \cdots * \sigma_{d_k}^{(100)}(h) \leq (\log T)^{D_0'} T^{-1/3}\Big(f(x) + B'\mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\textstyle\sum_{i=1}^{k} e^{2d_i}\Big) + B'\mathrm{vol}(Y)$$

where $D_0', B' \geq 1$ *are absolute.*

Proof. Again since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

Let us make the following two observations:

$$5\sum_{j=i+1}^{k} d_j \geq 0.05 \times 2^{-i-1}\log T \geq 0.01 \times 2^{-i}\log T = d_i \tag{A.7}$$

There is an absolute constant $M \geq 1$ so that the following holds

$$\sum_{j=1}^{i} C^{100(i-j)}e^{-d_j} \leq \sum_{j=1}^{k} C^{100(k-j)}e^{-d_j} \leq M \tag{A.8}$$

for all $1 \leq i \leq k$.

By Lemma A.4, for all $d \geq 3\log(4C)$, we have

$$\int f(hx)\,\mathrm{d}\sigma_d(h) \leq Ce^{-d/3}f(x) + cEe^{d}\mathrm{vol}(Y)\cdot(Ce^{-d/3}\mathrm{inj}(x)^{-1/3} + \bar{B}e^{d}). \tag{A.9}$$

Let $\lambda = cE\bar{B}$ and $\ell = 100$. Iterating (A.9), ℓ-times, we conclude that

$$\int f(h_k\cdots h_1 x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots\mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq C^{\ell}e^{-\ell d_k/3}\int f(h_{k-1}\cdots h_1 x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots\mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) \quad + \quad cEe^{d_k}\mathrm{vol}(Y)(\Xi_k + 2\bar{B}e^{d_k})$$

we used $Ce^{-d_k/3} \leq 1/4$ to bound the ℓ-terms geometric sum by $2\bar{B}e^{d_k}$, and

$$\Xi_k = \sum_{j=0}^{\ell-1}(Ce^{-d_k/3})^{\ell-j}\int \mathrm{inj}(h_k h_{k-1}\cdot\cdot h_1 x)^{-\frac{1}{3}}\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdot\cdot\,\mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1})\,\mathrm{d}\sigma_{d_k}^{(j)}(h_k).$$

Note that $cEe^{d_k}\mathrm{vol}(Y)(\Xi_k + 2\bar{B}e^{d_k}) \leq \lambda\mathrm{vol}(Y)e^{2d_k}(\Xi_k + 2)$, therefore,

$$\int f(h_k\cdots h_1 x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots\mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq C^{\ell}e^{-\ell d_k/3}\int f(h_{k-1}\cdots h_1 x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots\mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) \quad + \quad \lambda\mathrm{vol}(Y)e^{2d_k}(\Xi_k + 2). \tag{A.10}$$

We will apply Proposition A.2, to bound Ξ_k from above. Let us begin by applying Proposition A.2, j-times with d_k, then

$$\Xi_k \leq C^\ell e^{-\ell d_k/3} \int \operatorname{inj}(h_{k-1}\cdots h_1 x)^{-1/3}\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_{k-1}}^{(\ell)}(h_{k-1}) + \lambda e^{d_k}$$

where we used $Ce^{-d_k/3} \leq 1/4$ and $\lambda = cE\bar{B} \geq 2\bar{B}$ to estimate the ℓ-terms geometric sum.

The goal now is to inductively apply Proposition A.2, ℓ times with d_i for all $1 \leq i \leq k-1$, in order to simplify the above estimate. Applying Proposition A.2, ℓ-times with d_{k-1}, we obtain from the above that

$$\Xi_k \leq C^{2\ell} e^{-\ell(d_k+d_{k-1})/3} \int \operatorname{inj}(h_{k-2}\cdots h_1 x)^{-1/3}\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_{k-2}}^{(\ell)}(h_{k-2}) \quad + \\ C^\ell e^{-\ell d_k/3}\cdot(\lambda e^{d_{k-1}}) + \lambda e^{d_k}.$$

Put $\Theta_k = 0$, and for every $1 \leq i < k$, let $\Theta_i = \sum_{j=i+1}^k d_j$. Continuing the above inequalities inductively, we conclude

$$\begin{aligned}\Xi_k &\leq C^{\ell k} e^{-\ell(\sum_{i=1}^k d_i)/3} \operatorname{inj}(x)^{-1/3} + \lambda\Big(e^{d_k} + \sum_{i=1}^{k-1} C^{\ell(k-i)} e^{-\ell\Theta_i/3} e^{d_i}\Big)\\ &\leq C^{\ell k} e^{-\ell(\sum_{i=1}^k d_i)/3} \operatorname{inj}(x)^{-1/3} + \lambda\Big(e^{d_k} + \sum_{i=1}^{k-1} C^{\ell(k-i)} e^{-d_i}\Big)\\ &\leq C^{\ell k} e^{-\ell(\sum_{i=1}^k d_i)/3} \operatorname{inj}(x)^{-1/3} + \lambda(e^{d_k} + M)\end{aligned}$$

where we used $\ell\Theta_i/3 = 100\Theta_i/3 \geq 100 d_i/15$, see (A.7), in the second to last inequality and (A.8) in the last inequality.

Iterating (A.10) and the above analysis, we conclude

$$\int f(h_k\cdots h_1 x)\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq \\ C^{\ell k} e^{-\ell(\sum_{i=1}^k d_i)/3} f(x) + \lambda \operatorname{vol}(Y) \sum_{i=1}^{k} C^{\ell(k-i)} e^{-\ell\Theta_i/3} e^{2d_i}\big(\Xi_i + 2\big)$$

where for every $1 \leq i \leq k$, we have

$$\Xi_i = \sum_{j=0}^{\ell-1} (Ce^{-d_i/3})^{\ell-j} \int \operatorname{inj}(h_i h_{i-1}\cdots h_1 x)^{-\frac{1}{3}}\, \mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_{i-1}}^{(\ell)}(h_{i-1})\, \mathrm{d}\sigma_{d_i}^{(j)}(h_i).$$

Arguing as above, we have

$$\Xi_i \leq C^{\ell i} e^{-\ell(\sum_{j=1}^i d_j)/3} \operatorname{inj}(x)^{-1/3} + \lambda(e^{d_i} + M).$$

Recall that $\Theta_i = \sum_{j=i+1}^{k} d_j$; therefore, we conclude that

$$\int f(h_k \cdots h_1 x)\,\mathrm{d}\sigma_{d_1}^{(\ell)}(h_1)\cdots \mathrm{d}\sigma_{d_k}^{(\ell)}(h_k) \leq$$
$$C^{\ell k} e^{-\ell(\sum_{i=1}^{k} d_i)/3}\Big(f(x) + \lambda \mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\textstyle\sum_{i=1}^{k} e^{2d_i}\Big) +$$
$$(M+2)\lambda^2 \mathrm{vol}(Y)\sum_{i=1}^{k} C^{\ell(k-i)} e^{-\ell\Theta_i/3} e^{3d_i}$$

In view of (A.7), $\ell\Theta_i/3 = 100\Theta_i/3 \geq 100 d_i/15$. Hence, using (A.8), the last term above is $\leq B'\mathrm{vol}(Y)$ for an absolute constant $B' \geq \lambda$.

Moreover, $\ell\sum d_i = 100\sum d_i = \log T - O(1)$ where the implied constant is absolute, and $k \leq 2\log\log T$. Hence,

$$C^{\ell k} e^{-\ell(\sum_{i=1}^{k} d_i)/3} \leq (\log T)^{1+200\log C} T^{-1/3}$$

so long as T is large enough. The proof of the lemma is complete. □

A.5. **Lemma.** *Let the notation be as in Lemma A.4, in particular for every $T \geq T_0$ define $d_1, \ldots, d_k$ as in that lemma. Put $d(T) = 100\sum d_i$, then*

$$\int_0^1 f_Y(a_{d(T)} u_r x)\,\mathrm{d}r \leq$$
$$3(\log T)^{D_0'} T^{-1/3}\Big(f_Y(x) + B\mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\sum e^{2d_i}\Big) + B\mathrm{vol}(Y)$$

where $B \geq 1$ is absolute.

Proof. Again, since Y is fixed throughout the argument, we drop it from the index in the notation, e.g., we will denote f_Y by f etc.

By Lemma A.4, we have

(A.11)
$$\frac{1}{3^{100k}}\int_{-1}^{2}\cdots\int_{-1}^{2} f(a_{d_k}u_{r_{k,100}}\cdots a_{d_k}u_{r_{k,1}}\cdots a_{d_1}u_{r_{1,1}}x)\,\mathrm{d}r_{1,1}\cdots \mathrm{d}r_{k,100} \leq$$
$$(\log T)^{D_0'} T^{-1/3}\Big(f_Y(x) + B'\mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\sum e^{2d_i}\Big) + B'\mathrm{vol}(Y).$$

Now, for every $(r_{k,100}, \ldots, r_{1,2}, r_{1,1}) \in [-1,2]^{100k}$, we have

$$a_{d_k}u_{r_{k,100}}\cdots a_{d_k}u_{r_{k,1}}\cdots a_{d_1}u_{r_{1,1}} = a_{d(T)}u_{\varphi(\hat{r})+r_{1,1}}$$

where $\hat{r} = (r_{k,100}, \ldots, r_{1,2})$ and $|\varphi(\hat{r})| \leq 0.2$.

In view of (A.11), there is $\hat{r} = (r_{k,100}, \ldots, r_{1,2}) \in [-1,2]^{100k-1}$ so that

(A.12)
$$\frac{1}{3}\int_{-1+\varphi(\hat{r})}^{2+\varphi(\hat{r})} f(a_{d(T)}u_r x)\,\mathrm{d}r \leq$$
$$(\log T)^{D_0'} T^{-1/3}\Big(f_Y(x) + B'\mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\sum e^{2d_i}\Big) + B'\mathrm{vol}(Y).$$

Since $|\varphi(\hat{r})| \le 0.2$, we have $[0,1] \subset [-1,2]+\varphi(\hat{r})$. Therefore, (A.12) and the fact that $f \ge 0$ imply that

$$\frac{1}{3}\int_0^1 f(a_{d(T)}u_r x)\,\mathrm{d}x \le \\ (\log T)^{D_0'} T^{-1/3}\Big(f_Y(x) + B'\mathrm{vol}(Y)\mathrm{inj}(x)^{-1/3}\sum e^{2d_i}\Big) + B'\mathrm{vol}(Y).$$

The lemma follows with $B = 3B'$. □

Proof of Proposition 4.6. Let $R \ge 1$ be a parameter and assume that $\mathrm{vol}(Y) \le R$. Recall that for a periodic orbit Y, we put

$$f_Y(x) = \begin{cases} \sum_{w\in I_Y(x)} \|w\|^{-1/3} & I_Y(x) \ne \emptyset \\ \mathrm{inj}(x)^{-1/3} & \text{otherwise} \end{cases}.$$

Let $\psi(x_0) = \max\{d(x_0,Y)^{-1/3}, \mathrm{inj}(x_0)^{-1/3}\}$. Then

$$\psi(x_0) \ll f_{Y,d}(x_0) \ll \mathrm{vol}(Y)\psi(x_0), \tag{A.13}$$

where the implied constant depends only on X, see (A.1).

With the notation of Lemma A.4: let $T \ge T_0$ and $d_i = 0.01 \times 2^{-i}\log T$ for $1 \le i \le k$. Then

$$\log T - \bar{b} \le d(T) \le \log T \tag{A.14}$$

where $\bar{b}$ is absolute.

There exists $T_1 \ge T_0$ so that for all $T \ge T_1$ we have

$$(\log T)^{D_0'} T^{-1/3}\sum e^{2d_i} \le T^{-1/4}.$$

Let $T_1' = \max\{T_1, 3D_0'\}$, then $(\log T)^{D_0'}T^{-1/3}$ is decreasing on $[T_1', \infty)$. Let

$$T_2 = \inf\{T \ge \max\{T_1', \mathrm{inj}(x_0)^{-2}\} : (\log T)^{D_0'}T^{-1/3} \le d(x_0,Y)^{1/3}\}. \tag{A.15}$$

In view of (A.13) and since $\mathrm{vol}(Y) \le R$, thus for all $T \ge T_2$, we have

$$(\log T)^{D_0'}T^{-1/3} f_Y(x_0) \ll R(\log T)^{D_0'}T^{-1/3}\psi(x_0)$$

By the definition of T_2, we have $(\log T)^{D_0'}T^{-1/3}d(x_0,Y)^{-1/3} \le 1$, and

$$(\log T)^{D_0'}T^{-1/3}\mathrm{inj}(x_0)^{-1/3}\sum e^{2d_i} \le T^{-1/4}\mathrm{inj}(x_0)^{-1/3} \le 1.$$

In particular, using (A.13) again, we have $(\log T)^{D_0'}T^{-1/3}f_Y(x_0) \ll R$.

Altogether, we conclude that for all $T \ge T_2$, we have

$$\log(T)^{D_0'}T^{-1/3}\big(f_Y(x_0) + B\mathrm{vol}(Y)\mathrm{inj}(x_0)^{-1/3}\sum e^{2d_i}\big) \le B_2' R \tag{A.16}$$

where B_2' is absolute.

Let $T \ge T_2$, and let $d(T) = 100\sum d_i$ where d_i's are as above. Using (A.16) and Lemma A.5,

$$\int_0^1 f_Y(a_{d(T)}u_r x)\,\mathrm{d}r \le B_2 R \tag{A.17}$$

where $B_2 = 3B_2' + B$.

Let $D \geq 10$. Then by (A.17) we have

$$|\{r \in [0,1] : f_Y(a_{d(T)}u_r x_0) > B_2 R^D\}| \leq B_2 R / B_2 R^D \leq R^{-D+1}.$$

In view of (A.13), there is an absolute constant B_1 so that $d_X(a_s u_r x_0, Y) \leq B_1^{-1} R^{-3D}$ implies $f_Y(a_s u_r x_0) > B_2 R^D$ for all $s \geq 0$ and $r \in [0,1]$. Therefore, we conclude from the above that

$$\big|\big\{r \in [0,1] : d_X(a_{d(T)}u_r x_0, Y) \leq B_1^{-1} R^{-3D}\big\}\big| \leq R^{-D+1}. \tag{A.18}$$

Let now $s \geq \log T_2$, then by (A.14) there exists some $T \geq T_2$ so that

$$d(T) - 2\bar{b} \leq s \leq d(T) + 2\bar{b}$$

For every $s \geq \log T_2$ let T_s denote the minimum such T. Then (A.2) implies that is $\hat{B} \geq 1$ (absolute) so that if $s \geq \log T_2$ and $r \in [0,1]$ are so that

$$d_X(a_s u_r x_0, Y) \leq \hat{B}^{-1} R^{-3D},$$

then $d_X(a_{d(T_s)}u_r x_0, Y) \leq B_1^{-3} R^{-3D}$. This and (A.18), imply that

$$\big|\big\{r \in [0,1] : d_X(a_s u_r x_0, Y) \leq \hat{B}^{-1} R^{-3D}\big\}\big| \leq R^{-D+1} \tag{A.19}$$

Let C_4 be as in Proposition 4.2, increasing T_1 if necessary, we will assume $\log T_2 \geq |\log(\mathrm{inj}(x_0))| + C_4$. Using Proposition 4.2, thus, we conclude that

$$\big|\big\{r \in [0,1] : \mathrm{inj}(a_s u_r x) < \eta\big\}\big| < C_4 \eta^{1/2} \tag{A.20}$$

for any $\eta > 0$ and all $s \geq \log T_2$.

Altogether, from (A.19) and (A.20) it follows that for any $s \geq \log T_2$, we have

$$\left|\left\{r \in [0,1] : \begin{array}{c} \mathrm{inj}(a_s u_r x) < \eta \quad \text{or} \\ d_X(a_s u_r x_0, Y) \leq \hat{B}^{-1} R^{-3D} \end{array}\right\}\right| \leq C_4 \eta^{1/2} + R^{-D+1}. \tag{A.21}$$

In view of [MO20, Cor. 10.7], the number of periodic H-orbits with volume $\leq R$ in X is $\leq \hat{E} R^6$ where $\hat{E}$ depends on X. Let $D = 8$ and $C_1 = \max\{\hat{E}, \hat{B}, C_4\}$. Then (A.21) implies

$$\left|\left\{r \in [0,1] : \begin{array}{c} \mathrm{inj}(a_s u_r x) < \eta \text{ or there exists } x \text{ with} \\ \mathrm{vol}(Hx) \leq R \text{ s.t. } d_X(a_s u_r x_0, x) \leq \frac{1}{C_1 R^{24}} \end{array}\right\}\right| \leq C_1(\eta^{1/2} + R^{-1}). \tag{A.22}$$

We now show that (A.22) implies the proposition. Suppose

$$d_X(x_0, x) \geq S^{-1} (\log S)^{3D_0'}$$

for every x with $\mathrm{vol}(Hx) \leq R$. Then by (A.15), we have

$$T_2 \leq \max\{S, \mathrm{inj}(x_0)^{-2}, T_1'\}.$$

Therefore, the proposition follows from (A.22) if we let $D_0 = \max\{24, 3D_0'\}$ and put $s_0 = \log T_1'$. □

Appendix B. Proof of Proposition 4.8

In this section, we will give a detailed proof of Proposition 4.8. As it was mentioned, the proof is a slight modification of [LM21, Prop. 6.1].

Proof of Proposition 4.8. In what follows all the implied multiplicative constants depend only on X.

We begin by recalling Proposition 4.2: for all positive ε, every interval $J \subset [0,1]$, and every $x \in X$, we have

$$\big|\big\{r \in J : \mathrm{inj}(a_d u_r x) < \varepsilon^2\big\}\big| < C_4 \varepsilon |J|, \tag{B.1}$$

so long as $d \geq |\log(|J|^2 \mathrm{inj}(x))| + C_4$.

We also recall Lemma 4.4: Let $0 < \eta \leq \eta_X$ and let $g \in G$ be so that $g\Gamma \in X_\eta$. Then there exists some $\gamma \in \Gamma$ so that

$$\|g\gamma\| \leq C_5 \eta^{-D_2}. \tag{B.2}$$

For the rest of the argument, let

$$t \geq 100 D_2 |\log(\eta\, \mathrm{inj}(x_1))| + C_4 \tag{B.3}$$

Let $r_1 \in [0,1]$ be so that $x_2 = a_t u_{r_1} x_1 \in X_\eta$. Write $x_2 = g_2\Gamma$ where $|g_2| \ll \eta^{-D_2}$, see (B.2).

We will show that unless part (2) in the proposition holds, we have the following: for every x_2, there exists $J(x_2) \subset [0,1]$ with $|[0,1] \setminus J(x_2)| \leq 200 C_4 \eta^{1/2}$ so that for all $r \in J(x_2)$, we have:

(a) $a_{7t} u_r x_2 \in X_\eta$,
(b) the map $\mathsf{h} \mapsto \mathsf{h} a_{7t} u_r x_2$ is injective on E_t, and
(c) for all $z \in \mathsf{E}_t.a_{7t} u_r x_2$ we have $f_{t,\alpha}(z) \leq e^{Dt}$.

This will imply that part (1) in the proposition holds as

$$a_{7t} u_r a_t u_{r'} x_1 = a_{8t} u_{r' + e^{-t} r} x_1.$$

Assume contrary to the above claim that for some x_2 as above, there exists a subset $I'_{\mathrm{bad}} \subset [0,1]$ with $|I'_{\mathrm{bad}}| > 200 C_4 \eta^{1/2}$ so that one of (a), (b), or (c) above fails. Then in view of (B.1) applied with x_2 and $7t$, there is a subset $I_{\mathrm{bad}} \subset [0,1]$ with $|I_{\mathrm{bad}}| \geq 100 C_4 \eta^{1/2}$ so that for all $r \in I_{\mathrm{bad}}$ we have $a_{7t} u_r x_2 \in X_\eta$, but

- either the map $\mathsf{h} \mapsto \mathsf{h} a_{7t} u_r x_2$ is not injective on E_t,
- or there exists $z \in \mathsf{E}_t.a_{7t} u_r x_2$ so that $f_{t,\alpha}(z) > e^{Dt}$.

We will show that this implies part (2) in the proposition holds.

Finding lattice elements γ_r. We introduce the shorthand notation $h_r := a_{7t} u_r$, for any $r \in [0,1]$. Let us first investigate the latter situation. That is: for $r \in I_{\mathrm{bad}}$ (recall that $h_r x_2 \in X_\eta$) there exists some $z = \mathsf{h}_1 h_r x_2 \in \mathsf{E}_t.h_r x_2$, so that $f_{t,\alpha}(z) > e^{Dt}$. Since $h_r x_2 \in X_\eta$, we have

$$\mathrm{inj}(\mathsf{h} h_r x_2) \gg \eta e^{-t}, \quad \text{for all } \mathsf{h} \in \mathsf{E}_t. \tag{B.4}$$

Using the definition of $f_{t,\alpha}$, thus, we conclude that if $I_t(z) = \{0\}$, then $f_{t,\alpha}(z) \ll \eta^{-1}e^t$. Since $t \geq 100D_2|\log\eta|$, assuming t is large enough, we conclude that $I_t(z) \neq \{0\}$. Recall also that by virtue of Lemma 8.1 we have $\#I_t(z) \ll \eta^{-4}e^{4t}$, see also [LM21, Lemma 6.4].

Altogether, if $D \geq 6$ and t is large enough, there exists some $w \in I_t(z)$ with

$$0 < \|w\| \leq e^{(-D+5)t}.$$

The above implies that for some $w \in \mathfrak{r}$ with $\|w\| \leq e^{(-D+5)t}$ and $\mathsf{h}_1 \neq \mathsf{h}_2 \in \mathsf{E}_t$, we have $\exp(w)\mathsf{h}_1 h_r x_2 = \mathsf{h}_2 h_r x_2$. Thus

$$\exp(w_r)h_r^{-1}\mathsf{s}_r h_r x_2 = x_2 \tag{B.5}$$

where $\mathsf{s}_r = \mathsf{h}_2^{-1}\mathsf{h}_1$, $w_r = \mathrm{Ad}(h_r^{-1}\mathsf{h}_2^{-1})w$. In particular, $\|w_r\| \ll e^{(-D+13)t}$. Assuming t is large enough compared to the implied multiplicative constant,

$$0 < \|w_r\| \leq e^{(-D+14)t}. \tag{B.6}$$

Recall that $x_2 = g_2\Gamma$ where $|g_2| \ll \eta^{-D_2}$, thus, (B.5) implies

$$\exp(w_r)h_r^{-1}\mathsf{s}_r h_r = g_2\gamma_r g_2^{-1} \tag{B.7}$$

where $1 \neq \mathsf{s}_r \in H$ with $\|\mathsf{s}_r\| \ll e^t$ and $e \neq \gamma_r \in \Gamma$.

Similarly, if for some $r \in I_{\mathrm{bad}}$, $\mathsf{h} \mapsto \mathsf{h}h_r x_2$ is not injective, then

$$h_r^{-1}\mathsf{s}_r h_r = g_2\gamma_r g_2^{-1} \neq e.$$

In this case we actually have $e \neq \gamma_r \in g_2^{-1}Hg_2$ — we will not use this extra information in what follows.

Some properties of the elements γ_r. Recall that $\|g_2\| \ll \eta^{-D_2}$ and that $t \geq 100D_2|\log\eta|$. Therefore,

$$\|\gamma_r^{\pm 1}\| \leq e^{9t} \tag{B.8}$$

again we assumed t is large compared to $\|g_2\|$ hence the estimate $\ll e^{8.5t}$ is replaced by $\leq e^{9t}$.

Let $\xi > 0$ be so that $\|g\gamma g^{-1} - I\| \geq 20\xi\eta^{2D_2}$ for all $\gamma \in \Gamma \setminus \{1\}$ and $\|g\| \leq C_5\eta^{-D_2}$, see (B.2). Write $\mathsf{s}_r = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in H$ where $|a_i| \leq 10e^t$. Then by (B.7), we have

$$\|h_r^{-1}\mathsf{s}_r h_r - I\| = \left\| u_{-r}\begin{pmatrix} a_1 & e^{-7t}a_2 \\ e^{7t}a_3 & a_4 \end{pmatrix} u_r - I \right\| \geq 10\xi\eta^{2D_2}$$

which implies that

$$\max\{e^{7t}|a_3|, |a_1 - 1|, |a_4 - 1|\} \geq \xi\eta^{2D_2}. \tag{B.9}$$

Note also that if $e^{7t}|a_3| < \xi\eta^{2D_2}$, then $|a_2a_3| \leq 10\xi\eta^{2D_2}e^{-6t}$, thus $|a_1a_4 - 1| \ll \eta^{\star}e^{-6t}$. We conclude from (B.9) that $|a_1 - a_4| \gg \eta^{2D_2}$. Altogether,

$$\max\{e^{7t}|a_3|, |a_1 - a_4|\} \gg \eta^{2D_2}. \tag{B.10}$$

Since $|I_{\rm bad}| \geq 100C_4\eta^{1/2}$, there are two intervals $J, J' \subset [0,1]$ with $d(J,J') \geq \eta^{1/2}$, $|J|, |J'| \geq \eta^{1/2}$, and

$$|J \cap I_{\rm bad}| \geq \eta \quad \text{and} \quad |J' \cap I_{\rm bad}| \geq \eta. \tag{B.11}$$

Put $J_\eta = J \cap I_{\rm bad}$.

Claim: There are $\gg e^{29t/10}$ distinct elements in $\{\gamma_r : r \in J_\eta\}$.

Fix $r \in J_\eta$ as above, and consider the set of $r' \in J_\eta$ so that and $\gamma_r = \gamma_{r'}$. Then for each such r',

$$\begin{aligned} h_r^{-1}\mathsf{s}_r h_r &= \exp(-w_r)g_2\gamma_r g_2^{-1} = \exp(-w_r)\exp(w_{r'})h_{r'}^{-1}\mathsf{s}_{r'}h_{r'} \\ &= \exp(w_{rr'})h_{r'}^{-1}\mathsf{s}_{r'}h_{r'} \end{aligned}$$

where $w_{rr'} \in \mathfrak{g}$ and $\|w_{rr'}\| \ll e^{(-D+14)t}$.

Set $\tau = e^{7t}(r'-r)$. Assuming $D \geq 32$, we conclude that

$$u_\tau \mathsf{s}_r u_{-\tau} = h_{r'}h_r^{-1}\,\mathsf{s}_r\,h_r h_{r'}^{-1} = \exp(\hat{w}_{rr'})\mathsf{s}_{r'} \tag{B.12}$$

where $\|\hat{w}_{rr'}\| = \|\operatorname{Ad}(h_{r'})w_{rr'}\| \ll e^{(-D+21)}$.

Finally, we compute

$$u_\tau \mathsf{s}_r u_{-\tau} = \begin{pmatrix} a_1 + a_3\tau & a_2 + (a_4 - a_1)\tau - a_3\tau^2 \\ a_3 & a_4 - a_3\tau \end{pmatrix}.$$

In view of (B.10), for every $r \in J_\eta$ the set of $r' \in J_\eta$ so that

$$|a_2 e^{-7t} + (a_4 - a_1)(r' - r) - a_3 e^{7t}(r'-r)^2| \leq 10^4 e^{-6t} \tag{B.13}$$

has measure $\ll \eta^{-4D_2}e^{-3t}$ since at least one of the coefficients of this quadratic polynomial is of size $\gg \eta^{2D_2}$. Let $J_{\eta,r}$ be the set of $r' \in J_\eta$ for which (B.13) holds.

If $r' \in J_\eta \setminus J_{\eta,r}$, then $|a_2 + (a_4 - a_1)\tau - a_3\tau^2| > 10^4 e^t$ (recall that $\tau = e^{7t}(r'-r)$), thus for all $r' \in J_\eta \setminus J_{\eta,r}$, we have

$$\|u_\tau \mathsf{s}_r u_{-\tau}\| > 10^4 e^t > \|\exp(\hat{w}_{rr'})\mathsf{s}_{r'}\|,$$

in contradiction to (B.12).

In other words, for each $\gamma \in \Gamma$ the set of $r \in J_\eta$ for which $\gamma_r = \gamma$ has measure $\ll \eta^{-4D_2}e^{-3t}$ and so the set $\{\gamma_r : r \in J_\eta\}$ has at least $\gg \eta^{4D_1+1}e^{3t} \gg e^{29t/10}$ distinct elements (recall from (B.3) that $t \geq 100D_2|\log\eta|$); this establishes the claim.

Zariski closure of the group generated by $\{\gamma_r : r \in I_{\rm bad}\}$.

We now consider two possibilities for the elements $\{\gamma_r : r \in I_{\rm bad}\}$.

Case 1. The family $\{\gamma_r : r \in I_{\rm bad}\}$ is commutative.

Let $\mathbf{L}$ denote the Zariski closure of $\langle \gamma_r : r \in I_{\rm bad}\rangle$. Since $\langle\gamma_r\rangle$ is commutative, so is $\mathbf{L}$. Let $C_{\mathbf{G}}$ denote the center of $\mathbf{G}$. We claim that $\mathbf{L} = \mathbf{L}'\mathbf{C}'$ where $\mathbf{C}' \subset C_{\mathbf{G}}$ and $\mathbf{L}'$ is either a unipotent group or a torus. Indeed since $\mathbf{L}$ is commutative, we have $\mathbf{L} = \mathbf{T}\mathbf{V}$ where $\mathbf{T}$ is a (possibly finite) algebraic subgroup of a torus, $\mathbf{V}$ is a unipotent group and $\mathbf{T}$ and $\mathbf{V}$ commute. Therefore, if both $\mathbf{T}$ and $\mathbf{V}$ are non-central, then $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \Gamma_1 \times \Gamma_2$ is reducible. Moreover, $\mathbf{T} \subset \mathbf{T}'C_{\mathbf{G}}$ where $\mathbf{T}'$ is an algebraic subgroup of a torus, and $\mathbf{T}'$ and $\mathbf{V}$ belong to different $\mathrm{SL}_2(\mathbb{R})$ factors in G. Let us assume $\mathbf{V}$ belongs to the second factor. Recall from (B.5) that

$$\exp(w_r)h_r^{-1}\mathsf{s}_r h_r = g_2\gamma_r g_2^{-1} \tag{B.14}$$

where $\|w_r\| \le e^{(-D+14)t}$ with $D \ge 32$ and $h_r^{-1}\mathsf{s}_r h_r \in H = \{(h,h) : h \in \mathrm{SL}_2(\mathbb{R})\}$. Now if $\gamma_r = (\gamma_r^1, \gamma_r^2)$, then (B.14) together with the bound $\|h_r^{-1}\mathsf{s}_r h_r\| \ll e^{8t}$ implies that $|\mathrm{tr}(\gamma_r^1) - \mathrm{tr}(\gamma_r^2)| \ll e^{(-D+22)t}$; moreover, since $\gamma_r^2 \in \mathbf{V}C_{\mathbf{G}}$, we have $|\mathrm{tr}(\gamma_r^2)| = 2$. This and the fact that the length of closed geodesics in (finite volume) hyperbolic surfaces is bounded away from zero imply that $|\mathrm{tr}(\gamma_r^1)| = 2$ if t is large enough. This contradicts the fact that $\mathbf{T}$ is a non-central subgroup of a torus. Hence, the claim holds.

We now show that $\mathbf{L}'$ is indeed a unipotent group. In view of the above discussion, $\#\{\gamma_r : r \in J_\eta\} \ge e^{29t/10}$. Note also that that for every torus $T \subset G$, we have

$$\#(B_T(e,R) \cap \Gamma) \ll (\log R)^2,$$

where the implied constant is absolute. These, in view of the bound $\|\gamma_r\| \le e^{9t}$, see (B.8), imply that $\mathbf{L}'$ is unipotent.

Since $\mathbf{L}'$ is a unipotent subgroup of $\mathbf{G}$, we have that

$$\#\{\gamma_r : \|\gamma_r\| \le e^{4t/3}\} \ll e^{8t/3}.$$

Furthermore, there are $\gg e^{29t/10}$ distinct elements γ_r with $r \in J_\eta$. Thus

$$\#\{\gamma_r : \|\gamma_r\| > 100e^{4t/3} \text{ and } r \in J_\eta\} \gg e^{29t/10}.$$

For every $r \in I_{\rm bad}$, write

$$\mathsf{s}_r = \begin{pmatrix} a_{1,r} & a_{2,r} \\ a_{3,r} & a_{4,r}\end{pmatrix} \in H$$

where $|a_{j,r}| \le 10e^t$.

We will obtain an improvement of (B.9). Let $\xi\eta^{2D_2} \le \Upsilon \le e^{4t/3}$ and assume that $\|g_2\gamma_r g_2^{-1} - I\| \ge 20\Upsilon$ — by definition of ξ, this holds with $\Upsilon = \xi\eta^{2D_2}$ for all $r \in I_{\rm bad}$ and as we have just seen this also holds for with $\Upsilon = e^{4t/3}$ for many choices of $r \in J_{\rm bad}$. We claim

$$|a_{3,r}| \ge \Upsilon e^{-7t}. \tag{B.15}$$

Indeed by (B.7), we have

$$\|h_r^{-1}\mathsf{s}_r h_r - I\| = \left\| u_{-r}\begin{pmatrix} a_{1,r} & e^{-7t}a_{2,r} \\ e^{7t}a_{3,r} & a_{4,r}\end{pmatrix} u_r - I\right\| \geq 10\Upsilon.$$

This implies that $\max\{e^{7t}|a_{3,r}|, |a_{1,r}-1|, |a_{3,r}-1|\} \geq \Upsilon$. Assume contrary to our claim that $|a_{3,r}| < \Upsilon e^{-7t}$. Then

$$\max\{|a_{1,r}-1|, |a_{4,r}-1|\} \geq \Upsilon; \tag{B.16}$$

furthermore, we get $|a_{2,r}a_{3,r}| \ll \Upsilon e^{-6t}$. Thus,

$$|a_{1,r}a_{4,r} - 1| \ll \Upsilon e^{-6t} \ll e^{-14t/3}. \tag{B.17}$$

Moreover, since $h_r^{-1}\mathsf{s}_r h_r$ is very nearly $g_2\gamma_r g_2^{-1}$, and the latter is either a unipotent element or its minus, we conclude that

$$\min(|a_{1,r}+a_{4,r}-2|, |a_{1,r}+a_{4,r}+2|) \ll e^{(-D+22)t}. \tag{B.18}$$

Equations (B.17) and (B.18) contradict (B.16) if t is large enough (recall again from (B.3) that $t \geq 100D_2|\log\eta|$). Hence necessarily $|a_{3,r}| \geq \Upsilon e^{-7t}$.

Using this, we now show that Case 1 cannot occur. Since $\mathbf{L}'$ is unipotent, there exists some g so that $\mathbf{L}'(\mathbb{R}) \subset gNg^{-1}$; moreover g can be chosen to be in the maximal compact subgroup of G — for our purposes, we only need to know that the size of g can be bounded by an absolute constant.

It follows that

$$u_{-r}\begin{pmatrix} a_{1,r} & e^{-7t}a_{2,r} \\ e^{7t}a_{3,r} & a_{4,r}\end{pmatrix} u_r \in \exp(-w_r)(gNg^{-1})\cdot C_{\mathbf{G}} \tag{B.19}$$

for all $r \in I_{\mathrm{bad}}$. We show that this leads to a contradiction when $G = \mathrm{SL}_2(\mathbb{C})$, the proof in the other case is similar by considering first and second coordinates.

Recall the intervals J and J' from (B.11), and let $r_0 \in J' \cap I_{\mathrm{bad}}$. then $|r_0 - r| \geq \eta^{1/2}$ for all $r \in J_\eta$. Then, (B.19), yields that

$$u_{-r+r_0}\begin{pmatrix} a_{1,r} & e^{-7t}a_{2,r} \\ e^{7t}a_{3,r} & a_{4,r}\end{pmatrix} u_{r-r_0} \in \exp(-w_r')(u_{r_0}gNg^{-1}u_{-r_0})\cdot C_{\mathbf{G}} \tag{B.20}$$

for all $r \in I_{\mathrm{bad}}$.

Let us write $u_{r_0}g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then for all $z \in \mathbb{C}$ we have

$$u_{r_0}g\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} g^{-1}u_{-r_0} = \begin{pmatrix} 1-acz & a^2z \\ -c^2z & 1+acz\end{pmatrix}.$$

Let $z_0 \in \mathbb{C}$ be so that

$$\begin{pmatrix} a_{1,r_0} & e^{-7t}a_{2,r_0} \\ e^{7t}a_{3,r_0} & a_{4,r_0}\end{pmatrix} = \pm\exp(-w_r)\begin{pmatrix} 1-acz_0 & a^2z_0 \\ -c^2z_0 & 1+acz_0\end{pmatrix}.$$

By (B.15) applied with $\Upsilon = \xi\eta^{2D_2}$, $|a_{3,r_0}| \geq \xi\eta^{2D_2}e^{-7t}$. Since $|a|, |b|, |c|, |d| \ll 1$, comparing the bottom left entries of the matrices, we get $|z_0| \gg \eta^{2D_2}$. Now, since $|a_{2,r_0}| \leq 10e^t$, comparing the top right entries we conclude that $|a| \ll \eta^{-2D_2}e^{-3t} \ll e^{-29t/10}$. Since $\det(g) = 1$, it follows that $|c|$ is also $\gg 1$.

Let now $r \in J_\eta$ be so that $\|\gamma_r\| \geq 100e^{4t/3}$. We write $r_1 = r - r_0$, $a'_{2,r} = e^{-7t}a_{2,r}$ and $a'_{3,r} = e^{7t}a_{3,r}$. By (B.15), applied this time with $\Upsilon = e^{4t/3}$, we have that $|a'_{3,r}| \geq e^{4t/3}$; note also that $|a'_{2,r}| \ll e^{-6t}$. In view of (B.20), there exists $z_r \in \mathbb{C}$ so that

$$u_{-r_1}\begin{pmatrix} a_{1,r} & a'_{2,r} \\ a'_{3,r} & a_{4,r}\end{pmatrix}u_{r_1} = \begin{pmatrix} a_{1,r} - r_1 a'_{3,r} & a'_{2,r} + (a_{4,r} - a_{1,r})r_1 - a'_{3,r}r_1^2 \\ a'_{3,r} & a_{4,r} + r_1 a'_{3,r}\end{pmatrix}$$
$$= \pm\exp(-w'_r)\begin{pmatrix} 1 - acz_r & a^2 z_r \\ -c^2 z_r & 1 + acz_r\end{pmatrix}.$$

Recall that $|a'_{3,r}| \geq e^{4t/3}$, $|a_{1,r}|$ and $|a_{4,r}|$ are $\ll e^t$, and $|a'_{2,r}| \ll e^{-6t}$; moreover $\eta^{1/2} \leq |r_1| \leq 1$ and by (B.3) $e^{t/10} \geq \eta^{-1}$. We cocnlude

$$|a'_{3,r}|\eta/10 \leq |a'_{2,r} + (a_{4,r} - a_{1,r})r - a'_{3,r}r^2| \leq 2|a'_{3,r}|.$$

Hence, since w'_r is small, $|c^2 z_r|\eta \ll |a^2 z_r| \ll |c^2 z_r|$. On the other hand, using $r = r_0$, we already established $|a| \ll e^{-29t/10}$ and $|c| \gg 1$, thus $|a^2 z_r| \ll e^{-5t}|c^2 z_r|$, which is a contradiction, see (B.3) again.

Altogether, we conclude that Case 1 cannot occur.

Case 2. There are $r, r' \in I_{\text{bad}}$ so that γ_r and $\gamma_{r'}$ do not commute.

We first recall versions of [LM21, Lemma 6.2] and [LM21, Lemma 6.3]. The statements in those lemmas assume $g_2 \in \mathfrak{S}_{\text{cpt}}$. However, the arguments work without any changes and one has the following.

Let v_H be a unit vector on the line $\wedge^3\mathfrak{h} \subset \wedge^3\mathfrak{g}$.

B.1. **Lemma.** *Assume Γ is arithmetic. There exist C_9 and κ_8 depending on Γ, and C_{10} (absolute) so that the following holds. Let $\gamma_1, \gamma_2 \in \Gamma$ be two non-commuting elements. If $g \in G$ is so that $\gamma_i g^{-1}v_H = g^{-1}v_H$ for $i = 1, 2$, then $Hg\Gamma$ is a closed orbit with*

$$\mathrm{vol}(Hg\Gamma) \leq C_9\|g\|^{C_{10}}\big(\max\{\|\gamma_1^{\pm1}\|, \|\gamma_2^{\pm1}\|\}\big)^{\kappa_8}.$$

B.2. **Lemma.** *Assume Γ has algebraic entries. There exist κ_9, κ_{10}, C_{11} and C_{12} so that the following holds. Let $\gamma_1, \gamma_2 \in \Gamma$ be two non-commuting elements, and let*

$$\delta \leq {C_{11}}^{-1}\big(\max\{\|\gamma_1^{\pm1}\|, \|\gamma_2^{\pm1}\|\}\big)^{-\kappa_9}.$$

Suppose there exists some $g \in G$ so that $\gamma_i g^{-1}v_H = \epsilon_i g^{-1}v_H$ for $i = 1, 2$ where $\|\epsilon_i - I\| \leq \delta$. Then, there is some $g' \in G$ such that

$$\|g' - g^{-1}\| \leq C_{11}\|g\|^{C_{12}}\delta\big(\max\{\|\gamma_1^{\pm1}\|, \|\gamma_2^{\pm1}\|\}\big)^{\kappa_{10}}$$

and $\gamma_i g' v_H = g' v_H$ for $i = 1, 2$.

Let us now return to the analysis in Case 2. Recall that $\|g_2\| \leq \eta^{-D_1}$, we will assume t is large enough so that

$$e^t \geq \eta^{-2D_1\max\{C_{10}, C_{12}\}}.$$

Recall that $\exp(w_r)h_r^{-1}\mathsf{s}_r h_r = g_2\gamma_r g_2^{-1}$, thus

$$\gamma_r.g_2^{-1}v_H = \exp(\mathrm{Ad}(g_2^{-1})w_r).g_2^{-1}v_H.$$

Moreover, since $\|w_r\| \le e^{(-D+16)t}$,

$$\|\,\mathrm{Ad}(g_2^{-1})w_r\| \ll \eta^{-2D_1}e^{(-D+14)t} \ll e^{(-D+15)t}$$

similar statements also hold for r'.

Recall that $\|\gamma_r^{\pm1}\|, \|\gamma_{r'}^{\pm1}\| \le e^{9t}$. If D is large enough, we may apply Lemma B.2 and conclude that there exists some $g_3 \in G$ with

$$\|g_2 - g_3\| \le C_{11}\eta^{-D_1 C_{12}}e^{(-D+15+9\kappa_{10})t} \le C_{11}e^{(-D+16+9\kappa_{10})t},$$

so that $\gamma_r.g_3^{-1}v_H = g_3^{-1}v_H$ and $\gamma_{r'}.g_2^{-1}v_H = g_2^{-1}v_H$.

In view of Lemma B.1, thus, we have $Hg_3\Gamma$ is periodic and

$$\mathrm{vol}(Hg_3\Gamma) \le C_9\eta^{-D_2C_{10}}\big(\max\{\|\gamma_r^{\pm1}\|, \|\gamma_{r'}^{\pm1}\|\}\big)^{\kappa_8} \le C_9 e^{1+9\kappa_8 t}.$$

Then for t large enough, $\mathrm{vol}(Hg_2\Gamma) \le e^{D_0't}$ and $d_X(g_2\Gamma, g_2\Gamma) \ll e^{(-D+D_0')t}$ for $D_0' = 9\max\{\kappa_8, \kappa_{10}\} + 16$.

Since $g_2\Gamma = x_2 = a_t u_{r_1} x_1$, part (2) in the proposition holds with $x' = (a_t u_{r_1})^{-1} g_3\Gamma$ and $D_0 = \max\{D_0' + 2, 32\}$ if t is large enough (recall that we already assumed in several places that $D \ge 32$). $\square$

We note that the only place we used the arithmeticity of Γ is Lemma B.1. If we instead assume Γ has algebraic entries, the argument above goes through and yields *(2')* in §4.7.

Appendix C. Proof of Theorem 6.2

Theorem 6.2 will be proved using the following theorem. First note that replacing Θ by $\frac{1}{b_0}\Theta$ and Υ by $b_0^{-\alpha}\Upsilon$, we may assume $b_0 = 1$.

C.1. **Theorem.** *Let $0 < \alpha \le 1$. Let $\Theta \subset B_{\mathfrak{r}}(0,1)$ be a finite set satisfying*

$$\mathcal{G}_{\Theta,\mathsf{R}}(w) \le \Upsilon, \quad \textit{for every } w \in \Theta \textit{ and some } R \ge 1, \tag{C.1}$$

where $\Upsilon \ge 1$.

Let $0 < \mathsf{c} < 0.01\alpha$, and let $J \subset [0,1]$ be an interval with $|J| \ge 10^{-4}$. For every $b \ge \Upsilon^{-1/\alpha}$, there exists a subset $J_b \subset J$ with $|J \setminus J_b| \le L\mathsf{c}^{-L}b^{\mathsf{c}}$ so that the following holds. Let $r \in J_b$, then there exists a subset $\Theta_{b,r} \subset \Theta$ with

$$\frac{\#(\Theta \setminus \Theta_{b,r})}{\#\Theta} \le L\mathsf{c}^{-L}b^{\mathsf{c}}$$

such that for all $w \in \Theta_{b,r}$, we have

$$\#\{w' \in \Theta : |\xi_r(w) - \xi_r(w')| \le b\} \le L\mathsf{c}^{-L}\Upsilon^{1+7\mathsf{c}}b^{\alpha}$$

where L is an absolute constant and

$$\xi_r(w) = (\mathrm{Ad}(u_r)w)_{12} = -w_{21}r^2 - 2w_{11}r + w_{12}.$$

We prove the theorem for $J = [0,1]$, the proof in general is similar. We begin by fixing some notation. Let ρ denote the uniform measure on Θ.

Let

$$\Xi(w) = \big\{(r, \xi_r(w)) : r \in [0,1]\big\}$$

for every $w \in \Theta$, and let $\Xi = \bigcup_w \Xi(w)$.

For every $b > 0$ and every $w \in \Theta$, let

$$\Xi^b(w) = \big\{(q_1, q_2) \in [0,1] \times \mathbb{R} : |q_2 - \xi_{q_1}(w)| \le b\big\}.$$

Finally, for all $q \in \mathbb{R}^2$ and $b > 0$, define

$$m^b_\rho(q) := \rho\big(\{w' \in \mathfrak{r} : q \in \Xi^b(w')\}\big). \tag{C.2}$$

The assertion in the theorem may be rewritten in terms of the multiplicity function m^b_ρ as follows. We seek the set $J_b \subset [0,1]$, and for every $r \in J_b$, the set $\Theta_{b,r} \subset \Theta$ so that

$$m^b_\rho\big((r, \xi_r(w))\big) \le \tfrac{L\mathsf{c}^{-L}\Upsilon^{1+7\mathsf{c}}b^\alpha}{\#\Theta} \quad \text{for all } w \in \Theta_{b,r}. \tag{C.3}$$

The following lemma plays a crucial role in the proof of Theorem C.1. This is a more detailed version of [Sch03, Lemma 8] in the setting at hand, see also [Wol00, Lemma 1.4] and [Zah12a, Lemma 2.1], and [KOV17, Lemma 5.1]. The general case has recently been addressed in [PYZ22].

C.2. **Lemma.** *Let the notation be as in Theorem C.1. In particular, $\Theta \subset B_{\mathfrak{r}}(0,1)$ and (C.1) is satisfied. For every $0 < \mathsf{c} \le 0.01\alpha$, there exists $0 < D \ll \mathsf{c}^{-\star}\Upsilon/(\#\Theta)$ (implied constants are absolute) so that the following holds. Let $b \ge \Upsilon^{-1/\alpha}$. Then there exists a subset $\hat\Theta = \hat\Theta_b \subset \Theta$ with $\#(\Theta \setminus \hat\Theta) \le b^{\mathsf{c}} \cdot (\#\Theta)$ so that for every $w \in \hat\Theta$, we have*

$$\big|\Xi^b(w) \cap \big\{q \in \mathbb{R}^2 : m^b_\rho(q) \ge Db^{\alpha - 7\mathsf{c}}\big\}\big| \le b^{2\mathsf{c}/\alpha}|\Xi^b(w)|.$$

Proof. The proof of [LM21, Lemma B.2] goes through mutatis mutandis. ☐

Proof of Theorem C.1. Assume that the conclusion of the theorem fails for some L. That is, there exists a subset $\bar J \subset [0,1]$ with $|\bar J| > L\mathsf{c}^{-L}b^{\mathsf{c}}$ so that for all $r \in \bar J$ we have

$$\rho(\Theta'_r) > L\mathsf{c}^{-L}b^{-\mathsf{c}} \tag{C.4}$$

where $\Theta'_r = \{w \in \Theta : m^b_\rho\big((r, \xi_r(w))\big) \le L\mathsf{c}^{-L}\Upsilon^{1+7\mathsf{c}}b^\alpha/(\#\Theta)\}$.

We will get a contradiction if L is large enough. Let us write $C = L\mathsf{c}^{-L}$ and $\bar C = C \cdot (\#\Theta)^{-1}$. Let $\hat\Theta$ be as in Lemma C.2 applied with $8b$, then $\rho(\hat\Theta) \ge 1 - (8b)^{\mathsf{c}}$. This and (C.4) now imply that for every $r \in \bar J$, we have $\rho(\hat\Theta \cap \Theta'_r) \ge Cb^{\mathsf{c}}/2$ so long as $L \ge 16$.

We conclude that

$$\begin{aligned} 0.5C^2b^{2\mathsf{c}} &\le \int_{\bar J} \rho(\hat\Theta \cap \Theta'_r)\,\mathrm{d}r \\ &\le \int_{\hat\Theta} |\{r : m^b_\rho(r, \xi_r(w)) > \bar C\Upsilon^{1+7\mathsf{c}}b^\alpha\}|\,\mathrm{d}\rho. \end{aligned}$$

Therefore, there exists some $w_0 \in \hat\Theta$ so that

$$\Big|\big\{r \in [0,1] : m_\rho^b\big((r, \xi_r(w_0))\big) > \bar{C}\Upsilon^{1+7\mathsf{c}} b^\alpha\big\}\Big| \geq 0.5C^2 b^{2\mathsf{c}}. \tag{C.5}$$

For every $r \in [0,1]$, let $I \subset \{(r,s) : s \in \mathbb{R}\}$ be an interval of length b containing $(r, \xi_r(w_0))$. Put

$$I_{+,b} = \big\{(q_1,q_2) \in [r-b, r+b] \times \mathbb{R} : \exists (r,s) \in I, |q_2 - s| \leq b\big\}.$$

If $(q_1,q_2) \in I_{+,b}$, then $|q_1 - r| \leq b$ and $|q_2 - \xi_r(w_0)| \leq 2b$. Therefore,

$$|q_2 - \xi_{q_1}(w_0)| \leq |q_2 - \xi_r(w_0)| + |\xi_r(w_0) - \xi_{q_1}(w_0)| \leq 8b.$$

We conclude that $(q_1,q_2) \in \Xi^{8b}(w_0)$. This and $m_\rho^b\big((r,\xi_r(w_0))\big) > \bar{C}\Upsilon^{1+7\mathsf{c}} b^\alpha$ imply that for every $q \in I_{+,b}$, we have

$$m_\rho^{8b}(q) \geq \rho\big(\{w' \in E : (r, \xi_r(w')) \in I\}\big) \geq \bar{C}\Upsilon^{1+7\mathsf{c}} b^\alpha. \tag{C.6}$$

Combining (C.5) and (C.6), we obtain that

$$\begin{aligned}\Big|\Xi^{8b}(w_0) \cap \{q \in \mathbb{R}^2 : m_\rho^{8b}(q) \geq \bar{C}\Upsilon^{1+7\mathsf{c}} b^\alpha\}\}\Big| &\gg C^2 b^{1+2\mathsf{c}} \\ &\gg C^2 b^{2\mathsf{c}} |\Xi^{8b}(w_0)| > b^{2\mathsf{c}/\alpha} |\Xi^{8b}(w_0)|\end{aligned}$$

where the implied constant is absolute, and we assume L (and hence C) is large enough so that the final estimate holds — recall that $0 < \alpha \leq 1$.

This contradicts the fact that $w_0 \in \hat\Theta$ and finishes the proof. $\square$

Proof of Theorem 6.2. We will work with dyadic scales. Let $\ell_1 = \lfloor \frac{1}{\alpha} \log \Upsilon \rfloor$. Let L be as in Theorem C.1; put $C = L\mathsf{c}^{-L}$ and $\bar{C} = C \cdot (\#\Theta)^{-1}$.

Let $\ell_2 = 20 + \lfloor \mathsf{c} \log \Upsilon \rfloor$. Then

$$\sum_{\ell=\ell_2}^{\infty} 2^{-\mathsf{c}\ell} < 10^{-6} \Upsilon^{-\mathsf{c}^2}.$$

Let $J' = \bigcap_{\ell=\ell_2}^{\ell_1} J_{2^{-\ell}}$. Then the choice of ℓ_2 and Theorem C.1 imply that

$$|J \setminus J'| \leq C\Upsilon^{-\mathsf{c}^2}.$$

For every $r \in J'$, let $\Theta_r = \bigcap_{\ell=\ell_2}^{\ell_1} \Theta_{2^{-\ell},r}$. Then by Theorem C.1,

$$\rho(\Theta \setminus \Theta_r) \leq C\Upsilon^{-\mathsf{c}^2}.$$

Moreover, for all $w \in \Theta_r$ and all $\ell_2 \leq \ell \leq \ell_1$ we have

$$\rho(\{w' \in \Theta : |\xi_r(w') - \xi_r(w)| \leq 2^{-\ell}\}) \leq \bar{C}\Upsilon^{1+7\mathsf{c}} 2^{-\alpha\ell}. \tag{C.7}$$

Let $w \in \Theta_r$, and put $\Theta(w) = \Theta \setminus \{w' \in \Theta : |\xi_r(w') - \xi_r(w)| \leq 2^{-\ell_1}\}$. In view of (C.7), applied with $\ell = \ell_1$, we have

$$\#(\Theta \setminus \Theta(w)) \leq 2C\Upsilon^{7\mathsf{c}}. \tag{C.8}$$

Moreover, (C.7) applied with $\ell_2 \le \ell \le \ell_1$, implies that

$$\sum_{w'\in\Theta(w)} \|\xi_r(w)-\xi_r(w')\|^{-\alpha} \le (\#\Theta)\cdot\Big(\sum_{\ell=\ell_2}^{\ell_1} \bar{C}\Upsilon^{1+7\mathsf{c}}2^{-\alpha\ell}2^{\alpha\ell}+2^{\alpha\ell_2}\Big) \tag{C.9}$$
$$= \ell_1 C\Upsilon^{1+7\mathsf{c}} + 2^{\alpha\ell_2}\cdot(\#\Theta).$$

Recall that $\#\Theta \le \Upsilon$ and that $2^{\alpha\ell_2} \le 2^{20}\Upsilon^{\mathsf{c}}$. The claim in the theorem thus follows from (C.8) and (C.9). □

We also need the following theorem which was used in §13, in particular in the proof of Lemma 13.4. We will reduce this to the results proved in [LM21, App. B], these results have now been obtained in greater generality, see [PYZ22].

C.3. **Theorem.** *Let $0<\alpha\le 1$, and let $0<b_1<b_0\le 1$. Let $\Theta\subset B_{\mathfrak{r}}(0,b_0)$ be a finite set, and let θ denote a probability measure on Θ. Assume further that the following two properties hold*

$$K^{-1}\le\theta(w)\le K \tag{C.10a}$$

$$\theta(B_{\mathfrak{r}}(w,b))\le\bar{\Upsilon}\cdot(b/b_0)^{\alpha}\qquad\textit{for all } w \textit{ and all } b\ge b_1 \tag{C.10b}$$

where $\bar{\Upsilon}\ge 1$ and K is absolute.

Let $0<\mathsf{c}<0.01\alpha$, and let $J\subset[0,1]$ be an interval with $|J|\ge 10^{-4}$. For every $b\ge b_1$, there exists a subset $J_b\subset J$ with $|J\setminus J_b|\ll b^{\mathsf{c}}$ so that the following holds. Let $r\in J_b$, then there exists a subset $\Theta_{b,r}\subset\Theta$ with

$$\theta(\Theta\setminus\Theta_{b,r})\ll b^{\mathsf{c}}$$

such that for all $w\in\Theta_{b,r}$, we have

$$\theta\big(\{w'\in\Theta : |\zeta_r(w')-\zeta_r(w)|\le b\}\big)\le C(b/b_0)^{\alpha-7\mathsf{c}}$$

where $C\ll\mathsf{c}^{-\star}\bar{\Upsilon}$, the implied constants are absolute and $\zeta_r(w)$ is defined as follows:

$$u_r\exp(w)u_{-r}=\begin{pmatrix} d_{r,w} & 0\\ c_{r,w} & 1/d_{r,w}\end{pmatrix}\begin{pmatrix}1 & \zeta_r(w)\\ 0 & 1\end{pmatrix}.$$

Proof. In view of the assumption (C.10a), it suffices to prove the claim when θ is the uniform measure on Θ.

Define $f : B_{\mathfrak{r}}(0,0.01)\to G$ by

$$f\left(\begin{pmatrix} w_{11} & w_{12}\\ w_{21} & -w_{11}\end{pmatrix}\right)=\begin{pmatrix}1+w_{11} & w_{12}\\ w_{21} & \frac{1+w_{12}w_{21}}{1+w_{11}}\end{pmatrix}.$$

There exists an absolute constant δ_0 so that the map $g=f^{-1}\circ\exp$ is a diffeomorphism from $B_{\mathfrak{r}}(0,\delta_0)$ onto its image and

$$\|Dg-I\|\le 0.01. \tag{C.11}$$

We may, without loss of generality, assume that $\Theta \subset B_{\mathfrak{r}}(0,\delta_0)$. Let $\Theta' = g(\Theta)$. Then, in view of (C.10b) and (C.11), we have

$$\frac{\#B_{\mathfrak{r}}(w,b)\cap\Theta'}{\#\Theta'} \le 2\bar{\Upsilon}\cdot(b/b_0)^{\alpha} \qquad \text{for all } w \text{ and all } b\ge b_1. \tag{C.12}$$

Moreover, for any $w\in B_{\mathfrak{r}}(0,\delta_0)$, we have

$$u_r\exp(w)u_{-r} = u_r f(g(w))u_{-r}.$$

Therefore, it suffices to prove the theorem with exp replaced by g.

Altogether, it suffices to prove the theorem for $\check{\zeta}_r$ defined as follows

$$u_r\begin{pmatrix}1+w_{11} & w_{12}\\ w_{21} & \frac{1+w_{12}w_{21}}{1+w_{11}}\end{pmatrix}u_{-r} = \begin{pmatrix}d'_{r,w} & 0\\ c'_{r,w} & 1/d'_{r,w}\end{pmatrix}\begin{pmatrix}1 & \check{\zeta}_r(w)\\ 0 & 1\end{pmatrix},$$

and when θ is the counting measure.

The above definition, implies that

$$\check{\zeta}_r(w) = \frac{w_{12} + \frac{w_{12}w_{21}-2w_{11}-w_{11}^2}{1+w_{11}}r - w_{21}r^2}{1+w_{11}+w_{21}r};$$

define $\check{Z}(w) = \{(r,\check{\zeta}_r(w)) : r\in[0,1]\}$.

We also define $\Phi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ by

$$\Phi(x,y) = y_2(1+x_1) + \frac{(2x_1+x_1^2)y_1 + (x_2+x_1x_2)y_1^2}{1+x_1+x_2y_1}.$$

Note that $\Phi(0,y)=y_2$ and that

$$\check{Z}(w) = \{y\in\mathbb{R}^2 : y_1\in[0,1], \Phi(w_{11},w_{21},y) = w_{12}\}.$$

Assuming $|x_i|\le 0.1$ and $|y_i|\le 1$, a direct calculation shows that

$$\frac{\partial\Phi}{\partial y_1} = \frac{(1+x_1)(x_1^2+2x_1+2x_2(1+x_1)y_1+x_2^2y_1^2)}{(1+x_1+x_2y_1)^2}$$
$$\frac{\partial^2\Phi}{\partial y_1^2} = \frac{2(1+x_1)x_2}{(1+x_1+x_2y_1)^3}.$$

In particular, there exists some absolute constant C so that

$$\tfrac{1}{C}\max\{|x_1|,|x_2|\} \le |\tfrac{\partial\Phi}{\partial y_1}| + |\tfrac{\partial^2\Phi}{\partial y_1^2}| \le C\max\{|x_1|,|x_2|\}. \tag{C.13}$$

In view of [KW99, Eq. (21)], thus, the family $\check{Z}$ satisfies the *cinematic* curvature conditions [Zah12a, Eq. (1.5) and (1.6)].

For two curves $\check{Z} = \{y\in\mathbb{R}^2 : y_1\in[0,1], \Phi(w_{11},w_{21},y) = w_{12}\}$ and $\check{Z}' = \{y'\in\mathbb{R}^2 : y'_1\in[0,1], \Phi(w'_{11},w'_{21},y') = w'_{12}\}$, define

$$\Delta(\check{Z},\check{Z}') = \inf_{y\in\check{Z},y'\in\check{Z}'}\|y-y'\| + \Big|\frac{d_y\Phi(w_{11},w_{21},y)}{\|d_y\Phi(w_{11},w_{21},y)\|} - \frac{d_y\Phi(w'_{11},w'_{21},y)}{\|d_y\Phi(w'_{11},w'_{21},y')\|}\Big|;$$

this provides a quantitative tool to study incidence of $\check{Z}$ and $\check{Z}'$.

In view of (C.13), we may apply the results[4] in [Zah12b]. Therefore, the proof of the theorem goes through the same lines as the proof of [LM21, Thm. B.1] (see also the proof of Theorem C.1) if we replace the family Ξ there by the family $\check{Z}$ and Δ there by Δ above. □

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E.L.: The Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel

E-mail address: elon@math.huji.ac.il

A.M.: Department of Mathematics, University of California, San Diego, CA 92093

E-mail address: ammohammadi@ucsd.edu

Z.W.: Pennsylvania State University, Department of Mathematics, University Park, PA 16802

E-mail address: zhirenw@psu.edu