EFFECTIVE COUNTING OF SIMPLE CLOSED GEODESICS ON HYPERBOLIC SURFACES

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ABSTRACT. We prove a quantitative estimate, with a power saving error term, for the number of simple closed geodesics of length at most \( L \) on a closed hyperbolic surface of genus \( g \). The proof relies on the exponential mixing rate for the Teichmüller geodesic flow.

1. Introduction

Let \( g \geq 2 \) and let \( S \) be a compact Riemann surface of genus \( g \). Let \( \mathcal{T}(S) \) be the Teichmüller space of complete hyperbolic metrics on \( S \) and let

\[
\mathcal{M}(S) = \mathcal{T}(S)/\text{Mod}_g
\]

be the corresponding moduli space, where \( \text{Mod}_g \) is the mapping class group of \( S \).

Let \( M \in \mathcal{M}(S) \). Problems related to the asymptotic growth rate of the number of closed geodesics on \( M \) have been long studied. In particular, thanks to works of Delsart, Huber, and Selberg we have the following: There exists some \( \delta = \delta(M) > 0 \) so that the number of closed geodesics of length at most \( L \) on \( M \) equals

\[
\text{Li}(e^L) + O_M(e^{L-\delta}),
\]

where \( \text{Li}(x) = \int_2^x \frac{dt}{\log t} \), see [Bus] and references there.

More generally, the growth rate of the number of closed geodesics on a negatively curved compact manifold was studied by Margulis, [Mar]. His proof, which is different from the above mentioned works, is based on the mixing property of the Margulis measure for the geodesic flow. In the constant negative curvature, Margulis’ method combined with an exponential mixing rate for the geodesic flow, also provides an estimate like \([1]\) — albeit with a weaker power saving \( \delta \), see e.g. [MMO].

1.1. Simple closed geodesics. The aforementioned fundamental results do not provide any estimates for the number of simple closed geodesics on \( M \). Indeed, very few closed geodesics on \( M \) are simple, [BS2], and it is hard to discern them in \( \pi_1(M) \), [BS1]. More explicitly, it was shown in [Ri] that the number of simple closed geodesics of length at most \( L \) on \( M \) is bounded above and below by \( O_M(L^{6g-6}) \).

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In her PhD thesis, \cite{Mir1, Mir2}, Mirzakhani proved an asymptotic growth rate for the number of simple closed geodesics of a given topological type on $M$ — recall that two simple closed geodesics $\gamma$ and $\gamma'$ on $M$ are of the same topological type if there exists some $g \in \text{Mod}_g$ so that $\gamma' = g\gamma$.

By a multi-geodesic $\gamma$ on $M$ we mean $\gamma = \sum_{i=1}^{d} a_i \gamma_i$ where $\gamma_i$'s are disjoint, essential, simple closed geodesics, and $a_i > 0$ for all $1 \leq i \leq d$. In this case we define $\ell_M(\gamma) := \sum a_i \ell_M(\gamma_i)$, where $\ell_M$ denotes the hyperbolic length function on $M$. The multi-geodesic $\gamma$ will be called integral (resp. rational) if $a_i \in \mathbb{N}$ (resp. $a_i \in \mathbb{Q}$).

Given a rational multi-geodesic $\gamma_0$ on $M$ define

$$s_M(\gamma_0, L) := \# \{ \gamma \in \text{Mod}_g \cdot \gamma_0 : \ell_M(\gamma) \leq L \}.$$ 

Mirzakhani, \cite[Thm. 1.1]{Mir2}, proved that

$$s_M(\gamma_0, L) \sim n_{\gamma_0}(M)L^{6g-6},$$

where $n_{\gamma_0} : \mathcal{M}(S) \to \mathbb{R}^+$ (the Mirzakhani function) is a continuous proper function; geometric informations carried by $n_{\gamma_0}$ are also studied in \cite{Mir2}.

In this paper we proved the following.

**Theorem 1.1.** There exists some $\kappa = \kappa(g) > 0$ so that the following holds. Let $\gamma_0$ be a rational multi-geodesic on $M$. Then

$$s_M(\gamma_0, L) = n_{\gamma_0}(M)L^{6g-6} + O_{\gamma_0, M}(L^{6g-6-\kappa}).$$

The proof of Theorem 1.1 is based on the study of a related counting problem in the space of geodesic measured laminations on $S$, à la Mirzakhani. The space of measured laminations on $S$, which we denote by $\mathcal{ML}(S)$, is a piecewise linear integral manifold homeomorphic to $\mathbb{R}^{6g-6}$; but it does not have a natural differentiable structure, \cite{Th1}.

Train tracks were introduced by Thurston as a powerful technical device for understanding measured laminations. Roughly speaking, train tracks are induced by squeezing almost parallel strands of a very long simple closed geodesic to simple arcs on a hyperbolic surface; they provide linear charts for $\mathcal{ML}(S)$.

The mapping class group $\text{Mod}_g$ of $S$ acts naturally on $\mathcal{ML}(S)$. Moreover, there is a natural $\text{Mod}_g$-invariant locally finite measure on $\mathcal{ML}(S)$, the Thurston measure $\mu_{\text{Th}}$, given by the piecewise linear integral structure on $\mathcal{ML}(S)$, \cite{Th1}. For any open subset $U \subset \mathcal{ML}(S)$ and any $t > 0$, we have

$$\mu_{\text{Th}}(tU) = t^{6g-6}\mu_{\text{Th}}(U).$$

On the other hand, any complete hyperbolic structure $M$ on $S$ induces the length function $\lambda \mapsto \ell_M(\lambda)$ on $\mathcal{ML}(S)$, which satisfies $\ell_M(t\lambda) = t\ell_M(\lambda)$ for all $t > 0$. It is proved in \cite[App. A]{Mir1} that $\ell_M$ is a convex function on $\mathcal{ML}(S)$.

The source of the polynomially effective error term in Theorem 1.1 is the exponential mixing property of the Teichmüller geodesic flow proved by Avila, Gouëzel, and Yoccoz, \cite{AGY, AR, AG}. We combine this estimate with ideas developed by Margulis in his PhD thesis, \cite{Mar}, to prove the following theorem which is of independent interest — see Theorem 7.1 for a more general statement.
Let $\tau$ be a train track and let $U(\tau)$ be the corresponding train track chart. For every $\lambda \in U(\tau)$ we let $\|\lambda\|_\tau$ denote the sum of the weights of $\lambda$ in $U(\tau)$, see §5.

**Theorem 1.2.** There exists some $\kappa_1 = [\kappa_1(g)] > 0$ so the following holds. Let $\tau$ be a maximal train track. Let $L \geq 1$ and let $\gamma_0$ be a simple close curve on $M$. There exists a constant $c_{\gamma_0} > 0$ so that

$$\# \{\gamma \in U(\tau) \cap \text{Mod}_g : \|\gamma\|_\tau \leq L\} = c_{\gamma_0} \text{vol}_\tau L^{6g-6} + O_{\tau,\gamma_0}(L^{6g-6-\kappa_1})$$

where $\text{vol}_\tau = \mu_{\text{Th}}\{\lambda \in U(\tau) : \|\lambda\|_\tau \leq 1\}$.

It is worth noting that in view of Theorem 1.2, the asymptotic behavior of the number of points in one $\text{Mod}_g$-orbit in the cone $\{\lambda : \|\lambda\|_\tau \leq L\}$ and that of the number of integral points in this cone agree up to multiplicative constant.

Theorem 1.2, in the more general form Theorem 7.1, plays a crucial role in our analysis. Indeed, using the aforementioned convexity of the hyperbolic length function we will prove Theorem 1.1 using Theorem 7.1 in §8.

It is an intriguing problem to investigate the asymptotic behavior of functions similar to $s_M(\gamma_0, L)$ or the complexity considered in Theorem 1.2. For instance, for a suitable formulation of a combinatorial length — using intersection numbers — the count is exactly a polynomial, see [FLP]. We also refer the reader to [CMP] where a related problem is studied for trice punctured sphere.

1.2. Outline of the paper. In §2 we collect some preliminary results. In §3 we prove an equidistribution result, Proposition 3.2, which may be of independent interest; see, e.g., [KM], [LMir1]. The proof of this proposition is based on the exponential mixing rate for the Teichmüller geodesic flow, [AGY], and the so called thickening technique, see [Mar], [EMc]. In §4 we prove Proposition 4.1; this proposition is one of the main ingredients in the proof and could be compared to arguments in [Mar, Chap. 6]. We will recall some basic facts about $\mathcal{ML}(S)$, and study the relation between the linear structures on $\mathcal{ML}(S)$ and the space of quadratic differentials in §5 and §6. The orbital counting in sectors of $\mathcal{ML}(S)$ is studied in §7; the main result here is Theorem 7.1. We prove Theorem 1.1 in §8.

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2. Preliminaries and notation

Let $Q(S)$ denote the moduli space of quadratic differentials on $S$ and let $Q_1(S)$ be the moduli space of quadratic differentials with area one on $S$. For any $\alpha = (\alpha_1, \ldots, \alpha_k, \varsigma)$ with $\sum \alpha_i = 4g - 4$ and $\varsigma \in \{\pm 1\}$, define $Q_1(\alpha)$ to be the stratum of quadratic differentials consisting of pairs $(M, q)$ where $M \in T(S)$ and $q$ is a unit area quadratic differential on $M$. 
whose zeros have multiplicities $\alpha_1, \ldots, \alpha_k$ and the sign $\zeta$ is equal to 1 if $q$ is the square of an abelian differential and $-1$ otherwise. Then

$$Q_1(S) = \bigcup_{\alpha} Q_1(\alpha).$$

Put $Q(\alpha) := \{tq : t \in \mathbb{R}, q \in Q_1(\alpha)\}$. Also let $\pi : Q^1T(\alpha) \to Q_1(\alpha)$ be the universal covering map.

Similarly, let $\Omega(S)$ denote the moduli space of Abelian differentials on $S$ and let $\Omega_1(S)$ be the moduli space of area one Abelian differentials. For any $\alpha = (\alpha_1, \ldots, \alpha_k)$ we let $H(\alpha)$ denote the corresponding stratum and let $H_1(\alpha)$ denote the area one abelian differentials.

Note that passing to a branched double cover $\tilde{M}$ of $M$, we may realize $Q_1(\alpha)$ as an affine invariant submanifold in $H_1(\tilde{\alpha})$ corresponding to odd cohomology classes on $\tilde{M}$, see §2.1. However, even if $q$ belongs to a compact subset of $Q_1(S)$, the complex structure on $M$ may have very short closed curves in the hyperbolic metric, e.g. a short saddle connection between two distinct zeros on $(M, q)$ could lift to a short loop in $\tilde{M}$. Note however that if $(\tilde{M}, \omega)$ is the aforementioned double cover of $(M, q)$, then the length of the shortest saddle connection in $\omega$ is bounded by the length of the shortest saddle connection in $q$, i.e. compact subsets of $Q_1(\alpha)$ lift to compact subsets of $H_1(\tilde{\alpha})$.

2.1. Period coordinates. Let $x = (M, \omega) \in H(\alpha)$ and let $\Sigma \subset M$ be the set of zeros of $\omega$. Define the period map

$$\Phi : H(\alpha) \to H^1(M, \Sigma, \mathbb{C}).$$

Let us recall that $\Phi$ can be defined as follows. Let $\# \Sigma = k$. Fix a triangulation $T$ of the surface by saddle connections of $x$, that is: $2g + k - 1$ directed edges $\delta_1, \ldots, \delta_{2g+k-1}$ which form a basis for $H_1(M, \Sigma, \mathbb{Z})$. Define

$$\Phi(x) = \left( \text{hol}_{x}(\delta_i) \right)^{2g+k-1}.$$

Note that this map depends on the triangulation $T$. If $T'$ is any other triangulation, and $\Phi'$ is the corresponding period map, then $\Phi' \circ \Phi^{-1}$ is linear. For any $x \in H(\alpha)$, there is a neighborhood $B(x)$ of $x$ so that the restriction of $\Phi$ to $B(x)$ is a homeomorphism onto $\Phi(B(x))$, see §2.6. We always choose $B(x)$ small enough so that, using the Gauss-Manin connection, the triangulation at $y \in B(x)$ can be identified with the triangulation at $x$.

We define the period coordinates at $x = (M, q) \in Q(\alpha)$ as follows. If $\zeta = 1$, then $q$ is square of an abelian differential and we may define period coordinates as above. If $\zeta = -1$, we use the orienting double cover $H(\tilde{\alpha})$ to define the period coordinates: in this case there is a canonical injection from $Q(\alpha)$ into $H(\tilde{\alpha})$. The image is identified with the moduli space of Riemann surfaces with an involution. This way we get the period map from $Q(\alpha)$ to $H^1_{\text{odd}}(M, \Sigma, \mathbb{C})$ — the anti-invariant subspace of the cohomology for the involution.

Put $h + 1 := 2g + k - 1$ if $\zeta = 1$, and $h + 1 := 2g + k - 2$ if $\zeta = -1$; the number $h$ is the topological entropy of the Teichmüller geodesic flow on $Q_1(\alpha)$. 
2.2. \textit{SL}(2, \mathbb{R})\text{-}action on \( \mathcal{H}_1(\alpha) \). Let \( x \in \mathcal{H}_1(\alpha) \), we write \( \Phi(x) \) as a \( 2 \times n \) matrix. The action of \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \) in these coordinates is linear. We choose some fundamental domain for the action of mapping class group and think of the dynamics on the fundamental domain. Then, the \( \text{SL}(2, \mathbb{R}) \)-action becomes

\[
\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} A(g, x),
\]

where \( A(g, x) \in \text{Sp}(2g, \mathbb{Z}) \rtimes \mathbb{R}^{k-1} \) is the Kontsevich-Zorich cocycle. That is: \( A(g, x) \) is the change of basis one needs to perform to return the point \( gx \) to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection restricted to the orbit of \( \text{SL}(2, \mathbb{R}) \).

In the sequel we let \( a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \) and \( \bar{u}_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \).

We have the following.

\textbf{Theorem 2.1} (Veech-Masur). The space \( \mathcal{H}_1(\alpha) \) carries a natural measure \( \mu \) in the Lebesgue measure class such that

1. \( \mathcal{H}_1(\alpha) \) has finite measure,
2. \( \mu \) is \( \text{SL}(2, \mathbb{R})\)-invariant and ergodic,

More generally, for any affine invariant manifold, \( \mathcal{M} \subset \mathcal{H}_1(\alpha) \), we let \( \mu \) denote the \( \text{SL}(2, \mathbb{R})\)-invariant affine measure on \( \mathcal{M} \). In particular, all the strata in \( Q_1(S) \) are equipped with such invariant measures.

2.3. \textit{Modified hodge norm}. Let \( M \) be a Riemann surface. By definition, \( M \) has a complex structure. Let \( \mathcal{H}_M \) denote the set of holomorphic 1-forms on \( M \). One can define the \textit{Hodge inner product} on \( \mathcal{H}_M \) by

\[
\langle \omega, \eta \rangle = \frac{1}{2} \int_M \omega \wedge \bar{\eta}.
\]

We have a natural map \( r : H^1(M, \mathbb{R}) \to \mathcal{H}_M \) which sends a cohomology class \( c \in H^1(M, \mathbb{R}) \) to the holomorphic 1-form \( r(c) \in \mathcal{H}_M \) such that the real part of \( r(c) \) (which is a harmonic 1-form) represents \( c \). We can thus define the Hodge inner product on \( H^1(M, \mathbb{R}) \) by \( \langle c_1, c_2 \rangle = \langle r(c_1), r(c_2) \rangle \). Then

\[
\langle c_1, c_2 \rangle = \int_M c_1 \wedge \ast c_2,
\]

where \( \ast \) denotes the Hodge star operator and we choose harmonic representatives of \( c_1 \) and \( \ast c_2 \) to evaluate the integral. We denote the associated norm by \( \| \cdot \|_M \). This is the \textit{Hodge norm}, see \cite{FK}.

If \( x = (M, \omega) \in \mathcal{H}_1(\alpha) \), we will often write \( \| \cdot \|_{H,x} \) to denote the Hodge norm \( \| \cdot \|_M \) on \( H^1(M, \mathbb{R}) \). Since \( \| \cdot \|_{H,x} \) depends only on \( M \), we have \( \| c \|_{H,kx} = \| c \|_{H,x} \) for all \( c \in H^1(M, \mathbb{R}) \) and all \( k \in \text{SO}(2) \).
Let \( E(x) = \text{span}\{\text{Re}(\omega), \text{Im}(\omega)\} \) — the space \( E(x) \) is often referred to as the standard space. We let \( p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R}) \) denote the natural projection; \( p \) defines an isomorphism between \( E(x) \) and \( p(E(x)) \subset H^1(M, \mathbb{R}) \).

For our applications in the sequel (and in order to account for the loss of hyperbolicity in the thin part of the moduli space) we need to consider a modification of the Hodge norm.

The classes \( c_\alpha \) and \( *c_\alpha \). Let \( \alpha \) be a homology class in \( H_1(M, \mathbb{R}) \). We let \( *c_\alpha \in H^1(M, \mathbb{R}) \) be the cohomology class so that

\[
\int \alpha \omega = \int_M \omega \wedge *c_\alpha
\]

for all \( \omega \in H^1(M, \mathbb{R}) \). Then,

\[
\int_M *c_\alpha \wedge *c_\beta = i(\alpha, \beta),
\]

where \( i(\cdot, \cdot) \) denotes the algebraic intersection number. Let \( * \) denote the Hodge star operator, and let

\[
c_\alpha = *^{-1}(c_\alpha).
\]

Then, for any \( \omega \in H^1(M, \mathbb{R}) \) we have

\[
\langle \omega, c_\alpha \rangle = \int_M \omega \wedge *c_\alpha = \int_\alpha \omega,
\]

where \( \langle \cdot, \cdot \rangle \) is the Hodge inner product. We note that \( *c_\alpha \) is a purely topological construction which depends only on \( \alpha \), but \( c_\alpha \) depends also on the complex structure of \( M \).

Fix \( \epsilon_* > 0 \) (the Margulis constant) so that any two geodesics of hyperbolic length less than \( \epsilon_* \) must be disjoint.

Let \( \sigma \) denote the hyperbolic metric in the conformal class of \( M \). For any closed curve \( \alpha \) on \( M \), let \( \ell_M(\alpha) \) denote the length of the geodesic representative of \( \alpha \) in the metric \( \sigma \).

We recall the following.

**Theorem 2.2.** [ABEM, Thm. 3.1] For any constant \( L > 1 \) there exists a constant \( c > 1 \), such that for any simple closed curve \( \alpha \) with \( \ell_M(\alpha) < L \), we have

\[
\frac{1}{c} \ell_M(\alpha)^{1/2} \leq \|c_\alpha\|_M < c \ell_M(\alpha)^{1/2}.
\]

Furthermore, if \( \ell_M(\alpha) < \epsilon_* \) and \( \beta \) is the shortest simple closed curve crossing \( \alpha \), then

\[
\frac{1}{c} \ell_M(\alpha)^{-1/2} \leq \|c_\beta\|_M < c \ell_M(\alpha)^{-1/2}.
\]

**Short bases.** Suppose \( (M, \omega) \in \mathcal{H}_1(\alpha) \). Fix \( \epsilon_1 < \epsilon_* \) and let \( \alpha_1, \ldots, \alpha_k \) be the curves with hyperbolic length less than \( \epsilon_1 \) on \( M \). For every \( 1 \leq i \leq k \), let \( \beta_i \) be the shortest curve in the flat metric defined by \( \omega \) with \( i(\alpha_i, \beta_i) = 1 \). We can pick simple closed curves \( \gamma_r \), \( 1 \leq r \leq 2g - 2k \) on \( M \) so that the hyperbolic length of each \( \gamma_r \) is bounded by a constant \( L \) depending only on the genus, and so that the \( \alpha_j, \beta_j \) and \( \gamma_j \) form a symplectic basis \( S \) for \( H_1(M, \mathbb{R}) \). We will call such a basis short. A short basis is not unique, and in the following we fix some measurable choice of a short basis at each point of \( \mathcal{H}_1(\alpha) \).
We recall the definition of a modified Hodge norm from [EMM]; this is similar (but not the same) to the one defined in [ABEM]. The modified norm is defined on the tangent space to the space of pairs \((M, \omega)\) where \(M\) is a Riemann surface and \(\omega\) is a holomorphic 1-form on \(M\). Unlike the Hodge norm, the modified Hodge norm will depend not only on the complex structure on \(M\) but also on the choice of a holomorphic 1-form \(\omega\) on \(M\). Let \(\{\alpha_i, \beta_i, \gamma_r\}_{1 \leq i \leq k, 1 \leq r \leq 2g-2k}\) be a short basis for \(x = (M, \omega)\).

We can write any \(\theta \in H^1(M, \mathbb{R})\) as

\[
\theta = \sum_{i=1}^k a_i (\ast c_{\alpha_i}) + \sum_{i=1}^k b_i \ell_{\alpha_i}(\sigma)^{1/2}(\ast c_{\beta_i}) + \sum_{r=1}^{2g-2k} u_r (\ast c_{\gamma_r}).
\]

We then define

\[
\|\theta\|''_x = \|\theta\|_{H,x} + \left( \sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| + \sum_{r=1}^{2g-2k} |u_r| \right).
\]

Note that \(\| \cdot \|''\) depends on the choice of a short basis; however, switching to a different short basis can change \(\| \cdot \|''\) by at most a fixed multiplicative constant depending only on the genus. To manage this, we use the notation \(A \approx B\) to denote the fact that \(A/B\) is bounded from above and below by constants depending on the genus.

From (5) we have: for \(1 \leq i \leq k,\)

\[
\| \ast c_{\alpha_i} \|''_x \asymp 1,
\]

see §2.9 for the notation \(\asymp\). Similarly, we have

\[
\| \ast c_{\beta_i} \|''_x \asymp \frac{1}{\ell_M(\alpha_i)^{1/2}}.
\]

In addition, in view of Theorem 2.2 if \(\gamma\) is any other moderate length curve on \(M,\) \(\| \ast c_{\gamma} \|''_x \asymp \| \ast c_{\gamma} \|_{H,x} = O(1).\) Thus, if \(B\) is a short basis at \(x = (M, \omega),\) then for any \(\gamma \in B,\)

\[
\text{Ext}_\gamma(x)^{1/2} \asymp \| \ast c_{\gamma} \|_{H,x} \leq \| \ast c_{\gamma} \|''_x.
\]

By \(\text{Ext}_\gamma(x)\) we mean the extremal length of \(\gamma\) in \(M,\) where \(x = (M, \omega).\)

**Remark.** From the construction, we see that the modified Hodge norm is greater than the Hodge norm. Also, if the flat length of shortest curve in the flat metric defined by \(\omega\) is greater than \(\epsilon_1,\) then for any cohomology class \(c,\) for some \(N\) depending on \(\epsilon_1\) and the genus,

\[
\|c\|'' \leq N \|c\|_{H,x};
\]

i.e., the modified Hodge norm is within a multiplicative constant of the Hodge norm.

Note however that for a fixed absolute cohomology class \(c,\|c\|''\) is not a continuous function of \(x,\) as \(x\) varies in a Teichmüller disk; this is due to the dependence on the choice of a short basis. To remedy this, we pick a positive, continuous, \(\text{SO}(2)\)-bi-invariant function \(\phi\)
on $\text{SL}(2, \mathbb{R})$ which is supported on a neighborhood of the identity with $\int_{\text{SL}(2, \mathbb{R})} \phi(g) \, dg = 1$, and define
\[ \|c\|_x' = \|c\|_{H,x} + \int_{\text{SL}(2, \mathbb{R})} \|c\|_{qg}^p \phi(g) \, dg. \]

It follows from [EMM, Lemma 7.4] that for a fixed $c$, $\log \|c\|_x'$ is uniformly continuous as $x$ varies in a Teichmüller disk. In fact, there is a constant $m_0$ such that for all $x \in \mathcal{H}_1(\alpha)$, all $c \in H^1(M, \mathbb{R})$ and all $t > 0$,
\[ e^{-m_0 t} \|c\|_x' \leq \|c\|_{al,x} \leq e^{m_0 t} \|c\|_x'. \]

**Remark 2.3.** Even though $\|\cdot\|_x'$ is uniformly continuous as long as $x$ varies in a Teichmüller disk, it may be only measurable in general (because of the choice of short basis).

### 2.4. Relative cohomology.

For $c \in H^1(M, \Sigma, \mathbb{R})$ and $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, let $p_x(c)$ denote the harmonic representative of $p(c)$, where $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ is the natural map. We view $p_x(c)$ as an element of $H^1(M, \Sigma, \mathbb{R})$. Then, (similarly to [EMM §7], see also [ABEM] and [EMR]) we define the modified Hodge norm $\|\cdot\|'$ on $H^1(M, \Sigma, \mathbb{R})$ as follows.
\[ \|c\|_x' = \|p(c)\|_x' + \sum_{(z,z') \in \Sigma \times \Sigma} \left| \int_{\gamma_{z,z'}} (c - p_x(c)) \right|, \]
where $\gamma_{z,z'}$ is any path connecting the zeroes $z$ and $z'$ of $\omega$. Since $c - p_x(c)$ represents the zero class in absolute cohomology, the integral does not depend on the choice of $\gamma_{z,z'}$. Note that the $\|\cdot\|'$ norm on $H^1(M, \Sigma, \mathbb{R})$ is invariant under the action of $\text{SO}(2)$.

As above, we pick a positive continuous $\text{SO}(2)$-bi-invariant function $\phi$ on $\text{SL}(2, \mathbb{R})$ supported on a neighborhood of the identity such that $\int_{\text{SL}(2, \mathbb{R})} \phi(g) \, dg = 1$, and define
\[ \|c\|_x = \int_{\text{SL}(2, \mathbb{R})} \|c\|_{qg}^p \phi(g) \, dg. \]

Then, the $\|\cdot\|_x$ norm on $H^1(M, \Sigma, \mathbb{R})$ is also invariant under the action of $\text{SO}(2)$.

By [EMM] Lemma 7.5 there exists some $N_1$ so that
\[ e^{-N_1} \|c\|_x \leq \|c\|_{al,x} \leq e^{N_1} \|c\|_x. \]

### 2.5. The AGY-norm.

We will also denote by $\|\cdot\|_{\text{AGY},x}$ the norm defined in [AGY §2.2.2]; let us recall the definition. Let $x = (M, \omega) \in \mathcal{H}_1(\alpha)$, for any $c \in H^1(M, \Sigma, \mathbb{C})$ define
\[ \|c\|_{\text{AGY},x} = \sup_{\gamma} \frac{|c(\gamma)|}{|\Phi(x)(\gamma)|}, \]
where the supremum is taken over all saddle connections of $\omega$. This defines a norm and the corresponding Finsler metric is complete, [AGY].

We note that any two $\text{Mod}_q$-invariant norms, in particular, $\|\cdot\|_x$ and $\|\cdot\|_{\text{AGY},x}$, are commensurable to each other on compact subsets of $\mathcal{H}_1(\alpha)$.

For any $x = (M, q) \in \mathcal{Q}_1(\alpha)$ we define the norms $\|\cdot\|_x$ and $\|\cdot\|_{\text{AGY},x}$ using the branched double cover $\tilde{M}$.  

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2.6. **Period box.** Let \( x = (M, q) \in Q_1(\alpha) \). For every \( r > 0 \) define
\[
R_r(x) := \{ \Phi(x) + a' + ib' : a', b' \in H^1(M, \Sigma, \mathbb{R}), \|a' + ib'\|_{AGY,x} \leq r \}.
\]
Let now \( r > 0 \) be so that \( \Phi^{-1} \) is a homeomorphism on \( R_r(x) \cap \Phi(Q_1(\alpha)) \). Put
\[
B_r(x) = \Phi^{-1}(R_r(x)).
\]
The open subset \( B_r(x) \) will be called a *period box* of radius \( r \) centered at \( x \).

Thanks to [AG, Prop. 5.3](#), the above definition is well defined for any radius \( 0 < r \leq 1/2 \) and *any* \( x \).

2.7. **Horospherical foliation.** Given a point \( x = (M, q) \in Q_1(\alpha) \), the tangent space \( T_xQ_1(\alpha) \) decomposes as
\[
T_xQ_1(\alpha) = \mathbb{R}v(x) \oplus E^u(x) \oplus E^s(x)
\]
where \( v(x) \) is the direction of the Teichmüller geodesic flow,
\[
E^u(x) = T_xQ_1(\alpha) \cap \Phi(x)^{-1}(H^1(\xi, \mathbb{R})),
\]
\[
E^s(x) = T_xQ_1(\alpha) \cap \Phi(x)^{-1}(H^1(\xi, i\mathbb{R})).
\]
where \((\xi, \xi) = (M, \Sigma)\) if \( \zeta = 1 \) and \((\hat{\xi}, \xi) = (\hat{M}, \hat{\Sigma})\) if \( \zeta = -1 \) — recall that \( \hat{M} \) is the orienting double cover of \( M \) and we use \( \Phi \) to locally identify \( \mathbb{R}Q_1(\alpha) \) with \( H^1(M, \Sigma, \mathbb{C}) \) if \( \zeta = 1 \) and with the \( H^1_{od}(\hat{M}, \hat{\Sigma}, \mathbb{C}) \) if \( \zeta = -1 \).

If \( \Phi(x) = a + ib \) for some \( x \in Q_1(\alpha) \), then
\[
E^u(x) = \{ a' \in H^1(M, \Sigma, \mathbb{R}) : i(a', b) = 0 \}.
\]
Similarly \( E^s(x) = \{ b' \in H^1(M, \Sigma, i\mathbb{R}) : i(a, b') = 0 \} \) when \( \zeta = 1 \), and similarly one can define \( E^{u,s} \) in the case \( \zeta = -1 \).

\( E^{u,s}(x) \) depend on \( x \) in a smooth way, moreover, they are integrable; we denote the corresponding leaves by \( W^u(x) \) and \( W^s(x) \), respectively. Also put \( W^{cu}(x) := \{ a_tW^u(x) : t \in \mathbb{R} \} \) and \( W^{cs}(x) := \{ a_tW^s(x) : t \in \mathbb{R} \} \).

Let \( \mu^u_x \) and \( \mu^s_x \) denote the conditional measures of the natural measure \( \mu \) along \( W^u(x) \) and \( W^s(x) \), respectively. Then \( y \mapsto \mu^{u,s}_y \) is constant along \( W^{u,s}(x) \), respectively, and we have
\[
(a_t)_*\mu^u_x = e^{-ht}\mu^u_{a_tx} \quad \text{and} \quad (a_t)_*\mu^s_x = e^{ht}\mu^s_{a_tx}.
\]
Moreover, if \( B_r(x) \) is a period box centered at \( x \), then \( \mu|_{B_r(x)} \) has a product structure as \( d\text{Leb} \times d\mu^u_x \times d\mu^s_x \), see e.g. [AG, Prop. 4.1](#).

Given \( x \in Q_1(\alpha) \) and a period box \( B_r(x) \) with center \( x \) we let
\[
B^{u,s}_r(x) = \text{the connected component of } x \text{ in } B_r(x) \cap W^{u,s}(x).
\]
Define \( B^*_r(x) \) for \( * = \text{cu, cs} \) similarly.

We also denote functions which are supported on the leaves \( W^u, W^{cu} \), etc. using the same upper script, e.g., \( \phi^u \) denotes a function which is supported on a leaf \( W^u(x) \).
We use the norm \( \| \cdot \|_{AGY,x} \) to induce a metric \( d_{W^{u,s}(x)} \) on \( B_{W}^{u,s}(x) \) for \( 0 < r < 1/2 \). Hence notions such as diam etc. refer to this metric.

Given \( \tilde{x} \in Q^1 T(\alpha) \) we let \( B_{T}(\tilde{x}) \) be the ball of radius \( r \) centered at \( \tilde{x} \) in \( Q^1 T(\alpha) \). Let \( \tilde{W}^\bullet(\tilde{x}) \) denote the foliation \( \bullet \) in \( Q^1 T(\alpha) \) and define \( B^\bullet(\tilde{x}) \) accordingly.

### 2.8. Mapping class group action

We denote elements in \( \text{Mod}_g \) using bold letters, e.g., \( g \) denotes an element in \( \text{Mod}_g \). The action of \( \text{Mod}_g \) on \( Q^1 T(\alpha) \) commutes with the action of \( \text{SL}(2, \mathbb{R}) \), we will however denote both these actions as left action and write, e.g., \( g \cdot \tilde{x} \), \( g \cdot \tilde{W}^\bullet(\tilde{x}) \), and \( g \cdot a_t \tilde{W}^\bullet(\tilde{x}) \); often \( g \cdot \tilde{x} \) will be simply denoted by \( g \tilde{x} \).

### 2.9. The constants

In the sequel we will use \( \kappa_\bullet \) and \( N_\bullet \), \( \bullet = 1, 2, \ldots \) to denote various constants. Unless it is explicitly mentioned otherwise, these constants are allowed only to depend on the genus. The constants \( \kappa_\bullet \) are meant to indicate small positive numbers while \( N_\bullet \) are used for constants which are expected to be \( > 1 \).

We will also use the notation \( A \ll B \). This expression means: there exists a constant \( c > 0 \) so that \( A \leq cB \); the implicit constant \( c \) is permitted to depend on the genus, but (unless otherwise noted) not on anything else. We write \( A \asymp B \) if \( A \ll B \ll A \). If a constant (implicit or explicit) depends on another parameter others than the genus, we will make this clear by writing, e.g., \( \ll_{\epsilon}, C(x) \), etc.

We also adopt the following \( \ast \)-notation. We write \( B = A^{\pm \ast} \) if \( B = cA^\epsilon \) or \( B = cA^N \) where \( \kappa \) and \( N \) depend only on the genus. Similarly one defines \( B \ll A^\ast, B \gg A^\ast \). Finally we also write \( A \asymp B^\ast \) if \( A^\ast \ll B \ll A^\ast \) (possibly with different exponents). 

### 2.10. Smooth structure on affine manifolds

As it is done in [AG] §5.2 we use the affine structure to define a smooth structure on \( Q^1 T(\alpha) \) and \( Q_1(\alpha) \). Let us recall the definition of a \( C^k \)-norm from [AG], see also [AGY].

Let \( W \subset Q_1(\alpha) \) be an affine submanifold. For a function \( \varphi \) on \( W \) define

\[
c_k(\varphi) = \sup |D^k \varphi(x, v_1, \ldots, v_k)|,
\]

where the supremum is taken over \( x \) in the domain of \( \varphi \) and \( v_1, \ldots, v_k \in T_x W \) with AGY-norm at most 1. Define the \( C^k \)-norm of \( \varphi \) as \( \| \varphi \|_{C^k} = \sum_{j=0}^k c_j(\varphi) \).

By a \( C^k \) function we mean a function whose \( C^k \)-norm is finite. In the sequel we will only need \( C^1 \)-norm of functions. To avoid confusion between this norm and other relevant norms which will be used, and also since we often use the letter \( C \) to denote various constants, for any \( C^1 \) function \( \varphi \) we define

\[
C^1(\varphi) := \| \varphi \|_{C^1}.
\]

In the sequel we will need to replace the characteristic functions of certain sets with their smooth approximations. The following lemmas will provide such approximations.

**Lemma 2.4** (Cf. [AG], Prop. 5.8). There exist \( N_2 \) and \( N_3 \) so that the following holds. Let \( x \in Q_1(\alpha) \) and let \( D \subset W^u(x) \) be a compact set. There exists a finite collection \( \{ \varphi_i \} \) of \( C^\infty \) functions on \( W^u(x) \) with the following properties:
Lemma 2.5. There exists some $S$.

Similarly, if $0 \leq \epsilon$ then

Let $E$ be a compact subset. For any $0 < \epsilon < 0.01$ define

$E_{+\epsilon} = \{y \in Q_1(\alpha) : B_{\epsilon}(y) \cap E \neq \emptyset\}$.

Let $L, r > 0$. Let $S_{Q_1(\alpha)}(E, r, L)$, or simply $S(E, r, L)$, denote the class of Borel functions $0 \leq f \leq 1$ supported and defined everywhere in $E$ with the following properties. For all $\epsilon \leq r$ there exist $\varphi_{+\epsilon} \in C^\infty_c(E_{+\epsilon})$ so that

(S-1) $\varphi_{-\epsilon} \leq f \leq \varphi_{+\epsilon}$,
(S-2) $C^1(\varphi_{+\epsilon}) \leq \epsilon L$, and
(S-3) $\|\varphi_{+\epsilon} - \varphi_{-\epsilon}\|_2 \leq \epsilon \|f\|_2$.

Similarly, if $W$ is an affine submanifold in $Q_1(\alpha)$ and $E \subseteq W$ is a compact subset we define $S_{W}(E, r, L)$ using affine coordinate charts.

Lemma 2.5. There exists some $L$ depending only on $\alpha$ so that

$\text{1}_{B_{r}(x)} \in S_{W^{u,s}(x)}(B_{r}^{u,s}(x), r/10, L)$.

Similarly, $\text{1}_{B_{r}(x)} \in S(B_{r}(x), r/10, L)$.
Proof. To see the first claim in part (1) first apply Lemma 2.4 with $D = B_{r-2}\epsilon$; denote by \{\varphi_i, -\} the functions obtained from that lemma. Apply Lemma 2.4 again with $D = B_r(x)$; denote by \{\varphi_i, +\} the functions obtained from that lemma. Put
\[ \varphi_{\epsilon, -} = \sum \varphi_{i, -} \quad \text{and} \quad \varphi_{\epsilon, +} = \sum \varphi_{i, +}. \]
These functions satisfy (S-1) thanks to Lemma 2.4(1) and (5). They satisfy (S-2) and (S-3) thanks to Lemma 2.4(1)—(4).

The second claim in part (1) follows from the first claim, using the product structure of $B_r(x)$ and of the measure $\mu$.

We fix a large $L_0$ so that Lemma 2.5 holds true and drop $L_0$ from the notation. In particular, $S(E, r, L_0) \Rightarrow S(E, r)$.

Abusing the notation we will write $S(x, r)$ for $S(E, r)$ if the compact subset $E$ is not relevant except for the fact that it is a compact subset containing the point $x$.

2.11. **Non-divergence results.** Recall that $Q_1(\alpha)$ is realized as an affine invariant submanifold in $H_1(\tilde{\alpha})$, moreover, compact subsets of $Q_1(\alpha)$ lift to compact subsets of $H_1(\tilde{\alpha})$. Let $u : H_1(\tilde{\alpha}) \to [2, \infty]$ be the function constructed in [EMas] and [Ath].

**Theorem 2.6.** There exists a compact subset $K'_\alpha \subset Q_1(\alpha)$ and some $N_4 > 0$ with the following property. For every $t_0$ and every $x \in Q_1(\alpha)$, there exists $s \in [0, 1/2]$ and $t_0 \leq t \leq \max\{2t_0 N_4 \log u(x)\}$ such that $x' = a_t u s x \in K'_\alpha$.

**Proof.** The stratum $Q_1(\alpha)$ is an affine invariant submanifold in $H_1(\tilde{\alpha})$. The claim thus follows from [Ath, Thm. 2.2] and [AG, Lemma 6.3] applied with $\delta = 1/2$.

**Corollary 2.7.** Let $K'_\alpha$ be as in Theorem 2.6. There is a positive constant $N_5$ and for every $0 < \theta < 1$ there exists $\kappa_2(\theta)$, and a compact subset $K_\alpha(\theta) \supset K'_\alpha$ with the following properties. Let $x \in Q_1(\alpha)$ and let $B_r(x)$ be a period box centered at $x$. Put
\[ H^u_r(x, \theta) := \left\{ w \in B^u_r(x) : \text{\{t \in [0, t] : a_t w \in K_\alpha(\theta)\} \geq \theta t \right\} \]
Then for every $t \geq N_5 \log u(x)$ we have
\[ \mu_x^u (B^u_r(x) - H^u_r(x, \theta)) \leq e^{-\kappa_2(\theta)t} \mu_x^u (B^u_r(x)). \]

**Proof.** See [AG, Prop. 6.1].

We apply the above with $\theta = 1/2$ and put
\[ K_\alpha = K_\alpha(1/2), \quad \kappa_2 := \kappa_2(1/2), \quad \text{and} \quad H^u_r(x) := H^u_r(x, 1/2) \]
for the rest of the paper.
Proposition 2.8 (Cf. [AG], Prop. 5.3). Let $x \in Q_1(\alpha)$ and let
\[
y = \Phi^{-1}(\Phi(x) + s\nu(x) + w)
\]
for some $w \in E^s(x)$ with $\|w\|_{\text{AGY},x} \leq 0.1$ and $|s| \leq 0.1$. Then
\[
a_t y = \Phi^{-1}(\Phi(a_t x) + s\nu(a_t x) + w)
\]
for all $t \geq 0$. That is: the trajectories stay in the same period box.

Proof. First note that $\|w\|_{\text{AGY},x} \leq 0.1$ and $|s| \leq 0.1$ imply $\|w + s\nu(x)\|_{\text{AGY},x} \leq 1/2$. Since $w \in E^s(x)$ and $\nu(x)$ is the direction of the geodesic flow, we get
\[
\|w + s\nu(a_t x)\|_{\text{AGY},a_t x} \leq \|w + s\nu(x)\|_{\text{AGY},x}.
\]
The claim thus follows from [AG, Prop. 5.3]. It is worth mentioning that [AG, Prop. 5.3] is stated for translations with vectors in $E^s(x)$; the proof however works in the more general setting at hand where we translate by elements in $E^s(x) + \mathbb{R}\nu(x)$. □

Proposition 2.9. Let $K \subset Q_1(\alpha)$ be a compact subset. There exist some $\kappa_3(K)$ and some $t_0 = t_0(K)$ with the following property. Let $t \geq t_0$; suppose that $x, a_t x \in K$, moreover, assume
\[
|\{\tau \in [0, t] : a_\tau x \in K\}| \geq t/3.
\]
Then
\[
\|w\|_{\text{AGY},a_t x} \leq e^{-\kappa_3(K)t}\|w\|_{\text{AGY},x} \quad \text{and} \quad \|w\|_{a_t x} \leq e^{-\kappa_3(K)t}\|w\|_x
\]
for all $w \in E^s(x)$ and all $t \geq t_0$.

Proof. Let $\|w\|_{\text{ABEM},x}$ denote the modified Hodge norm defined in [ABEM, §3]. Let $C$ be a constant so that
\[
C^{-1}\|v\|_{\text{ABEM},y} \leq \|v\|_{\text{AGY},y} \leq C\|v\|_{\text{ABEM},y}
\]
for all $y \in K$.
In view of [ABEM, Thm. 3.15], there exists some $\kappa_4(K)$ so that under our assumptions in this proposition we have
\[
\|w\|_{\text{ABEM},a_t x} \leq e^{-\kappa_4(K)}\|w\|_{\text{ABEM},x}.
\]
We now compute
\[
\|w\|_{\text{AGY},a_t x} \leq C\|w\|_{\text{ABEM},a_t x} \leq Ce^{-\kappa_4(K)}\|w\|_{\text{ABEM},x} \leq C^2e^{-\kappa_4(K)}\|w\|_{\text{AGY},x}
\]
since $a_t x \in K$ by (19) since $x \in K$.
The claim thus holds with $\kappa_3 = \kappa_4/2$ and $t_0 = \frac{4\log C}{\kappa_3}$. □
3. Translates of horospheres

In this section we will use a fundamental result of Avila, Gouëzel, and Yoccoz, \cite{AGY, AR, AG}, together with Margulis’ thickening technique, \cite{Mar, EMc, KM}, to study translations of pieces of the horospherical foliations along the geodesic flow.

**Theorem 3.1** (Exponential Mixing, \cite{AGY, AR, AG}). Let \((M, \mu)\) be an affine invariant manifold. There exists a positive constant \(\kappa = \kappa(M, \mu)\) so that the following holds. Let \(\Psi_1, \Psi_2 \in C^\infty_c(M)\), then

\[
\left| \int \Psi_1(a_t x) \Psi_2(x) \, d\mu(x) - \mu(\Psi_1) \mu(\Psi_2) \right| \ll e^{-\kappa t} C^1(\Psi_1) C^1(\Psi_2)
\]

where the implied constant depends on \((M, \mu)\).

It is worth mentioning that the Sobolev norm in Theorem 3.1 may be taken to include derivatives only in the direction of \(SO(2) \subset SL(2, \mathbb{R})\). Our choice, \(C^1\), is more restrictive; this is tailored to our applications later, e.g. we will use the estimate \(\|\phi\|_\infty \ll C^1(\phi)\).

**Proposition 3.2.** There exists some \(\kappa_5 = \kappa_5(\alpha)\) with the following property. Let \(x \in \mathcal{Q}_1(\alpha)\) and let \(B_r(x)\) be a period box centered at \(x\). Let \(\psi^u \in C_c^\infty(\mathcal{B}_r^u(x))\), then for any \(\phi \in C_c^\infty(\mathcal{Q}_1(\alpha))\) we have

\[
\left| \int_{W^u(x)} \phi(a_t y) \psi^u(y) \, d\mu_x^u(y) - \int_{\mathcal{Q}_1(\alpha)} \phi \, d\mu \int_{W^u(x)} \psi^u \, d\mu_x \right| \ll_{x} C^1(\phi) C^1(\psi^u) e^{-\kappa_5};
\]

the implied constant may be taken to be uniform on compact subsets of \(\mathcal{Q}_1(\alpha)\).

**Proof.** The idea is to related the integral \(\int_{W^u(x)} \phi(a_t y) \psi^u(y) \, d\mu_x^u(y)\) to correlations of the function \(a_{-t} \phi\) with a thickening of \(\psi^u\) in the missing directions of \(W^{cs}(x)\). Then we may use Theorem 3.1 to conclude the proof.

Let \(0 < \epsilon < 0.01\) be a parameter which will be fixed later. In particular, it will be taken to be of the form \(e^{-\kappa t}\). Let \(\tilde{\psi}_\epsilon^s\) be a smooth function supported in

\[
\{ w \in E^s(x) : \|w\|_{x, \text{AGY}} \leq \epsilon \},
\]

so that \(\int_{W^s(x)} \tilde{\psi}_\epsilon^s \circ \Phi \, d\mu_x^s = 1\). We can choose such a function such that it moreover satisfies \(C^1(\tilde{\psi}_\epsilon^s) \ll \epsilon^{-N_6}\) for some \(\frac{N_5}{2} - N_6\)\(\alpha\). Similarly, let \(\tilde{\psi}_\epsilon^c\) be a smooth function supported in the interval \((-\epsilon, \epsilon)\) so that \(\int \psi^c_{\epsilon} \, d\text{Leb} = 1\) and \(C^1(\tilde{\psi}_\epsilon^c) \ll \epsilon^{-N_6}\).

Let \(y \in B_r(x)\), then \(\Phi(y) = \Phi(x) + \tau_y v(x) + w_x^s + w_x^u\) where \(w_x^{u,s} \in E_{u,s}(x)\) and \(\tau_y \in \mathbb{R}\). Put

\[
\Psi_\epsilon(y) = \tilde{\psi}_\epsilon^s(w_x^s) \tilde{\psi}_\epsilon^c(s_y) \tilde{\psi}_\epsilon^u(w_x^u)
\]

whenever \(y \in B_r(x)\), where \(\tilde{\psi}_\epsilon^u(w_x^u) = \tilde{\psi}_\epsilon^u(\Phi^{-1}(\Phi(x) + w_x^u))\). Extend \(\Psi_\epsilon\) to a smooth function on \(\mathcal{Q}_1(\alpha)\) by defining \(\Psi_\epsilon(y) = 0\) for all \(y \not\in B_r(x)\); note that \(\mu(\Psi) = \mu_x^u(\psi^u)\). Throughout the argument, \(\epsilon\) is fixed; therefore, we simply write \(\Psi\) for \(\Psi_\epsilon\).
For any $y \in B_\varepsilon(x)$ put $y^u = \Phi^{-1}(\Phi(x) + w^u_y)$. Recall the definition of $H^l_\varepsilon(x)$ from (16). In view of Corollary 2.7 and the bound $\| \cdot \|_\infty \ll C^1(\cdot)$, we have

\begin{align*}
(21) \quad & \left| \int_{W^u(x)} \phi(a_1y)\psi^u(y) \, d\mu^u_{x}(y) - \int_{Q_1(\alpha)} \phi(a_1z)\Psi(z) \, d\mu(z) \right| \ll C^1(\phi)C^1(\Psi)e^{-\kappa_6t} + \\
& \quad \left| \int_{H^l_\varepsilon(x)} \phi(a_1y)\psi^u(y) \, d\mu^u_{x}(y) - \int_{\{z:z^u \in H^l_\varepsilon(x)\}} \phi(a_1z)\Psi(z) \, d\mu(z) \right|
\end{align*}

Let now $z$ be so that $z^u \in H^l_\varepsilon(x)$. Then by Proposition 2.8 we have

$$a_1z = \Phi^{-1}(\Phi(a_1z^u) + D(a_1)w^u_y + \tau_z v(x)).$$

Moreover, in view of the definition of $H^l_\varepsilon(x)$, we have $a_1z^u \in K_\alpha$. Apply Proposition 2.9 with $K = K_\alpha$. Then using the definition of $C^1(\phi)$, we have

$$|\phi(a_1z) - \phi(a_1z^u)| \ll e^{\kappa_7}C^1(\phi)$$

for some positive constant $\kappa_7$. In consequence, we may replace $\phi(a_1z)$ by $\phi(a_1z^u)$ in (21). Using again the bound $\| \cdot \|_\infty \ll C^1(\cdot)$, we get that

\begin{align*}
(22) \quad & \left| \int_{H^l_\varepsilon(x)} \phi(a_1y)\psi^u(y) \, d\mu^u_{x}(y) - \int_{z^u \in H^l_\varepsilon(x)} \phi(a_1z)\Psi(z) \, d\mu \right| \ll C^1(\phi)C^1(\Psi)e^{\kappa_7} + \\
& \quad \left| \int_{H^l_\varepsilon(x)} \phi(a_1y)\psi^u(y) \, d\mu^u_{x}(y) - \int_{z^u \in H^l_\varepsilon(x)} \phi(a_1z^u)\Psi(z) \, d\mu(z) \right|
\end{align*}

Now, there exists some $\kappa_8$ so that the following holds on the support $\Psi$.

$$d\mu = (1 \pm O(\kappa_8)) \, d\text{Leb} \times d\mu_x \times d\mu^u_{x}.$$ 

Moreover, recall from (20) that $\Psi = \tilde{\psi}^s\tilde{\psi}^c\tilde{\psi}^a$, also recall that $\int \tilde{\psi}^s = \int \tilde{\psi}^c = 1$ and $\tilde{\psi}^u(w^u_y) = \psi^u(\Phi^{-1}(\Phi(x) + w^u_y))$. Therefore,

\begin{align*}
(23) \quad & \int_{z^u \in H^l_\varepsilon(x)} \phi(a_1z^u)\Psi(z) \, d\mu(z) = \int_{z^u \in H^l_\varepsilon(x)} \phi(a_1z^u)\tilde{\psi}^s(w^u_y)\tilde{\psi}^c(s_y)\psi^u(z^u) \, d\mu(z) \\
& \quad = \int_{z^u \in H^l_\varepsilon(x)} \phi(a_1z^u)\psi^u(z^u) \, d\mu^u_x \pm C^1(\phi)C^1(\Psi)O(\kappa_8)
\end{align*}

We now combine the estimates in (21), (22), and (23) to get the following.

\begin{align*}
(24) \quad & \left| \int_{W^u(x)} \phi(a_1y)\psi^u(y) \, d\mu^u_{x}(y) - \int_{Q_1(\alpha)} \phi(a_1z)\Psi(z) \, d\mu(z) \right| \ll C^1(f)C^1(\Psi)e^{-\kappa_9t} + \\
& \quad \left| C^1(\phi)C^1(\Psi)e^{\kappa_8} \right| + \left| C^1(\phi)C^1(\Psi)e^{\kappa_8} \right|
\end{align*}

Optimizing the choice of $\epsilon$ to be of size $e^{-\kappa_9t}$ for some small $0 < \kappa_{10} < 1$; the proposition follows from (24) and Theorem 3.1 applied with $\Psi_1 = \phi$ and $\Psi_2 = \Psi$. \qed
Remark 3.3. It is worth mentioning that Proposition 3.2 and its proof hold for any affine invariant manifold, \((M, \mu)\). In the sequel, however, we will only need this result for \(Q_1(\alpha)\); and even more specifically, in our application to counting problems, we will need this result for the principle stratum \(Q_1(1, \ldots, 1)\). The main result in [AGY] was generalized to \(Q_1(\alpha)\) in [AR].

Corollary 3.4. There exist \(\kappa_{11}, \kappa_{12},\) and \(N_7\) so that the following holds. Let \(x, z \in Q_1(\alpha)\) and suppose \(0 < r, r' \leq 0.01\). Let \(B \subset B_{r'}(z)\) be so that \(1_B \in \mathcal{S}(z, r')\) and let \(\psi^u \in C_c^\infty(B_{r'}^u(x))\). Then for any \(\epsilon < r'\) we have

\[
\left| \frac{1}{\mu(B)} \int_{W^u(x)} 1_B(a_t y)\psi^u(y) \, d\mu^u(y) - \int \psi^u \, d\mu^u \right| \ll_{x, z} \epsilon^{-1} + C^1(\psi^u) \epsilon^{-11} + C^1(\psi^u) \epsilon^{-12}
\]

where the implied constant is uniform on compact subsets of \(Q_1(\alpha)\).

Proof. This follows from Proposition 3.2 by approximating \(1_B\) with \(\varphi_{x, \epsilon}\) and using properties (S-1)—(S-3). \(\square\)

4. A COUNTING FUNCTION

Let \(x, z \in Q_1(\alpha)\) and \(0 < r, r' \leq 0.01\). Let \(\psi^u\) be a function which is supported and defined everywhere in \(B_{r'}^u(x) = B_r(x) \cap W^u(x); \) let \(\phi^{cs}\) be a function which is supported and defined everywhere in \(B_{r'}^{cs}(z) = W^{cs}(z) \cap B_r(z)\). For any \(t > 0\) and \(\psi^u\) and \(\phi^{cs}\) as above define

\[
\mathcal{N}_{nc}(t, \psi^u, \phi^{cs}) := \sum \psi^u(y)\phi^{cs}(a_t y)
\]

where the sum is taken over all \(y \in B_{r'}^u(x)\) so that \(a_t y \in B_{r'}^{cs}(z)\) — note that the sum is indeed over all \(y \in \text{supp}(\psi^u)\) so that \(a_t y \in \text{supp}(\phi^{cs})\).

Alternatively, the sum is taken over connected components of \(a_t \text{supp}(\psi^u) \cap \text{supp}(\phi^{cs})\); this point will be made more explicit in the course of the proof — see also Lemma 4.2 below and recall that \(W^u\) and \(W^{cs}\) are complimentary foliations.

The function \(\mathcal{N}_{nc}\) may be thought of as a bisector counting function — one studies the asymptotic behavior of the number of translates of a piece of \(W^u\) by \(\text{Mod}_g\) which intersect a cone in the Teichmüller space.

The following proposition is the main result of this section and provides an asymptotic behavior for \(\mathcal{N}_{nc}\). This proposition plays a prime role in the proof of Theorem 1.2 in [7].

Proposition 4.1. There exist \(\kappa_{13}\) with the following property. Let \(K_\alpha\) be as in Corollary 2.7. Let \(x, z\) be so that \(B_{0.01}(\bullet) \subset K_\alpha\) for \(\bullet = x, z\) and let \(0 < r, r' \leq 0.01\). Let \(\psi^u \in C_c^\infty(B_r^u(x))\) with \(0 \leq \psi^u \leq 1\) and let \(\phi^{cs} \in C_c^\infty(B_{r'}^{cs}(z))\). Then

\[
|\mathcal{N}_{nc}(t, \psi^u, \phi^{cs}) - e^{ht} \mu_x^u(\psi^u)\mu_z^{cs}(\phi^{cs})| \leq C^1(\psi^u)C^1(\phi^{cs})e^{(h-\kappa_{13}t)}
\]

where \(h = \frac{1}{2}(\dim \mathbb{Q}(\alpha) - 2)\).

The proof of this proposition is based on Lemma 4.5 which in turn relies on Proposition 3.2. In particular, the main term is given by Proposition 3.2. However, we need to control the contribution of two types of exceptional points as we now describe.
Similar to Corollary 2.7, given a compact subset $K \supset K_\alpha$, define
\begin{equation}
H_t^\alpha(x, K) := \left\{ y \in \mathcal{B}_t^\alpha(x) : a_t y \in K_\alpha, \text{ and } \{|\tau \in [0, t] : a_\tau y \in K\}| \geq t/2 \right\}.
\end{equation}
The first (and more difficult to control) type of exceptional points are $y \in \mathcal{B}_t^\alpha(x)$ so that $a_t y \in \mathcal{B}_r^\alpha(z)$, however, $y \notin H_t^\alpha(x, K)$. The contribution coming from these points is controlled using [EMR, Thm. 1.7], see Theorem 4.4 below.

We also need to control the contribution of points $y \in \mathcal{B}_t^\alpha(x)$ which are exponentially close to the boundary of $\mathcal{B}_r^\alpha(x)$. This set has a controlled geometry, and we use a simple covering argument and Proposition 3.2 to control this contribution. The argument here is standard and will be presented after we establish the estimate (36).

Let us begin with some preliminary statements; these assertions are essentially consequences of the fact that $\hat{W}^u$ and $\hat{W}^{cs}$ are complimentary foliations in the universal cover $Q^1 T(1, \ldots, 1)$ of $Q_1(1, \ldots, 1)$.

**Lemma 4.2.** Let $\tilde{x}, \tilde{x}' \in Q^1 T(1, \ldots, 1)$ and let $r > 0$. Assume there are $\tilde{y}_1, \tilde{y}_2 \in \hat{W}^u(\tilde{x})$ and some $t \in \mathbb{R}$ so that $a_t \tilde{y}_1$ and $a_t \tilde{y}_2$ belong to $\mathcal{B}_r^{cs}(\tilde{x}')$. Then $\tilde{y}_1 = \tilde{y}_2$.

**Proof.** By the assumption, we have $a_t \tilde{y}_i \in \hat{W}^{cs}(\tilde{x}')$ which implies that $\tilde{y}_i \in \hat{W}^{cs}(\tilde{x}')$ for $i = 1, 2$.

Recall now that $\tilde{y}_1, \tilde{y}_2 \in \hat{W}^u(\tilde{x})$, hence, by (14) the corresponding abelian differentials at $\tilde{y}_1$ and $\tilde{y}_2$ differ from each other by some $c \in H^1_{\text{odd}}(\tilde{M}, \tilde{\Sigma}, \mathbb{R})$. However, since $\tilde{y}_1, \tilde{y}_2 \in \hat{W}^{cs}(\tilde{x}')$, they differ from each other by some $c \in H^1_{\text{odd}}(\tilde{M}, \tilde{\Sigma}, i\mathbb{R}) \oplus \mathbb{R} v(\tilde{x}')$. Therefore, $\tilde{y}_1 = \tilde{y}_2$. \□

Recall that for any $\tilde{x} \in Q^1 T(1, \ldots, 1)$, $\mathcal{B}_t^\bullet(\tilde{x})$ denotes a ball in $\hat{W}^\bullet(\tilde{x})$ for $\bullet = u, s, cs, cu$.

**Corollary 4.3.** Let $g_1, g_2 \in \text{Mod}_g$ be so that $g_1 \cdot \hat{W}^u(\tilde{y}) = \hat{W}^u(\tilde{x}) = g_2 \cdot \hat{W}^u(\tilde{y})$. Let $\tilde{x}_1$ and $\tilde{x}_2$ in $\hat{W}^u(\tilde{x})$. Assume for some $r, b > 0$ that
$$\mathcal{B}_t^{cs}(\tilde{x}') \cap g_i \cdot a_t \mathcal{B}_b^{cs}(\tilde{x}_i) \neq \emptyset$$
for $i = 1, 2$ and some $t \in \mathbb{R}$.

Then $\mathcal{B}_t^{cs}(\tilde{x}') \cap g_1 \cdot a_t \mathcal{B}_b^{cs}(\tilde{x}_1) = \mathcal{B}_t^{cs}(\tilde{x}') \cap g_2 \cdot a_t \mathcal{B}_b^{cs}(\tilde{x}_2)$. In particular, we have
$$g_1 \cdot \mathcal{B}_b^{cs}(\tilde{x}_1) \cap g_2 \cdot \mathcal{B}_b^{cs}(\tilde{x}_2) \neq \emptyset.$$

**Proof.** Let $\tilde{y}_i \in \mathcal{B}_t^{cs}(\tilde{x}') \cap g_i \cdot a_t \mathcal{B}_b^{cs}(\tilde{x}_i)$ for $i = 1, 2$. Then $\tilde{y}_1, \tilde{y}_2 \in \mathcal{B}_r^{cs}(\tilde{x}') \cap a_t \hat{W}^u(\tilde{x})$. Hence, by Lemma 4.2 we have $\tilde{y}_1 = \tilde{y}_2$ which implies the claim. \□

As was discussed above, there are two types of exceptional points. The first type will be controlled using the following theorem.

**Theorem 4.4** (Cf. [EMR], Theorem 1.7). There exists a compact subset $\bar{K}_\alpha \supset K_\alpha$ so that
$$\#\{y \in \mathcal{B}_t^u(x) - \mathcal{B}_b^u(x, \bar{K}_\alpha) : a_t y \in \mathcal{B}_r^{cs}(z)\} \ll e^{(b-0.5)t}.$$
Proof. Let $K \supset K\alpha$ be any compact subset. For simplicity in notation put

$$E_t(x, K) := \{y \in B^u_{2r}(x) - \mathcal{H}^u_t(x, K) : a_t y \in B^s_{\alpha}(z)\}.$$ 

In $Q^1 T(\alpha)$ fix lifts $B^u_{2r}(\tilde{x})$ and $B^s_{\alpha}(\tilde{z})$ for the sets $B^u_{2r}(x)$ and $B^s_{\alpha}(z)$, respectively. For every element $y \in B^u_{2r}(x)$ we fix a lift $\tilde{y} \in B^u_{2r}(\tilde{x})$. Then for every $y \in E_t(x, K)$ there exists some $g_y \in \text{Mod}_g$ and some $\tilde{z}_y \in B^s_{\alpha}(\tilde{z})$ so that $a_t \tilde{y} = g_y \tilde{z}_y$.

Hence, for every $y \in E_t(x, K)$ we have

1. $\tilde{y}$ is within Teichmüller distance $2r$ from $\tilde{x}$ and $a_t \tilde{y} = g_y \tilde{z}_y$ is within Teichmüller distance $r'$ of $g_y \tilde{z}$, and

2. $|\{\tau \in [0, t] : \pi(a_t \tilde{y}) \in K\}| < t/2$.

It is shown in [EMR] Thm. 1.7, see also [EMir], that there exists some $K_0$ so that if $K \supset K_0$, then the number of $\{g \tilde{z}\}$ for which such a $\tilde{y}$ exists is $\ll e^{(h-0.5)t}$.

We now claim that there exists some $C$ which depends on $\alpha$ and $K$ so that the following holds. Then the map $y \mapsto g_y \tilde{z}$ from $E_t(x, K)$ to $\{g \tilde{z} : g \in \text{Mod}_g\}$ is at most $C$-to-one.

First note that the above discussion together with the claim implies that

$$\#E_t(x, K) \leq C e^{(h-0.5)t},$$

as we wanted to show.

To see the claim, let $y_1, y_2 \in E_t(x, K)$. Then there exists $g_1, g_2 \in \text{Mod}_g$ so that

$$g_i \cdot a_t y_i \in B^s_{\alpha}(\tilde{z}).$$

Therefore, by Corollary 1.3, applied with $\tilde{x}_i = \tilde{x}$ and $b = 2r$, we have

- either $g_1 \cdot W^u(\tilde{x}) \neq g_2 \cdot W^u(\tilde{x})$ which in particular implies that $g_1 \neq g_2$,
- or $g_1 \cdot B^u_{2r}(\tilde{x}) \cap g_2 \cdot B^u_{2r}(\tilde{x}) \neq \emptyset$ which implies $g_1^{-1}g_2$ belongs to a fixed finite subset of $\text{Mod}_g$.

The claim thus follows and the proof is complete. \hfill $\square$

The following lemma will play a crucial role in the proof of Proposition 4.1.

Lemma 4.5. There exists $\kappa_{14}$ with the following property. Let $K\alpha$ be as in Corollary 2.7. Let $x, z$ be so that $B_{0.01}(\bullet) \subset K\alpha$ for $\bullet = x$, $z$ and let $0 < r, r' \leq 0.01$. Let

- $\psi^u \in C^\infty_c(B^u_{\kappa}(x))$ with $0 \leq \psi^u \leq 1$, and
- $\phi^u \in C^\infty_c(B^u_{\kappa}(z))$ and $\phi^s \in C^\infty_c(B^s_{\kappa}(z))$.

Put $\phi := \phi^u \phi^s$. Define

$$N^\prime_{nc}(t, \psi^u, \phi) := \sum \psi^u(y) \mu^u_{\alpha y}(\phi)$$

where the sum is taken over all $y \in B^u_{\kappa}(x)$ so that $a_t y \in B^s_{\alpha}(z)$. Then

$$|N^\prime_{nc}(t, \psi^u, \phi) - e^{ht} \mu^u_{\alpha}(\psi^u) \mu(\phi)| \leq C(\psi^u) C(\phi) e^{(h-\kappa_{14})t},$$

where $h = \frac{1}{2}(\dim_{\mathbb{R}} Q(\alpha) - 2)$. 

Proof. We will compute
\[ \int_{W^u(x)} \phi(a_t y) \psi^u(y) \, d\mu^u_x(y) \]
in terms of \( \mathcal{N}'_{nc} \). The claim will then follow from Proposition 3.2.

First note that since \( K_\alpha \) is a compact set we have
\[ r' \ll \text{diam}(W^u(z') \cap B_{r}(z)) \ll r' \]
where the diam is measured with respect to \( \| \|_{z', \text{AGY}} \) for all \( z' \in B_{r}(z) \), see also [AG Prop. 5.3].

Let \( \tilde{K}_\alpha \) be given by Theorem 4.4 and put \( H^u_t(x) := H^u_t(x, \tilde{K}_\alpha) \), see (26) for the notation. Since \( K_\alpha \subset \tilde{K}_\alpha \), it follows from Corollary 2.7 that
\[ \mu^u_x(B^u_r(x) - H^u_t(x)) \leq e^{-\kappa_{16}} \mu^u_x(B^u_r(x)) \]
for every \( t \geq t_0 \) where \( t_0 \) depends only on \( K_\alpha \).

It is more convenient for the proof to treat points in \( H^u_t(x) \) which are too close to the boundary of \( B^u_r(x) \) separately. Define
\[ H^u_{t, \text{int}} := \{ y \in H^u_t(x) : B^u_{10e^{-15}}(y) \subset B^u_r(x) \} \]
where \( \kappa_{15} := [\kappa_3, \tilde{K}_\alpha]/2 \), see Proposition 2.9 for the definition of \( \kappa_3 \). The precise radius which is used in the definition of \( H^u_{t, \text{int}} \) is motivated by estimates for uniform hyperbolicity of the Teichmüller geodesic flow, see Claim 4.6 below.

Using (30) and the definition of \( H^u_{t, \text{int}} \) we have
\[ \mu^u_x(B^u_r(x) - H^u_{t, \text{int}}) \leq e^{-\kappa_{16}} \mu^u_x(B^u_r(x)) \]
for some \( \kappa_{16} \) depending on \( \tilde{K}_\alpha \). The estimate in (31) implies the following:
\[ \int_{W^u(x)} \phi(a_t y) \psi^u(y) \, d\mu^u_x(y) = O(e^{-\kappa_{16}}) \mu^u_x(B^u_r(x)) C^1(\psi^u) C^1(\phi) + \int_{H^u_{t, \text{int}}} \phi(a_t y) \psi^u(y) \, d\mu^u_x(y). \]

We now compute the term \( \int_{H^u_{t, \text{int}}} \phi(a_t y) \psi^u(y) \, d\mu^u_x(y) \) appearing in (32).

For every \( y \in H^u_{t, \text{int}} \) so that \( a_t y \in B_r(z) \), there is an open neighborhood \( C_y \) of \( y \) such that \( a_t C_y \) is a connected component of \( a_t B^u_r(x) \cap B_r(z) \) containing \( a_t y \). We note that \( C = \{ C_y \} \) is a disjoint collection of open subsets in \( B^u_r(x) \). Further, in view of (15) we have
\[ \mu^u_x(\phi) = e^{ht} \mu^u_x(a_{-t} \phi) = e^{ht} \mu^u_x(a_{-t} \phi); \]
recall that \( a_{-t} \phi(y') = \phi(a_t y') \).

Claim 4.6. Let \( y \in H^u_{t, \text{int}} \), then \( C_y \subset B^u_{10e^{-15}}(y) \). If we further assume that \( y \in H^u_{t, \text{int}} \), then \( C_y \subset B^u_{10e^{-15}}(y) \subset B^u_r(x) \).

Proof of Claim 4.6. Let \( y \in H^u_{t, \text{int}} \), then \( C_y \subset B^u_{10e^{-15}}(y) \subset B^u_r(x) \).
Proof of the claim. Let \( y' \in C_y \). It follows from the definition of \( C_y \) that \( a_t y' \in W^u(a_t y) \cap B_r(z) \). Let us write \( a_t y' = \Phi^{-1}(\Phi(a_t y) + w) \). Hence, by (29) we have
\[
\|w\|_{a_t y} \ll r'.
\]
This, in view of Proposition 2.9 implies that
\[
\|w\|_y \leq e^{-\kappa_{15}}\|w\|_{a_t y} \ll e^{-\kappa_{15}} r'.
\]
The claim follows from this estimate if we assume \( t \) is large enough so that
\[
e^{-\kappa_{15}} r' < e^{-\kappa_{15}};
\]
recall that \( \kappa_{15} = \kappa_3/2 \). The final claim follows from the definition of \( H_{t,\text{int}} \).

Claim 4.6 in particular implies that
\[
|\psi^u(y') - \psi^u(y)| \ll e^{-\kappa_{15}} C^1(\psi^u) \text{ for all } y' \in C_y.
\]
Returning to (32), we get from (33) and (34) that
\[
\int_{H_{t,\text{int}}^u} \phi(a_t y) \psi^u(y) d\mu^u_x(y) = O(e^{-\kappa_{15}}) C^1(\psi^u) C^1(\phi) + e^{-ht} \sum_{C_y \in C} \psi^u(z_y) \mu^u_{a_t y}(\phi).
\]
where \( z_y \) is the unique point of intersection in \( (a_t C_y) \cap W^{cs}(z) \cap B_r(z) \) for every \( C_y \).
Combining (32) and (35) we get the following from Proposition 3.2
\[
\left| \sum_C \psi^u(z_y) \mu^u_{a_t y}(\phi) - \mu^u_x(\psi^u) \mu(\phi) e^{ht} \right| \leq C^1(\psi^u) C^1(\phi) e^{(h-\kappa_{17})t}
\]
for some \( \kappa_{17} \) depending on \( \alpha \). Thus, in order to get the conclusion we need to control the difference between \( N_{\kappa}^u(t, \psi^u, \phi) \) and the summation appearing on the left side of (36). That is: the contribution of points \( y \notin H_{t,\text{int}}^u \).

Contribution from points in \( H_{t,\text{int}}^u(x) \) which are not in \( H_{t,\text{int}}^u \). Let \( y \in H_{t,\text{int}}^u(x) - H_{t,\text{int}}^u \) be so that \( a_t y \in B_r(z) \), and let \( z_y \in (a_t C_y) \cap W^{cs}(z) \cap B_r(z) \). We note that \( C_y \) is not necessarily contained in \( B_{10 e^{-\kappa_{15}}}(x) \); however, in view Claim 4.6 we have \( C_y \) is contained in \( B_{10 e^{-\kappa_{15}}}(y) \).

We first note the following consequence of the definition.
\[
\bigcup_{y \in \overline{H_{t,\text{int}}^u(x)}} B_{10 e^{-\kappa_{15}}}(y) \subset B_{r + O(e^{-\kappa_{15}})}(x) - B_{r - O(e^{-\kappa_{15}})}(x) =: G(x)
\]
where the implicit multiplicative constant depends on \( \tilde{K}_\alpha \); this constant can be taken to be uniform over \( \tilde{K}_\alpha \) in view of \( \text{AG Prop. 5.3} \).

Let \( 0 < \kappa < \kappa_{15} \) be a small constant which will be optimized later. We can cover \( G(x) \) with period balls \( \{ B(y_i) : 1 \leq i \leq I \} \) centered at \( y_i \) and of radius \( e^{-\kappa t} \) with multiplicity depending only on the dimension, e.g., since \( K_\alpha \) is a compact set, this can be done by choosing a maximal \( e^{-\kappa t}/2 \) separated net in \( G(x) \). We have
\[
I \ll e^{(h-1)\kappa t}.
\]
To see (37), note that
\[ e^{-\hat{\kappa}t} I \ll \sum_i \mu_x^u(B^u(y_i)) \ll \mu_z^u(\cup B^u(y_i)) \ll \epsilon^{-1} e^{-\hat{\kappa}t} \ll e^{-\hat{\kappa}t}. \]

For every i let \( \hat{B}(y_i) \) denote the the period ball with the same center \( y_i \) and with radius \( 4e^{-\hat{\kappa}t} \). Note that since \( \hat{\kappa} < \frac{\kappa_{15}}{2} \) we have
\[ 2e^{-\hat{\kappa}t} > e^{-\hat{\kappa}t} + 10e^{-\hat{\kappa}t}. \]

Let \( 0 \leq \hat{\psi}_u \leq 1 \) be a smooth function which is supported in \( \hat{B}^u(y_i) \) which equals 1 on \( B^u_{2e^{-\hat{\kappa}t}}(y_i) \) and
\[ C^1(\hat{\psi}_u) \leq \frac{\kappa_{19}}{2}. \]

Let \( I_i \) be the contribution coming from \( B(y_i) \) to \( N_{nc}(t, \psi^u, \phi) \). Then by Proposition 3.2 and the choice of \( \hat{\psi}^u \) we have the following.
\[ I_i \leq e^{ht} \int_{W^u(x)} \phi(a_t y)\hat{\psi}_u(y) \, d\mu_x(y) \leq e^{ht} \mu(\phi) \int \hat{\psi}_u \, d\mu_x + C^1(\hat{\psi}_u)C^1(\phi)e^{(h-\hat{\kappa})t}. \]

Summing (39) over all \( 1 \leq i \leq I \) and using (38), (37), and \( \int \hat{\psi}_u \, d\mu_x \ll e^{-\hat{\kappa}t} \) we get
\[ \sum_i I_i \ll (e^{(h-\hat{\kappa})t} + C^1(\phi)e^{(h-\hat{\kappa})t})e^{(h-\hat{\kappa})t} = e^{(h-\hat{\kappa})t} + C^1(\phi)e^{(h-\hat{\kappa})t + \hat{\kappa}(h-1)\hat{\kappa}}. \]

We now choose \( \hat{\kappa} \) so that \( (h-1)\hat{\kappa} + \frac{\kappa_{19}}{2} = \frac{\kappa_{18}}{2} \) and get
\[ \sum_i I_i \ll C^1(\phi)C^1(\psi^u)e^{(h-\hat{\kappa})t} \]
for some \( \kappa_{18} \) depending only on \( \alpha \) and \( \hat{K}_\alpha \).

**Contribution from points in \( B^u_\alpha(x) - H_\alpha^u(x) \).** Let \( J \) denote the contribution to \( N'_{nc}(t, \psi^u, \phi) \) coming from points \( y \in B^u_\alpha(x) - H_\alpha^u(x) \). Then by (17) there is a unique \( z_y \in B^u_{r+\epsilon}(x) - H_\alpha^u(x, \hat{K}_\alpha) \) such that \( a_t z_y \in B^u_{r+\epsilon}(z) \). In consequence, by Theorem 4.4 we have
\[ J \ll \|\phi\|_\infty \|\psi^u\|_\infty e^{(h-0.5)t} \ll C^1(\phi)C^1(\psi^u)e^{(h-0.5)t}. \]

The proposition now follows from (36) in view of (40) and (41). \( \square \)

**Proof of Proposition 4.1.** Let \( \varrho = e^{-\kappa t} \) and let \( \epsilon = \varrho^N \) for two constants \( \kappa, N > 0 \) which will be optimized later. Put \( \phi = 1_{B^u_\varrho(z)}\phi^{cs} \) and
\[ \mu(\phi) = \varrho^h \mu_{cs}(\phi^{cs}) \]

In view of Lemma 2.5, properties (S-1), (S-2), and (S-2) hold with \( \epsilon \) and \( f = 1_{B^u_{\varrho^{-2}}(z)} \). Let \( \phi_1 = \varphi^{\varrho_1} \phi^{cs} \) for these choices. Let \( \phi_1 = 1_{B^u_{\varrho^{-2}}(z)} \phi^{cs} \) so that\( \mu(\phi_1) = \mu^1(\phi^{cs}) \leq C_{19} \mu_{cs}(\phi^{cs}) \) and
\[ \mu(\phi_1) = \mu(\phi_1) \leq C_{19} \mu_{cs}(\phi^{cs}). \]
By Lemma 4.5, we have
\[ N_{nc}^\prime(t, \psi^n, \phi_1) = e^{ht} \mu_x^n(\psi^n) \mu(\phi_1) + O(\mathcal{C}^1(\psi^n)C^1(\phi_1)e^{(h-k_{14})t}) \]
\[ = e^{ht} \mu_x^n(\psi^n) \mu(\phi) + O(e^{h/2}e^{ht} \mu_x^n(\psi^n) + C^1(\psi^n)C^1(\phi^{cs})e^{-h}e^{(h-k_{14})t}) \]
(44)
\[ = e^{ht} e^h \mu_x^n(\psi^n) \mu_x(\phi^{cs}) + O(e^{h/2}e^{ht} \mu_x^n(\psi^n) + C^1(\psi^n)C^1(\phi^{cs})e^{-h}e^{(h-k_{14})t}). \]

Let now \( \phi_1^n = \varphi_{+, \epsilon} \) for \( \epsilon \) and \( f = 1_{B^2_\xi(z)} \). Put \( \phi_2 = \phi_2^n \phi^{cs} \). Then similar to the above estimate, using Lemma 4.5, we get that
\[ N_{nc}^\prime(t, \psi^n, \phi) = e^{ht} e^h \mu_x^n(\psi^n) \mu_x(\phi^{cs}) + O(e^{h/2}e^{ht} \mu_x^n(\psi^n) + C^1(\psi^n)C^1(\phi^{cs})e^{-h}e^{(h-k_{14})t}). \]
(45)
Since \( \phi_1 \leq \phi \leq \phi_2 \), we have
\[ N_{nc}(t, \psi^n, \phi_1) \leq N_{nc}^\prime(t, \psi^n, \phi) \leq N_{nc}(t, \psi^n, \phi_2). \]
(46)
Moreover, using the definitions of \( N_{nc} \) and \( N_{nc}^\prime \) we have
\[ N_{nc}^\prime(t, \psi^n, \phi) = \sum \psi^n(y) \mu_{x,y}(\phi) \]
\[ = \sum \psi^n(y) \phi^{cs}(ay) \mu_x(B^n_\phi(z)) = e^h \sum \psi^n(y) \phi^{cs}(ay) \]
\[ = e^h N_{nc}(t, \psi^n, \phi^{cs}). \]
This and (46) imply that
\[ \varrho^{-h} N_{nc}(t, \psi^n, \phi_1) \leq N_{nc}(t, \psi^n, \phi^{cs}) \leq \varrho^{-h} N_{nc}(t, \psi^n, \phi_1). \]
Hence, using (44) and (45), we get that
\[ N_{nc}(t, \psi^n, \phi^{cs}) = e^{ht} \mu_x^n(\psi^n) \mu_x(\phi^{cs}) + O(e^{h/2}e^{ht} \mu_x^n(\psi^n) + C^1(\psi^n)C^1(\phi^{cs})e^{-h}e^{(h-k_{14})t}). \]

We choose \( N \) large enough so that \( k_{19}N - h > k_{19}N/2 \) then choose \( \kappa \) small enough so that \( e^{-h}e^{(h-k_{14})t} = e^{(h-k_{14})/2}t \). The proof is complete. \( \square \)

We end this section with the following corollary.

**Corollary 4.7.** There exist \( \kappa_{20}, \kappa_{21}, \) and \( N_8 \) with the following property. Let \( K_\alpha \) be as in Corollary 2.7. Let \( x, z \) be so that \( B_{0,0.1}(\bullet) \subset K_\alpha \) for \( \bullet = x, z \) and let \( 0 < r, r' \leq 0.01 \). Let \( \psi^n \in C^\infty_c(B^n_\alpha(x)) \) with \( 0 \leq \psi^n \leq 1 \) and let \( \phi^{cs} \in \mathcal{S}_{W^{cs}(z)}(r', r) \). Then for any \( \delta < r' \) we have
\[ |N_{nc}(t, \psi^n, \phi^{cs}) - e^{ht} \mu_x^n(\psi^n) \mu_x(\phi^{cs})| \ll C^1(\psi^n) \delta^{-\kappa_{20}} e^{(h-k_{20})t} + \delta^{k_{21}} \delta^{1} e^{ht}. \]
where \( h = \frac{1}{2}(\dim_\mathbb{R} Q(\alpha) - 2) \).

In particular, there exists some \( \kappa_{22} \) so that
\[ |N_{nc}(t, \psi^n, \phi^{cs}) - e^{ht} \mu_x^n(\psi^n) \mu_x(\phi^{cs})| \ll C^1(\psi^n) e^{(h-k_{22})t}. \]
(47)

**Proof.** The corollary follows from Proposition 1.1 by approximating \( \phi^{cs} \) with smooth functions. Let \( \delta < r' \) and let \( \phi^{cs}_{+,-\delta} \) be smooth functions satisfying (S-1), (S-2), and (S-2) with \( \delta \) and \( \phi^{cs} \). Hence, we have
\[ \psi_{-,-\delta} \leq \phi^{cs} \leq \phi_{+,-\delta} \quad \text{and} \quad C^1(\phi_{+,-\delta}) \ll \delta^{-r}; \]
(48)
furthermore, property (S-3) implies that
\[ |\mu_z^{\text{cs}}(\phi_+^{\text{cs}}) - \mu_z^{\text{cs}}(\phi_-^{\text{cs}})| \ll \delta^*. \]

With this notation and in view of the first estimate in (48), we have
\[ N_{\text{nc}}(t, \psi^u, \phi_+^{\text{cs}}, \delta) \leq N_{\text{nc}}(t, \psi^u, \phi_-^{\text{cs}}, \delta) \leq N_{\text{nc}}(t, \psi^u, \phi_-^{\text{cs}}, \delta). \]

In addition we may apply Proposition 4.1 with \( \psi^u \) and \( \phi^{\pm}_+ \) and get that
\[ N_{\text{nc}}(t, \psi^u, \phi^{\pm}_- \pm) = e^{ht \mu_u^{\text{cs}}(\psi^u) \mu_z^{\text{cs}}(\phi^{\pm}_-) + O(C_1(\psi^u)C_1(\phi^{\pm}_-) e^{x_1 t}). \]

This together with (50), (49), and the second estimate in (48) implies the first claim.

The second claim follows from the first claim by optimizing the choice \( \delta = e^{-xt} \). \( \square \)

5. The space of measured laminations

In this section we recall some basic facts about the space of geodesic measured laminations and train track charts. The basic references for these results are [Th1] and [HP].

The space of geodesic measured laminations on \( S \) is denoted by \( \mathcal{ML}(S) \); it is a piecewise linear manifold homeomorphic to \( \mathbb{R}^{6g-6} \), but it does not have a natural differentiable structure [Th1]. Train tracks were introduced by Thurston as a powerful technical device for understanding measured laminations. Roughly speaking train tracks are induced by squeezing almost parallel strands of a very long simple closed geodesic to simple arcs on a hyperbolic surface. A train track \( \tau \) on a surface \( S \) is a finite closed 1 complex \( \tau \subset S \) with vertices (switches) which is

- embedded on \( S \),
- away from its switches, it is \( C^1 \),
- it has tangent vectors at every point, and
- for each component \( R \) of \( S - \tau \), the double of \( R \) along the interiors of the edges of \( \partial(R) \) has negative Euler characteristic.

The vertices (or switches), \( V \), of a train track are the points where 3 or more smooth arcs come together. Each edge of \( \tau \) is a smooth path with a well defined tangent vector. That is: all edges at a given vertex are tangent. The inward pointing tangent of an edge divides the branches that are incident to a vertex into incoming and outgoing branches.

A train track \( \tau \) is called maximal (or generic) if at each vertex there are two incoming edges and one outgoing edge.

5.1. Train track charts. A lamination \( \lambda \) on \( S \) is carried by a train track \( \tau \) if there is a differentiable map \( f : S \rightarrow S \) so that

- \( f \) is homotopic to the identity,
- the restriction of \( df \) to a tangent line of \( \lambda \) is nonsingular, and
- \( f \) maps \( \lambda \) onto \( \tau \).
Every geodesic lamination is carried by some train track. Let $\lambda$ be a measured lamination with invariant measure $\mu$. If $\lambda$ is carried by the train track $\tau$, then the carrying map defines a counting measure $\mu(b)$ to each branch line $b$: $\mu(b)$ is just the transverse measure of the leaves of $\lambda$ collapsed to a point on $b$. At a switch, the sum of the entering numbers equals the sum of the exiting numbers.

The piecewise linear integral structure on $\mathcal{ML}(S)$ is induced by train tracks as follows. Let $\mathcal{V}(\tau)$ be the set of measures on a train track $\tau$; more precisely, $u \in \mathcal{V}(\tau)$ is an assignment of positive real numbers to the edges of the train track satisfying the switch condition:

$$\sum_{\text{incoming } e_i} u(e_i) = \sum_{\text{outgoing } e_j} u(e_j).$$

Also, let $\mathcal{W}(\tau)$ be the vector space of all real weight systems on edges of $\tau$ satisfying the switch condition, i.e., $u(e_i)$ need not be positive for $u \in \mathcal{W}(\tau)$. Then $\mathcal{V}(\tau)$ is a cone on a finite-sided polyhedron where the faces are of the form $\mathcal{V}(\sigma) \subset \mathcal{V}(\tau)$ where $\sigma$ is a sub train track of $\tau$.

If $\tau$ is bi-recurrent, then the natural map $\iota_\tau: \mathcal{V}(\tau) \to \mathcal{ML}(S)$ is continuous and injective, see [HP, §1.7]. Let

$$U(\tau) = \iota_\tau(\mathcal{V}(\tau)) \subset \mathcal{ML}(S).$$

Moreover, we have the following.

**Lemma 5.1.** Let $U_1 \subset \mathcal{V}(\tau_1)$ and $U_2 \subset \mathcal{V}(\tau_2)$ be such that $\iota_{\tau_1}(U_1) = \iota_{\tau_2}(U_2)$. Then the map $\iota_{\tau_2}^{-1} \circ \iota_{\tau_1}: U_1 \to U_2$ is a piecewise linear map and hence it is bilipschitz.

For the proof see [HP, §2 and Thm. 3.1.4].

5.2. **Thurston symplectic form on $\mathcal{ML}(S)$**. We can identify $\mathcal{W}(\tau)$ with the tangent space of $\mathcal{ML}(S)$ at a point $u \in \mathcal{V}(\tau)$, see [HP].

For any train track $\tau$, the integral points in $\mathcal{V}(\tau)$ are in one to one correspondence with the set of integral multicurves in $U(\tau) \subset \mathcal{ML}(S)$. The natural volume form on $\mathcal{V}(\tau)$ defines a mapping class group invariant volume form $\mu_{Th}$ in the Lebesgue measure class on $\mathcal{ML}(S)$.

In fact, the volume form on $\mathcal{ML}(S)$ is induced by a mapping class group invariant two form $\omega$ as follows. Suppose $\tau$ is maximal, for $u_1, u_2 \in \mathcal{W}(\tau)$ the symplectic pairing is defined as follows.

$$\omega(u_1, u_2) = \frac{1}{2} \left( \sum u_1(e_1) u_2(e_2) - u_1(e_2) u_2(e_1) \right),$$

the sum is over all vertices $v$ of the train track where $e_1$ and $e_2$ are the two incoming branches at $v$ such that $e_1$ is on the right side of the common tangent vector.

This form defines an antisymmetric bilinear form on $\mathcal{W}(\tau)$.

**Lemma 5.2.** Let $\tau$ be maximal. The Thurston form $\omega$, defined in (52), is non-degenerate. Therefore it gives rise to a symplectic form on the piecewise linear manifold $\mathcal{ML}(S)$.

See [HP, §3] for a proof and also the relationship between the intersection pairing of $H^1(S, \mathbb{R})$ and Thurston intersection pairing.
5.3. **Combinatorial type of measured laminations and train tracks.** Each component of $S - \lambda$ is a region bounded by closed geodesics and infinite geodesics; further, any such region can be doubled along its boundary to give a complete hyperbolic surface which has finite area.

We say a filling measured lamination $\lambda$ is of type $a = (a_1, \ldots, a_k)$ if and only if $S - \lambda$ consists of ideal polygons with $a_1, \ldots, a_k$ sides. By extending the measured lamination $\lambda$ to a foliation with isolated singularities on the complement, we see that $\sum_{i=1}^{k} a_i = 4g - 4 + 2k$, see [Th1] and [Le].

Similarly, each component of the complement of a filling train track $\tau$ is a non-punctured or once-punctured cusped polygon of negative Euler index. We say a train track $\tau$ is of type $a = (a_1, \ldots, a_k)$, if and only if $S - \tau$ consists of $k$ polygons with $a_1, \ldots, a_k$ sides. Every measured lamination of type $a = (a_1, \ldots, a_k)$ can be carried by a train track of type $a$.

**Lemma 5.3.** For any filling train track $\tau$ of type $a = (a_1, \ldots, a_k)$ we have

$$\dim(V(\tau)) = 2g + k - 1 \quad \text{if } \tau \text{ is orientable;}$$

$$\dim(V(\tau)) = 2g + k - 2 \quad \text{if } \tau \text{ is not orientable.}$$

More generally, a measured lamination $\lambda$ is said to be of type $a$ if there exists a quadratic differential $q \in \mathbb{Q}(a_1 - 2, \ldots, a_k - 2)$ such that $\lambda = \Re(q)$. It is easy to check that if $\lambda$ is filling, the above can happen only if $S - \lambda$ consists of ideal polygons with $a_1, \ldots, a_k$ sides.

In general, see [HP, §3], we have:

**Proposition 5.4.** Given a measured lamination $\lambda$ of type $a$, there exists a birecurrent train track of type $a$ such that $\lambda$ is an interior point of $U(\tau)$.

For every $a = (a_1, \ldots, a_k)$ so that $\sum_{i=1}^{k} a_i = 4g - 4 + 2k$, we can fix a collection $\tau_{a,1}, \ldots, \tau_{a,c_a}$ of train tracks with the following property. Every $\lambda$ which can be carried by a train track of type $a$ can be carried by at least one $\tau_{a,i}$ for some $i$.

5.4. **The Hubbard-Masur map.** Let $\mathcal{MF}(S)$ denote the space of measured foliations on $S$. Define

$$\hat{P} : QT(S) \to \mathcal{MF}(S) \times \mathcal{MF}(S) - \Delta$$

by $\hat{P}(q) = (\Re(q^{1/2}), \Im(q^{1/2}))$ where

$$\Delta = \{(\eta, \lambda) : \text{there exists } \sigma \text{ so that } i(\sigma, \lambda) + i(\sigma, \eta) = 0\}.$$

**Theorem 5.5** (Hubbard-Masur, Gardiner). The map $\hat{P}$ is a Mod$_g$ equivariant homeomorphism.

This gives rise to an equivariant homeomorphism from $QT(S)$ onto $\mathcal{ML}(S) \times \mathcal{ML}(S) - \Delta$ which we continue to denote by $\hat{P}$, see [Th1] and [Le].

Recall that $\mathcal{PM}(S)$ denotes the space of projective measured lamination. The map $\hat{P}$ also gives rise to an equivariant homeomorphism

$$\hat{P}_1 : Q_1 T(S) \to \mathcal{PM}(S) \times \mathcal{ML}(S) - \Delta$$
where \( \tilde{P}_1(q) = ([R(q^{1/2}]), J(q^{1/2})) \) and \( \Delta = \{(\eta, \lambda): \exists \sigma \text{ so that } i(\sigma, \eta) + i(\sigma, \lambda) = 0\} \).

Recall that \( \pi \) is the natural projection from \( Q_1T(S) \) to \( Q_1(S) \), then
\[
\pi \circ \tilde{P}_1^{-1} : PM\mathcal{L}(S) \times M\mathcal{L}(S) - \Delta \rightarrow Q_1(S)
\]

Let \( U^1 \) and \( U^2 \) be as above. Then for any \( \lambda \in U^2 \) and \( [\eta], [\eta'] \in U^1 \) we have \( \tilde{P}_1^{-1}(\eta, \lambda) \) and \( \tilde{P}_1^{-1}(\eta', \lambda) \) belong to the same leaf of the strong unstable foliation, i.e., the leaf
\[\{q: J(q) = \lambda\}\]
similarly, \( \pi \circ \tilde{P}_1^{-1}([\eta], \lambda) \) and \( \pi \circ \tilde{P}_1^{-1}([\eta'], \lambda) \) lie in \( \pi(\{q: J(q) = \lambda\}) \).

5.5. Convexity of the hyperbolic length function. Let \( \beta_1, \beta_2 \in U(\tau) = \iota_\tau(V(\tau)) \), see §5.1 for the definition of \( \iota_\tau \). The sum
\[
\beta_1 \oplus_\tau \beta_2 = \iota_\tau(i_\tau^{-1}(\beta_1) + i_\tau^{-1}(\beta_2))
\]
could depend on \( \tau \). However, it is proved in [Mir1, App. A] that given a closed curve \( \gamma \), \( i(\gamma, .) : U(\tau) \rightarrow \mathbb{R}_+ \) defines a convex function; from this one gets the following.

**Theorem 5.1** ([Mir1], Thm. A.1). For any hyperbolic surface \( M \), the hyperbolic length function
\[
\ell_M : U(\tau) \rightarrow \mathbb{R}^+
\]
is convex. That is: if \( \beta_1 \) and \( \beta_2 \) are carried by \( \tau \), then \( \ell_M(\beta_1 \oplus_\tau \beta_2) \leq \ell_M(\beta_1) + \ell_M(\beta_2) \).

Let \( C \subset \mathbb{R}^n \) be a cone and \( f : C \rightarrow \mathbb{R} \) be a convex function. Let \( K \) be a closed and bounded set contained in the relative interior of the domain of \( f \). Then \( f \) is Lipschitz continuous on \( K \). That is: there exists a constant \( L = L(K) \) such that for all \( x, y \in K \) we have
\[
|f(x) - f(y)| \leq L|x - y|.
\]
Therefore, we have the following.

**Corollary 5.2.** For any hyperbolic surface \( M \),
\[
\ell_M : M\mathcal{L}(S) \rightarrow \mathbb{R}^+
\]
is locally Lipschitz. In other words, and in view of the fact that \( \ell_M(t\cdot) = t\ell_M(\cdot) \) for all \( t > 0 \), we can cover \( M\mathcal{L}(S) \) with finitely many cones such that \( \ell_M \) is Lipschitz in each cone.

The Lipschitz constant depends on \( M \). See also [LS].

6. Linear structure of \( M\mathcal{L}(S) \) and \( QT(S) \)

Our arguments are based on relating the counting problems in \( M\mathcal{L}(S) \) to dynamical results in \( Q_1(1, \ldots, 1) \). To that end, we need to compare the linear structure on \( Q_1(1, \ldots, 1) \), arising from period coordinates, with the piecewise linear structure on \( M\mathcal{L}(S) \), which arises from train track charts. This section establishes required results in this direction.

From this point to the end of the paper we will be concerned with the principal stratum, i.e., \( Q_1(1, \ldots, 1) \). Also \( a = (3, \ldots, 3) \) for the rest of the discussion.
Fix once and for all a collection $\tau_1, \ldots, \tau_c$ of train tracks so that every $\lambda$ can be carried by at least one $\tau_i$ for some $i$, see \cite{5.3}.

Given a point $x = (M, q) \in \mathcal{Q}_1(1, \ldots, 1)$ we sometimes use $q$ to denote $x$. We fix a fundamental domain for $\mathcal{Q}_1(1, \ldots, 1)$, and unless explicitly stated otherwise, by a lift $\tilde{q}$ of $q \in \mathcal{Q}_1(1, \ldots, 1)$ we mean a representative in this fundamental domain.

Let $x = (M, q) \in \mathcal{Q}_1(1, \ldots, 1)$. We denote by $\mathfrak{R}(q^{1/2})$ (resp. $\mathfrak{I}(q^{1/2})$) the real (resp. imaginary) foliation induced by $q$; abusing the notation we will often simply denote these foliations by $\mathfrak{R}(q)$ and $\mathfrak{I}(q)$. Note that $W^{u,s}(x)$, which we also sometimes denote by $W^{u,s}(q)$, may also be defined as follows.

$$W^u(q) := \{q' \in \mathcal{Q}_1(1, \ldots, 1) : \mathfrak{I}(q') = \mathfrak{I}(q)\},$$

and

$$W^s(q) := \{q' \in \mathcal{Q}_1(1, \ldots, 1) : \mathfrak{R}(q') = \mathfrak{R}(q)\}.$$

Similarly, we will write $B_r(q)$ and $B^*(q)$ for $B_r(x)$ and $B^*_r(x)$, respectively.

Let $\tau$ be a maximal train track, i.e., a train track of type $(3, \ldots, 3)$; and let $U(\tau)$ be a train track chart, i.e., the set of weights on $\tau$ satisfying the switch conditions. Recall from \cite{5.1} that $U(\tau)$ has a linear structure, indeed $U(\tau)$ is a cone on a finite-sided polyhedron. We use the $L^1$-norm on $W(\tau)$ to define a norm on $U(\tau)$. That is: for any measured lamination $\lambda \in U(\tau)$ we define $\|\lambda\|_\tau$ to be the sum of the weights of $\lambda$. Let us define

$$P(\tau) := \{\lambda \in U(\tau) : \|\lambda\|_\tau = 1\}.$$

For every $\lambda \in U(\tau)$ define

$$\bar{X}^\tau := \frac{1}{\|\lambda\|_\tau} \lambda \in P(\tau);$$

if $\tau$ is fixed and clear from the context, we sometimes drop the subscript and the superscript $\tau$ and simply write $\|\lambda\|$ and $\bar{X}$ for $\|\lambda\|_\tau$ and $\bar{X}^\tau$, respectively.

By a polyhedron $U \subset U(\tau)$ we mean a polyhedron of dimension $\dim U(\tau) - 1$ where the angles are bounded below and the number of facets are bounded, both by absolute constants depending only on the genus. We will mainly be concerned with $\dim U(\tau) - 1$ dimensional cubes in the sequel.

**Lemma 6.1** (Cf. \cite{LMir}, Thm. 6.4). Let $\eta \in \mathcal{ML}(S)$ be maximal. There is a compact subset $K \subset \mathcal{Q}_1(1, \ldots, 1)$, depending on $\tau$ and $\eta$, so that $\pi \circ \mathcal{P}^{-1}_1([\eta], P(\tau)) \subset K$.

**Proof.** Recall that we fixed a collection $\tau_1, \ldots, \tau_c$ of train tracks so that every lamination $\lambda$ is carried by some $\tau_i$. In view of Lemma 5.1 there exists some $L = L(\tau)$ so that

$$P(\tau) \subset \bigcup_{i=1}^c \{\lambda \in U(\tau_i) : 1/L \leq \|\lambda\|_i \leq L\};$$

where $\|\| = \|\|_{\tau_i}$.

Since $\eta$ is a maximal measured lamination, for any $\lambda \in U(\tau_i)$ we have $\pi \circ \mathcal{P}^{-1}_1([\eta], \lambda) \in \mathcal{Q}_1(1, \ldots, 1)$. Let $1 \leq i \leq c$, and put $U_i := \{\lambda \in U(\tau_i) : 1/L \leq \|\lambda\|_i \leq L\}$. Define

$$K := \bigcup_i \pi \circ \mathcal{P}^{-1}_1([\eta]) \times U_i.$$
Then \( K \subset \mathcal{Q}_1(1, \ldots, 1) \) is a compact subset with the desired property. \( \square \)

**Lemma 6.2.** There is some \( N_9 \) so that the following holds. Let \( q \in \mathcal{Q}_1(1, \ldots, 1) \). There exists a 1-complex \( T \subset M \) with the following properties.

1. Every edge of \( T \) is a saddle connection.
2. \( |\mathcal{I}(e)| \geq 0.1 \ell_q(e) \) for any \( e \in T \).
3. \( S - T \) is a union of triangles.
4. There is a constant \( A_q \) so that \( A_q^{-1} \leq \ell_q(e) \leq A_q \) for every edge \( e \in T \); moreover, \( A_q \) may be taken to be uniform on compact sets of \( \mathcal{Q}_1(1, \ldots, 1) \).
5. There is a period box \( B_q(q) \) containing \( q \) with the following properties.
   - (a) \( \text{dist}(q, \partial B_q(q)) \geq u(q)^{-1/N_4} \) where \( u(q) \) is as in Theorem 2.6.
   - (b) the parallel translate of \( T \) to \( q' = B_q(q) \) satisfies (1), (2), and (3) above,
   - (c) the restriction of \( \pi \circ \mathcal{P}_1^{-1} \) to \( B_q(q) \) is a diffeomorphism.

Similar statement holds if we replace \( \mathcal{I}(e) \) in (3) above by \( \mathcal{R}(e) \).

**Proof.** We find such a \( T \) with \( |\mathcal{I}(e)| > 0.1 \ell_q(e) \), the proof for \( T \) with \( |\mathcal{R}(e)| > 0.1 \ell_q(e) \) is similar by replacing \( a_tu_s \) by \( a_{-t}u_s \) in the following argument.

Let \( K \) be the compact set given by Theorem 2.6. Then for every \( q' \in K \), there is a graph \( T' \) of saddle connections in \( q' \) each of length bounded by \( L_0 = L_0(K) \) so that \( S - T' \) is a union of triangles. We will always assume \( L_0 > 2 \). Increasing \( L_0 \), if necessary, we may and will assume that \( L_0 \) also bounds the lengths of saddle connections obtained by parallel transporting \( T' \) to \( q'' \in B_0.1(q') \) for all \( q' \in K \).

Set \( R_q := \{ \text{saddle connections } \gamma \text{ of } q \text{ with } |\mathcal{R}(\gamma)| > 0.9 \ell_q(\gamma) \} \). Note that for all \( \gamma \in R_q \) and any \( 0 \leq s \leq 1 \) we have \( |\mathcal{R}(u_s \gamma)| \geq \ell_q(\gamma)/2 \). Define the function

\[
f(q) := \max\{1, \max\{1/\ell_q(\gamma) : \gamma \in R_q\}\}.
\]

Apply Theorem 2.6 with \( t_0 = L_0 \log f(q) \). There exists some

\[(55) \quad t_0 < t \leq \max\{2t_0, N_4 \log u(q)\}\]

and some \( u_s \in [0, 1] \) so that \( q' = a_tu_s q \in K \).

Let now \( T' \) be a graph of saddle connections for \( q' \) defined as above. We claim that for any \( e \in T' \) we have \( e \notin a_tu_s R_q \). To see the claim note that for every \( \gamma \in R_q \) we have

\[
\ell_q'(a_tu_s \gamma) \geq e^t \mathcal{R}(u_s \gamma)
\]

\[
\geq e^t \ell_q(\gamma)/2 \quad |\mathcal{R}(u_s \gamma)| \geq \ell_q(\gamma)/2
\]

\[
\geq e^{L_0} f(q) \ell_q(\gamma)/2 > L_0 \quad t > L_0 \log f(q) \& f(q) \ell_q(\gamma) \geq 1.
\]

Hence \( a_tu_s \gamma \) is not contained in \( T' \). In consequence, \( T = u_{-s}a_{-t}T' \) satisfies (1), (2), (3), and (4) in the lemma.

We now turn to the proof of part (5). Let \( r_0 \) be so that

\[
\Phi^{-1}(\Phi(q') + v) \in B_{0.1}(q')
\]

for all \( q' \in K \) and all \( \|v\|_{q'} \leq r_0 \).
First note that there is a constant $N_{10}$ so that $u(q)^{N_{10}} \geq f(q)^{2L_0}$; put $N_{11} := \max\{N_4, N_{10}, N, N_4\}$.

Let $N_0 > N_{11}$ be so that

$$e^{N(t)N_0 - N_0} \leq r_0.$$  

Let $r$ be small enough so that $z \in B_r(q)$ implies that $z = \Phi^{-1}(\Phi(q') + v)$ for some $\|v\|_q \leq u(q)^{-N_0}$. We claim that (5) above holds for $B_r(q)$.

To see the claim, let $t \leq \max\{2L_0f(q), N_4\log u(q)\}$ and $0 \leq s \leq 1$ be so that $q' = a_tu_sq \in K$; see the preceding discussion. Note that in view of the choice of $t$ and since $N_{11} = \max\{N, N_{10}, N, N_4\}$ we have

$$e^{N(t)N_0} \leq u(q)^{-N_{11}}.$$  

Now for all $v$ so that $\Phi^{-1}(\Phi(q') + v) \in B(q)$ we have

$$\|v\|_{a_tu_sq} \leq e^{N\|v\|_{u_sq}}$$ by (12),

$$\leq e^{N\|v\|_{u_sq}N_0}e^{-N_0}\|v\|_q$$ by (12), and $|s| \leq 1$,

$$\leq e^{N_{11}u(q)^{N_{11}}N_0}\|v\|_q$$ by (57),

$$\leq e^{N_{11}u(q)^{N_{11}}N_0}\|v\|_q \leq u(q)^{-N_0}$$ by the choice of $r$,

$$\leq e^{N_{11}u(q)^{N_{11}}N_0} \leq r_0$$ since $u(q) \geq 2$ and using (56).

Hence $a_tu_sq \in B_0(q)$ which gives the claim. 

\[
\textbf{Lemma 6.3 (Cf. Mir3, Lemma 4.3). Let } q \in Q_1(1, \ldots, 1) \text{ and let } \tilde{q} \text{ be a lift of } q \text{ in our fixed fundamental domain. There exists a period box } B_r(\tilde{q}), \text{ which homeomorphically maps onto a period box } B_r(q) \subset Q_1(1, \ldots, 1), \text{ and a maximal train track } \sigma \text{ whose dependence on } q \text{ will explicate in the proof with the following properties:}
\]

\begin{enumerate}
\item $\text{dist} (\tilde{q}, \partial B_r(\tilde{q})) \geq u(q)^{-N_0}$
\item The restriction of $\mathfrak{P}_1$ to $B_r(\tilde{q})$ is a homeomorphism.
\item $\{\mathfrak{J}(\tilde{p}) : \tilde{p} \in B_r(\tilde{q})\}$ is contained in one train track chart $U(\sigma)$.
\item The linear structure on $U_2(\tilde{q}) := \{\mathfrak{J}(\tilde{p}) : \tilde{p} \in B_r(\tilde{q}), \mathfrak{R}(\tilde{p}) = \mathfrak{R}(\tilde{q})\}$ as a subset of $U(\sigma)$ agrees with the linear structure on $U_2(\tilde{q})$ which is induced by the restriction of $\mathfrak{P}_1$ to $\{\tilde{p} \in B_r(\tilde{q}) : \mathfrak{R}(\tilde{p}) = \mathfrak{R}(\tilde{q})\} \subset W^s(\tilde{q})$.
\end{enumerate}

Moreover, the radius $r$ of $B_r(\tilde{q})$ can be taken to be uniform on compact subsets of $Q_1(1, \ldots, 1)$.

\textbf{Proof.} Let $T$ be a triangulation of $q$ given by Lemma 6.2. In particular,

\begin{enumerate}
\item every edge of $T$ is a saddle connection,
\item $|\mathfrak{J}(e)| \geq 0.1L_q(e)$ for any $e \in T$,
\item $S - T$ is a union of triangles, and
\item there is a constant $A_q$ so that $A_q^{-1} \leq L_q(e) \leq A_q$ for every edge $e \in T$; moreover, $A_q$ may be taken to be uniform on compact sets of $Q_1(1, \ldots, 1)$.
\end{enumerate}

Our construction of the train track $\sigma$ will depend on $T$. 

Let $B_r(q)$ be as in Lemma 6.2 and let $B_r(\tilde{q})$ be the corresponding lift at $\tilde{q}$. Therefore, $B_r(\tilde{q})$ satisfies (1) and (2) in the lemma by Lemma 6.2.5.

We will always assume that the radius $r$ of $B_r(q)$ is $\leq 0.01 \lambda_q^{-2}$. Let $\sigma'$ be the null-gon dual graph to $T$, in particular, there is one triangle of $\sigma'$ in each component of $S - T$. Let $\sigma$ be the train track obtained from $\sigma'$ as follows. If $\Delta$ is a triangle in $T$ with edges $e_1^\Delta, e_2^\Delta, e_3^\Delta$, then there is a permutation $\{i_1, i_2, i_3\}$ of $\{1, 2, 3\}$ so that

$$|J(e_{i_1}^\Delta)| = |J(e_{i_2}^\Delta)| + |J(e_{i_3}^\Delta)|;$$

put $\sigma := \sigma' - \bigcup\{\text{the edge corresponding to } e_{i_1}^\Delta \text{ in } \sigma'\}$. We claim the lemma holds with $\sigma$. To see the claim, first note that $\sigma$ is a maximal train track. Assign the weight $|J(e_{i_b})|$ to each branch $b \in \sigma$ where $e_{i_b} \in T$ is the edge which intersects $b$. In view of (58) and the fact that $|J(\gamma)| = i(\gamma, \mathcal{R}(\tilde{q}))$ for any saddle connection $\gamma$, we get that $\lambda = J(\tilde{q})$ is carried by $\sigma$.

By Lemma 6.2, for any $\tilde{p} \in B_r(\tilde{q})$ we identify $T$ with its image (under parallel transport) on $\tilde{p}$. Let $\tilde{p} \in B(q)$ and write $\tilde{p} = \tilde{q} + w$ for some $w$ with $\|w\|_q \leq 0.01 \lambda_q^{-2}$, see (2.1). Then

$$|J(\text{hol}_l(e_{i_b}))| = |J(\text{hol}_q(e_{i_b})) + J(w(e_{i_b}))|.$$

Further, we have $|w(e_{i_b})| \leq 0.01 A_q^{-2} \lambda_q(e_{i_b}) < 0.01 A_q^{-1} \leq 0.1 |J(\text{hol}_q(e_{i_b}))|$; in particular, $J(\tilde{p})$ is carried by the train track $\sigma$.

Taking $w \in iH_1(M, \Sigma, \mathbb{R})$, the above discussion also implies that $\sigma$ satisfies (3) and (4). □

7. Counting integral points in $\mathcal{ML}(S)$

Let the notation be as in (66). In particular, $\tau$ is a maximal train track. Also recall that $P(\tau)$ denotes the finite-sided polyhedron in $U(\tau)$ corresponding to laminations with $\|\lambda\|_\tau = 1$.

The smallest $t$ so that a lamination $\lambda \in U(\tau)$ lies in

$$[0, e^t]P(\tau) = \{\lambda' \in U(\tau) : \|\lambda'\|_\tau \leq e^t\}$$

can be thought of as a measure of complexity (or length) for the lamination $\lambda$. In this section we obtain an effective counting result with respect to this complexity. In 88 we will use the convexity of the hyperbolic length function in $U(\tau)$ to relate the counting problem in Theorem 1.1 to this counting problem.

Let $U \subset P(\tau)$ be a cube. For any $t \geq 0$, define

$$\mathcal{O}_\tau(\gamma_0, e^t, U) := \{\gamma \in \text{Mod}_g \cdot \gamma_0 \cap [0, e^t]\|U\}.\tag{59}$$

The following is the main result of this section which is a strengthening of Theorem 1.2.

**Theorem 7.1.** There exist $\kappa_{23}$ and $\kappa_{24}$ so that the following holds. Let $t \geq 1$ and let $U \subset P(\tau)$ be a cube of size $\geq e^{\kappa_{23} \tau}$. Then

$$\#\mathcal{O}_\tau(\gamma_0, e^t, U) = v(\gamma_0)\mu_{\text{Th}}([0, 1]\|U\|)e^{ht} + O_{\tau, \gamma_0}(e^{(h - \kappa_{24})t})$$

where $v(\gamma_0)$ is defined as in (62) and $h = 6g - 6$. 

The basic tool in the proof of Theorem \ref{thm:effective-counting} is Proposition \ref{prop:local-counting}. We relate the counting problem in Theorem \ref{thm:effective-counting} to a counting problem for translations of $W^u(q_0)$ in Lemma \ref{lem:local-counting}. Proposition \ref{prop:local-counting} studies a more local version of this latter counting problem. That is: one works with translations of a small region in $W^u(q_0)$. Using Corollary \ref{cor:local-counting} we will reduce to this local analysis. The main step in the proof of Theorem \ref{thm:effective-counting} is Lemma \ref{lem:local-counting} below.

Let us begin with some preparation. Recall that $\mathcal{ML}(S)$ does not have a natural differentiable structure, in particular, $\tilde{\varphi}$ is only a homeomorphism. The situation however drastically improves so long as we restrict to one train track chart and fix a transversal lamination. Therefore, we fix a maximal lamination $\eta$ which is transversal to $\tau$ for the rest of the discussion.

Let $\delta > 0$ and let $U \subset P(\tau)$ be a cube of size $\geq \delta$ centered at $\lambda$; and let $\epsilon \leq \delta$. We always assume $\tilde{\varphi}^{-1}$ is a homeomorphism on $\{[\eta]\} \times \{e^U : |r| \leq \delta\}$. Put $\tilde{W}^{cs}_{U,\epsilon} = \tilde{\varphi}^{-1}_1([\eta]) \times U$ and

$$\tilde{\varphi}_1^{-1}([\eta]) \times \{e^U : \epsilon < r \leq 0\};$$

Let $\gamma_0 \in U(\tau)$ be a rational multicurve. For all $t \geq 0$ and $0 < \epsilon < 1$ define

$$O_\tau(\gamma_0, t, U, \epsilon) := \{\gamma \in U(\tau) \cap \text{Mod}_g: \gamma_0 : \epsilon^{t-\epsilon} \leq \|\gamma\|_r \leq \epsilon^t \text{ and } \gamma^r \in U\}.$$ 

Put $\tilde{q}_0 := \tilde{\varphi}_1^{-1}([\eta], \tau_0)$. Without loss of generality we assume $\gamma_0$ and $\eta$ are so that $\tilde{q}_0$ belongs to our fixed fundamental domain.

**Lemma 7.2.** Let $\delta > 0$ and let $U \subset P(\tau)$ be a cube of size $\geq \delta$; let $\lambda$ denote the center of $U$. For all $\epsilon \leq \delta$ and all large enough $t \geq 0$ we have:

$$g\gamma_0 \in O_\tau(\gamma_0, t, U, \epsilon) \text{ if and only if } \tilde{W}^{cs}_{U,\epsilon} \cap g \cdot a_t \tilde{W}^u(\tilde{q}_0) \neq \emptyset.$$

**Proof.** Since $\tau$ is fixed throughout, we drop it from the subscript and superscript for the norm and the normalization.

Suppose $\gamma = g\gamma_0 \in O_\tau(\gamma_0, t, U, \epsilon)$ for some $g \in \text{Mod}_g$; such $g$ is not unique, however, for any other $g' \in \text{Mod}_g$ with $g\gamma_0 = g'\gamma_0$ we have $g \cdot \tilde{W}^u(\tilde{q}_0) = g' \cdot \tilde{W}^u(\tilde{q}_0)$. Put $\tilde{q} = g \cdot \tilde{q}_0$. Then $g\gamma = \tilde{\varphi}(\tilde{q})$, moreover,

$$g \cdot a_t \tilde{W}^u(\tilde{q}_0) = a_t \tilde{W}^u(\tilde{q}).$$

Recall that $\gamma \in U$ and put $\tilde{p}' := \tilde{\varphi}^{-1}_1([\eta], \gamma)$. Then, $\tilde{p}' \in \tilde{W}^{cs}_{U,\epsilon}$; moreover, it follows from the definition that $\tilde{\varphi}(\tilde{p}') = \gamma$. Hence, $\tilde{p}' \in a_t \tilde{W}^u(\tilde{q})$ where $t_1 = \log \|\gamma\|_r$.

Put $s = t_1 - t$; since $\gamma \in O_\tau(\gamma_0, t, U, \epsilon)$ we have $-\epsilon \leq s \leq 0$. We get from the above and the definition of $\tilde{W}^{cs}_{U,\epsilon}$ that $a_s \tilde{p}' \in a_t \tilde{W}^u(\tilde{q}) \cap \tilde{W}^{cs}_{U,\epsilon}$. In particular,

$$\tilde{W}^{cs}_{U,\epsilon} \cap a_t \tilde{W}^u(\tilde{q}) = \tilde{W}^{cs}_{U,\epsilon} \cap g \cdot a_t \tilde{W}^u(\tilde{q}_0) \neq \emptyset.$$

Conversely, suppose that for some $g \in \text{Mod}_g$ we have $\tilde{W}^{cs}_{U,\epsilon} \cap g \cdot a_t \tilde{W}^u(\tilde{q}_0) \neq \emptyset$. Put $\gamma = g\gamma_0$; we claim that $\gamma \in O_\tau(\gamma_0, t, U, \epsilon)$.

Set $\tilde{q} = g \cdot \tilde{q}_0$. Then $\tilde{\varphi}(\tilde{q}) = \gamma$, and as above we have $g \cdot a_t \tilde{W}^u(\tilde{q}_0) = a_t \tilde{W}^u(\tilde{q})$. Let now $\lambda \in U$ and $-\epsilon \leq s \leq 0$ be so that

$$\tilde{\varphi}_1^{-1}([\eta], e^s\lambda) \in \tilde{W}^{cs}_{U,\epsilon} \cap a_t \tilde{W}^u(\tilde{q}).$$
Let us write $\tilde{P}_1^{-1}([\eta], e^s\lambda) = a_0 q'$ where $q' \in \tilde{W}^u(q)$. Then, we have

$$e^{-t} \gamma = \mathcal{J}(a_0 q') = e^s \lambda \in e^s \mathcal{U}.$$ 

This gives $\tilde{\gamma} = \lambda$, hence, $\tilde{\gamma} \in \mathcal{U}$ and $\|\gamma\| = e^{t+s}$; we get $\gamma \in O_t(\gamma_0, t, \mathcal{U}, \epsilon)$ as we claimed. \qed

### 7.1. Strebel differentials

Problems related to the existence and uniqueness of Jenkins-Strebel differentials have been extensively studied.

**Theorem 7.3** (Cf. [Str], Thm. 20.3). Let $\gamma = \sum_{i=1}^d a_i \gamma_i$ be a rational multi-geodesic on $M$ and let $r_1, \ldots, r_d$ be positive real numbers. Then there exists a unique holomorphic quadratic differential $q$ on $M$ (Jenkins-Strebel differential) with the following properties.

1. If $\Gamma$ is the critical graph\(^1\) of $q$, then $M - \Gamma = \bigcup_{i=1}^d \Omega_i$, where $\Omega_i$ is either empty or a cylinder whose core curve is $\gamma_i$.

2. If $\Omega_i$ is not empty, it is swept out by trajectories whose $q_i$ length is $r_i$.

The following lemma will be used in the sequel.

**Lemma 7.4.** Let $\gamma \in U(\tau)$ be rational and let $\tilde{q} = \tilde{P}_1^{-1}([\eta], \gamma) \in Q^1 \mathcal{T}(\alpha)$ be a quadratic differential so that $\mathcal{J}(\tilde{q}) = \gamma$; put $q := \pi(\tilde{q})$. Then

1. $W^u(q) \subset Q_1(1, \ldots, 1)$ is a properly immersed, affine submanifold which carries a natural finite Borel measure $\nu$.

2. There exists some $\epsilon_0 = \epsilon_0(\tau, \eta, \|\gamma\|_\tau) > 0$ so that the following holds. Let $0 < \hat{\epsilon} < \epsilon_0$ and let

$$K(\hat{\epsilon}) = \{q : \text{all saddle connections on } q \text{ are } \geq \hat{\epsilon}\}.$$ 

Put $D_{\text{cusp}}(\hat{\epsilon}) := W^u(q) \cap K(\hat{\epsilon})^c$. There are constants $\kappa_{25}$ and $N_{12}$, and a smooth function $0 \leq \psi^u_\tau \leq 1$ supported on $W^u(q)$ so that

1. $\mathcal{C}^1(\psi^u_\tau) \ll \hat{\epsilon}^{-N_{12}}$

2. $\|\psi^u_\tau\|_{2, \nu} \ll \frac{1}{\kappa_{25}}$

3. $\psi^u_\tau(\nu, D(\hat{\epsilon})) = 1$, and $\|1_{\mathcal{D}(\hat{\epsilon})} - \psi^u_\tau\|_{2, \nu} \ll \frac{1}{\kappa_{25}}$

In particular, we have $\nu(D(\hat{\epsilon})) \leq \frac{1}{\kappa_{25}}$ for all small enough $\hat{\epsilon}$.

**Proof.** We first show that $W^u(q)$ is a properly immersed submanifold of $Q_1(1, \ldots, 1)$. This is equivalent to showing the following two statements.

1. $g_1 \cdot \tilde{W}^u(\tilde{q}) \cap g_2 \cdot \tilde{W}^u(\tilde{q}) \neq \emptyset$ if and only if $g_1 \cdot \tilde{W}^u(\tilde{q}) = g_2 \cdot \tilde{W}^u(\tilde{q})$.

2. $\bigcup_{g \in \text{Mod}_q} g \cdot W^u(\tilde{q}) \subset Q^1 \mathcal{T}(\alpha)$ is closed.

Recall that $\tilde{W}^u(\tilde{p}) = \{\tilde{p}' : \mathcal{J}(\tilde{p}') = \mathcal{J}(\tilde{p})\}$ and that $g \cdot \tilde{W}^u(\tilde{p}) = \tilde{W}^u(g \cdot \tilde{p})$ for all $\tilde{p} \in Q^1 \mathcal{T}(\alpha)$. These imply (i). To see (ii), note further that the set

$$\bigcup_{g \in \text{Mod}_q} g \cdot \tilde{W}^u(\tilde{q})$$

\(^1\)Recall that the critical graph of a quadratic differential is the union of the compact leaves of the measured foliation induced by $q$ which contain a singularity of $q$. 


is the set of quadratic differentials $\tilde{p} \in Q_1^1 T(1, \ldots, 1)$ so that $T(\tilde{p}) \in \text{Mod}_g \cdot \gamma$. Since $\gamma$ is rational, $\text{Mod}_g \cdot \gamma$ is a discrete $\text{Mod}_g$-invariant set; (ii) follows.

Let $\gamma$ be as in the statement; write $\gamma = \sum a_i \gamma_i$ where each $\gamma_i$ is a simple closed curve and $a_i \in \mathbb{Q}$. By Theorem 7.3 we have: the locus $W^u(q) \cap Q_1(1, \ldots, 1)$ is identified with a linear subspace $W = \{(x_{i,j}) : \sum x_{i,j} = r_i, x_{i,j} > 0\}$ in the period coordinates, where $r_1, \ldots, r_d$ are positive real numbers. Moreover, the measure $\nu$ is the pull back of the Lebesgue measure from $W$ to $W^u(q)$. This finishes the proof of (1).

To see part (2) let $\epsilon_0$ be so that $\pi \circ \bar{D}_1^{-1}(\eta_i, \gamma) \in K(\epsilon_0)$, recall from Lemma 6.1 that $\epsilon_0$ depends only on $\tau, \eta$, and $||\gamma||_\tau$. For any $0 < \hat{\epsilon} < \epsilon_0$ put $W(\hat{\epsilon}) = \{(x_{i,j}) \in W : 0 < x_{i,j} < \hat{\epsilon} \text{ for some } i, j\}$.

Using Theorem 7.3, we have $W^u(q) \cap K(\hat{\epsilon}) \subset \bar{D}^{-1}(W(\hat{\epsilon}))$. The claims in part (2) now follow from Lemma 2.4. Indeed apply Lemma 2.4 with $\hat{\epsilon} = D(2\hat{\epsilon}) - D(\hat{\epsilon}/2)$ and let $\{\phi_i\}$ be the collection of functions obtained by that lemma. Define

$$
\psi_i^u(p) = \begin{cases} 
\sum \phi_i(p) & \text{if } p \in W^u(q) - D(\hat{\epsilon}/2) \\
1 & \text{if } p \in D(\hat{\epsilon}/2) \n\end{cases},
$$

This function satisfies the claims. \[ \square \]

Let $\gamma_0$ and $\tilde{q}_0 \in Q_1^1 T(1, \ldots, 1)$ be as in Lemma 7.2 and put $q_0 := \pi(\tilde{q}_0)$. Then by Lemma 7.4 we have $W^u(q_0)$ is an affine submanifold of $Q_1(1, \ldots, 1)$. We will put

$$
(62) \quad \nu(\gamma_0) = \nu(W^u(q_0))
$$

where $\nu$ is the finite measure in Lemma 7.4.

Let $b > 0$; this choice will be optimized later. Apply Lemma 7.4 with $\hat{\epsilon} = 10b$ and let $D_{\text{cusp}}(10b)$ be as in that lemma. Put $D_b := W^u(q) - D_{\text{cusp}}(10b)$.

**Lemma 7.5.** For every $b$ there exists some $N(b) \ll b^{-N_{14}}$ so that the following holds. There exists a collection of functions $\{\psi_i^u : 0 \leq i \leq N(b)\}$ with the following properties:

1. $\psi_0^u = \psi_b^u$ where $\psi_b^u$ is given by Lemma 7.4(2).
2. $0 \leq \psi_i^u \leq 1$ for all $i \geq 0$.
3. for all $i \geq 1$, $\psi_i^u$ is supported in $B_b(y_i)$ where $y_i \in D_b$; furthermore, the multiplicity of $\{B_b(y_i)\}$ is at most $N_2$.
4. $\sum_{i=1}^{N(b)} \psi_i^u = 1$ on $\bigcup_{i=1}^{N(b)} B_b(y_i)$.

Moreover, we have

$$
(63) \quad C^1(\psi_i^u) \leq N_{15} b^{-N_{14}} \text{ for all } 0 \leq i \leq N(b)
$$

where $N_{14}$ is an absolute constant and $N_{15}$ is allowed to depend on $q_0$.

**Proof.** This follows from Lemma 2.4 applied with $D = D_b$ and Lemma 7.4. \[ \square \]
Let us also fix a fundamental domain \( \tilde{D} \subset \tilde{W}^u(\tilde{q}_0) \) which projects to \( W^u(q_0) \). For each \( i \geq 1 \) we let \( \tilde{y}_i \in \tilde{D} \) be a lift of \( y_i \), see Lemma \( \text{7.5} \). Let \( N(b)' \) be so that
\[
B^u_i(\tilde{y}_i) \subset \tilde{D} \text{ for all } N(b)' < i \leq N(b).
\]
For simplicity in notation, let \( B^u_i(\tilde{y}_0) \subset \tilde{D} \) denote the lift of \( D_{\text{cusp}}(10b) \).

7.2. Counting in linear sectors in \( \mathcal{ML}(S) \). Recall from the beginning of this section that \( U \subset P(\tau) \) is a box of size \( \geq \delta \). Let \( \lambda \) be the center of \( U \) and let \( \epsilon \leq \delta \). Let \( \eta \in \mathcal{ML}(S) \) be fixed as in the beginning of this section. We always assume \( 0 < \delta < 1/2 \) and \( \eta \) are so that \( \tilde{P}^{-1}_1 \) is a homeomorphism on \( \{[\eta]\} \times \{e^r U : |r| < \delta\} \). Recall also our notation \( \tilde{W}^\text{cs}_U = \tilde{P}^{-1}_1(\{[\eta]\} \times U) \) and
\[
\tilde{W}^\text{cs}_{U,\epsilon} = \tilde{P}^{-1}_1(\{[\eta]\} \times \{e^r U : -\epsilon < r \leq 0\}).
\]
Abusing the notation, we denote by \( \mu_{\text{Th}}(U) \) the measure induced from \( \mu_{\text{Th}} \) on \( P(\tau) \). The following lemma is a crucial step in the proof of Theorem \( \text{7.1} \).

**Lemma 7.6.** There exist \( \kappa_{26} \) and \( \kappa_{27} \) so that the following holds. Let \( t \geq 0 \) and in the above notation, define
\[
\mathcal{N}(\tilde{q}_0, t, U, \epsilon) := \{ g \cdot \tilde{W}^u(\tilde{q}_0) : g \in \text{Mod}_g \text{ and } \tilde{W}^\text{cs}_{U,\epsilon} \cap g \cdot \tilde{W}^u(\tilde{q}_0) \neq \emptyset \}.
\]
Suppose \( \epsilon \geq e^{-\kappa_{26}} \), then
\[
\# \mathcal{N}(\tilde{q}_0, t, U, \epsilon) = v(\gamma_0) \mu_{\text{Th}}(U) \left( \frac{1-e^\gamma h}{h} \right) e^{ht} + O_{r,\gamma_0}((1-e^\gamma h) e^{(h+\kappa_{27})t}).
\]
We will prove Lemma \( \text{7.6} \) using Proposition \( \text{4.1} \) more precisely Corollary \( \text{4.7} \). In order to use those results we need to control the geometry of \( \tilde{W}^\text{cs}_{U,\epsilon} \).

**Lemma 7.7.** The characteristic function of
\[
\tilde{W}^\text{cs}_{U,\epsilon} = \tilde{P}^{-1}_1(\{[\eta]\} \times \{e^s U : |s| \leq \epsilon\})
\]
belongs to \( \mathcal{S}_{\tilde{W}^\text{cs}(\tilde{q}_j)}(\tilde{p}, \epsilon) \) where \( \tilde{p} = \tilde{P}^{-1}_1(\eta, \lambda) \).

**Proof.** Apply Lemma \( \text{6.1} \) with \( \tau \) and let \( K = K(\tau) \) be defined as in \( \text{54} \). Then
\[
\pi \circ \tilde{P}^{-1}_1(\eta, P(\tau)) \subset K.
\]
Let \( \{B_p(p) : p \in K \} \) be the covering of \( K \) by period boxes given by Lemma \( \text{6.3} \). Let \( B_1(q_1), \ldots, B_b(q_{b'}) \) be a finite subcover of this covering. Consider all lifts of \( B(q_j) \) to period boxes based at lifts \( \tilde{q}_j \) of \( q_j \) in our fixed (weak) fundamental domain. Denote these lifts by \( B_{r_1}(\tilde{q}_j), \ldots, B_{r_b}(\tilde{q}_0) \) — note that we only fixed a weak fundamental domain, hence there might be more than one lift, however, there is a universal bound on the number of lifts.

For every \( 1 \leq j \leq b \) let \( \sigma_j \) be a train track obtained by applying Lemma \( \text{6.3} \) to \( B_{r_j}(\tilde{q}_j) \). Assume \( \epsilon \) is smaller than the radius of \( B_{r_j}(\tilde{q}_j) \) for all \( j \). Write \( U = \cup U_i \) where
\[
U_i = U \cap U(\sigma_j).
\]
By Lemma 5.1 each \( \hat{U}_t \) is a piecewise linear subset of \( U_t \). The claim now follows from Lemma 6.3(4) if we ignore those \( \hat{U}_t \)'s which have size less than \( e^N \) for some \( N > 1 \) depending only on the dimension. \( \square \)

**Proof of Lemma 7.6.** Recall that \( \lambda \) is the center of \( U \); put \( \hat{p} = \hat{P}_1^{-1}(\eta, \lambda) \) and \( p = \pi(\hat{p}) \). Let \( \tilde{\phi}^c(u) \) be the characteristic function of \( \bar{W}_u^c \subset W^c(p) \). Define

\[
\phi^c := \tilde{\phi}^c \circ (\pi^{-1}|_{\pi(\text{supp}(\tilde{\phi}^c))})
\]

— the push-forward of \( \tilde{\phi}^c \) to \( W^c(p) \). Recall from Lemma 7.7 that \( \phi^c \in S_{W^c(p)}(\mu, \epsilon) \).

Recall from §2 that \( \mu \) denotes the SL(2, \( \mathbb{R} \))-invariant probability measure on \( Q_1(1, \ldots, 1) \) which is in the Lebesgue measure class. The measures \( \mu_x^c \) and \( \mu_x^a \) are the conditional measures of \( \mu \) along \( \bar{W}_u(x) \) and \( W^c(x) \); \( \mu_x^c \) and \( \mu_x^a \) are defined accordingly.

Recall that \( \mu_{\text{Th}}(\{e^U : -\epsilon < s \leq 0\}) = \frac{1-e^{-ht}}{h}\mu_{\text{Th}}(U) \). Therefore, we have

\[
(65) \quad \mu_p^c(\phi^c) = \frac{1-e^{-ht}}{h}\mu_{\text{Th}}(U).
\]

For simplicity in notation, let us write \( \bar{W}^c = \bar{W}_{U, \epsilon}^c \) and put

\[
\mathcal{N} = \mathcal{N}(\tilde{q}_0, t, U, \epsilon).
\]

Let \( g \in \text{Mod}_g \) be so that \( \bar{W}^c \cap g \cdot \bar{W}_u^c(\tilde{q}_0) \neq \emptyset \). Recall that \( \{B_u^n(\tilde{q}_i) : 0 \leq i \leq N(b)\} \) cover \( \hat{D} \subset \bar{W}_u^c(\tilde{q}_0) \), see Lemma 7.5 and the paragraph following that lemma; there exists some \( g' \in \text{Mod}_g \) so that \( g' \cdot \bar{W}_u^c(\tilde{q}_0) = \bar{W}_u^c(\tilde{q}_0) \) and some \( 0 \leq i \leq N(b) \) so that

\[
(66) \quad \bar{W}^c \cap gg' \cdot a_iB_0^u(\tilde{q}_i) \neq \emptyset.
\]

Let \( N(b)' \) be defined in (64). We claim that the following holds:

\[
(67) \quad \#(g \cdot \bar{W}_u^c(\tilde{q}_0) : (66) \text{ holds for some } 0 \leq i \leq N(b)' \} \ll e^{-\epsilon b^*v(\gamma_0)}e^{(h-\epsilon)\epsilon t} + b^*v(\gamma_0)e^{ht}.
\]

Let us assume (67) and finish the proof. Let

\[
\mathcal{N}' := \{g \cdot \bar{W}_u^c(\tilde{q}_0) \in \mathcal{N} : (66) \text{ does not hold for any } 0 \leq i \leq N(b)' \}
\]

i.e., the contribution to \( \mathcal{N} \) coming from \( N(b)' < i \leq N(b) \). We claim that

\[
(68) \quad |\#\mathcal{N}' - \sum_i \sum_y \psi_i^u(y)| \ll e^{-\epsilon b^*v(\gamma_0)}e^{(h-\epsilon)\epsilon t} + b^*v(\gamma_0)e^{ht}
\]

where the outer summation is over all \( N(b)' < i \leq N(b) \) and the inner summation is over all \( y \in B_u^n(\gamma_i) \) so that \( a_iy \in \pi(\bar{W}^c) \).

To see the claim, first note that by the definition of \( \mathcal{N}' \), if \( g \cdot \bar{W}_u^c(\tilde{q}_0) \in \mathcal{N}' \), then (66) holds with some \( N(b)' < i \leq N(b) \). Let now \( g_1, g_2 \in \text{Mod}_g \) and \( N(b)' \leq i_1, i_2 \leq N(b) \) be so that

\[
\bar{W}^c \cap gg_j \cdot a_iB_0^u(\tilde{y}_{i_j}) \neq \emptyset.
\]
Then $\mathbf{g}_i \hat{W}^u(\eta_0) = \hat{W}^u(\eta_0)$ for $j = 1, 2$, see the discussion preceding (66); hence by Corollary 4.3 we have

$$\hat{W}^u \cap \mathbf{g}_1 \cdot \mathbb{A}_t \mathbb{B}^u_\mathbb{F}(\eta_1 \cap \eta_2) = \hat{W}^u \cap \mathbf{g}_2 \cdot \mathbb{A}_t \mathbb{B}^u_\mathbb{F}(\eta_1 \cap \eta_2).$$

In particular, $\mathbf{g}_1 \mathbb{B}^u_\mathbb{F}(\eta_1 \cap \eta_2) \neq \emptyset$. Since $\mathbb{B}^u_\mathbb{F}(\eta_j) \subseteq \mathbb{D}$ for $j = 1, 2$ — recall that $N(b)'/i \leq i \leq N(b)$ — we get that $\mathbf{g}_1 = \mathbf{g}_2$. Therefore,

$$\hat{W}^u \cap \mathbf{g}_1 \cdot \mathbb{A}_t \mathbb{B}^u_\mathbb{F}(\eta_1 \cap \eta_2)$$

corresponds to points lying in the intersection $\mathbb{B}^u_\mathbb{F}(\eta_1 \cap \eta_2)$ but not in $\bigcup_{i=1}^{N(b)'} \mathbb{B}^u_\mathbb{F}(\eta_i)$. Recall from Lemma 7.5 that $\sum_i \psi^u = 1$ on $\bigcup_{i=1}^{N(b)} \mathbb{B}_\mathbb{F}(\eta_i)$, hence $\sum_{N(b)<i\leq N(b)} \psi^u = 1$ on $\mathbb{D}_b \setminus \bigcup_{i=1}^{N(b)} \mathbb{B}^u_\mathbb{F}(\eta_i)$. In particular, since $\psi^u > 0$, we get that

$$\# \mathcal{N}' \leq \sum_i \sum_y \psi^u(y)$$

where the outer summation is over all $N(b)'/i \leq i \leq N(b)$ and the inner summation is over all $y \in \mathbb{B}^u_\mathbb{F}(\eta_i)$ so that $a_i y \in \pi(\hat{W}^u)$. Moreover, in view of Lemma 7.5 (2) and (3) we have

$$\sum_i \sum_y \psi^u(y) - \# \mathcal{N}' \leq N_2 \cdot \# \{ \mathbf{g} \cdot \hat{W}^u(\eta_0) : (66) \text{ holds for some } 0 \leq i \leq N(b) \}.$$

The claim in (68) thus follows in view of the estimate in (67).

Using the definition of $\mathcal{N}_{nc}$ in (28), we have

$$\mathcal{N}_{nc}(t, \psi^u, \phi^{cs}) = \sum_i \psi^u(y) \phi^{cs} \mathbb{A}_t y$$

where the sum is over all $y \in \mathbb{B}^u_\mathbb{F}(\eta_i)$ so that $a_i y \in \pi(\hat{W}^u) = \text{supp}(\phi^{cs})$. Now apply Corollary 4.7, see in particular (47), with $\psi^u$ and $\phi^{cs}$; we thus get that

$$\sum_i \psi^u(y) \phi^{cs} \mathbb{A}_t y = 0, 1$$

for all $y \in \mathbb{B}^u_\mathbb{F}(\eta_i)$.

In view of (65) and the estimate $C^4(\psi^u) \leq N_{15} N_{31}$, see (63), we get the following from (69).

$$\sum_i \psi^u(y) - \phi^{cs} \mathbb{A}_t y < N_{15} e^{-b \nu} e^{(h \log 2)^t}.$$
Moreover, \( \phi \)

Putting this estimate and (71) together we get that

We now use these estimates to get an estimate for \( \#\mathcal{N} \). First note that

This estimate and (73) imply that

Let \( \kappa > 0 \) be small enough so that \( 10\rho \)-neighborhood of \( \text{supp}(\phi^{cs}) \) embeds in \( Q(1, \ldots, 1) \). Let \( \kappa > 0 \) be a constant which will be chosen later. In view of Lemma 2.5, we have

Therefore, properties (S-1), (S-2), and (S-3) hold with \( \epsilon = 0.1\rho e^{-\kappa t} \) and \( f = 1_{B_0(p)} \). Let \( \phi_1^{cs} = \varphi_{+0.1\rho e^{-\kappa t}} \) for these choices.

Similarly, using Lemma 2.5, which is applicable to the function \( \phi^{cs} \) with \( \epsilon = 0.1\rho e^{-\kappa t} \) this time, we let \( \phi_1^{cs} = \varphi_{+0.1\rho e^{-\kappa t}} \).

Put \( \phi := \phi_1^{cs} \). Note that \( 1_{B_0(p)} \phi^{cs} \leq \phi_1 \leq 1_{B_0(p)} \phi^{cs} \). Therefore,

Moreover, \( \mu_p(\phi_1) \geq \mu_p^{cs}(\phi^{cs}) \).
Therefore, it suffices to show that 
\[ \#\{ g \cdot W^u(q_0) : (66) \text{ holds with } 0 \leq i \leq N(b) \} \ll \frac{e^{ht}}{\mu^u(B_{v}(p))} \sum_1^i \int_{W^u(q_0)} \phi_1(a_i y) \psi_i(y) d\mu_{q_0}(y). \]

Moreover, by Proposition 3.2 we have
\[ \int_{W^u(q_0)} \phi_1(a_i y) \psi_i(y) d\mu_{q_0}(y) = \mu(\phi_1) \mu^u(q_0) + O(C^1(\psi_i)C^1(\phi_1)e^{(h-\kappa_5) t}) \]
for all \( 0 \leq i \leq N(b)' \).

Combining these two estimates and using the fact that in view of the estimates in (75) we have \( \mu(\phi_1)/\mu^u(B_{v}(p)) \ll 1 \) we conclude that
\[ \#\{ g \cdot W^u(q_0) : (66) \text{ holds with } 0 \leq i \leq N(b) \} \ll e^{ht} \sum \mu^u(q_0) + O(C^1(\psi_i)C^1(\phi_1)e^{(h-\kappa_5) t}) N(b). \]

In view of (63) we have \( C^1(\psi_i) \ll b^{-*}v(\gamma_0) \); moreover, \( C^1(\phi) \ll \epsilon^{-*} \) and \( N(b)' \ll N(b) \ll b^{-*} \). Recall also that \( \sum_{i=0}^{N(b)'} \mu^u(q_0) \ll e^{29}v(\gamma_0) \) — this estimate was also used in (72).

If we now choose \( \kappa \) small enough, (76) follows from (76). This finishes the proof of the lemma. \( \square \)

**Corollary 7.8.** There exist some \( \kappa_{28} \) and \( \kappa_{29} \) so that the following holds. Let \( t \geq 0 \) and let \( \epsilon \geq e^{-29/2} \). Then
\[ \#O_{\tau}(\gamma_0, t, U, \epsilon) = v(\gamma_0) \mu_{\text{Th}}(U) \left( \frac{1-e^{-h\epsilon}}{h} \right) e^{ht} + O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{29}) t}) \]
where as in (61) we have
\[ O_{\tau}(\gamma_0, t, U, \epsilon) = \{ \gamma \in \text{Mod}_g \cdot \gamma_0 \cap ([0, \epsilon]U - [0, \epsilon^{-*}][U]) \}. \]

**Proof.** We will show this holds with \( \kappa_{28} = \kappa_{26}/2 \). By Lemma 7.2 we have \( \gamma \in O_{\tau}(\gamma_0, \epsilon, U, \epsilon) \) if and only if
\[ g \cdot a_t W^u(q_0) \cap \tilde{W}_{U, \epsilon}^{\kappa_{28}} \neq \emptyset. \]

Therefore, it suffices to show that
\[ \#N(\tilde{q}_0, t, U, \epsilon) = v(\gamma_0) \mu_{\text{Th}}(U) \left( \frac{1-e^{-h\epsilon}}{h} \right) e^{ht} + O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{29}) t}). \]

This last statement is proved in Lemma 7.6. \( \square \)

**Proof of Theorem 7.7.** Let \( \epsilon \geq e^{-29/2} \), and for every \( n \geq 0 \) define \( t_n := t - n\epsilon \). Then (77) applied with \( t = t_n \) implies that
\[ \#O_{\tau}(\gamma_0, t_n, U, \epsilon) = v(\gamma_0) \mu_{\text{Th}}(U) \left( \frac{1-e^{-h\epsilon}}{h} \right) e^{ht_n} + O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{29}) t_n}) \]
\[ = v(\gamma_0) \mu_{\text{Th}}(U) \left( \frac{e^{-nh\epsilon} - e^{(n-1)h\epsilon}}{h} \right) e^{ht} + O_{\gamma_0}((1-e^{-h\epsilon})e^{(h-\kappa_{29}) t_n}). \]
Summing these up over all $n \geq 0$ so that $t_n \geq \frac{h-1}{n} t$ we get that

$$\#\{\gamma \in \text{Mod}_g \gamma_0 \cap ([0, e^{t} U] - [0, e^{\frac{h-1}{n} t} U])\} = v(\gamma_0) \mu_{\text{Th}}(U)(1 - e^{\frac{h-1}{n} t}) e^{ht} + O_{\gamma_0}(e^{(h-1)t}).$$

This implies the proposition — note that by basic lattice point count in Euclidean spaces we have the number of integral points $\gamma \in U(\tau)$ so that $\|\gamma\| \leq e^{\frac{h-1}{n} t}$ is $\ll e^{(h-1)t}$.

\[\square\]

8. Proof of Theorem 1.1

We are now in the position to prove Theorem 1.1. The proof relies on Theorem 7.1. We cover $\mathcal{M}L(S)$ with finitely many train track charts $U(\tau_1), \ldots, U(\tau_c)$. Using the convexity of the hyperbolic length function, we can reduce the counting problem in Theorem 1.1 to an orbital counting in sectors on $U(\tau_i)$, with respect to linear structure, where the hyperbolic length function is well approximated by the $\|\|_{\tau_i}$. Theorem 7.1 is then brought to bear in the study of the latter counting problem.

Let $M \in \mathcal{M}(S)$ and recall that $\ell_M : \mathcal{M}L(S) \to \mathcal{M}L(S)$ denotes the hyperbolic length function. It satisfies $\ell_M(t\lambda) = t\ell_M(\lambda)$ for any $t > 0$.

Let $\tau$ be a maximal train track. By Corollary 5.2, $\ell_M$ is Lipschitz in $U(\tau)$. Let $\ell_{\tau}$ be the Lipschitz constant, hence

$$|\ell_M(\lambda) - \ell_M(\lambda')| \leq \ell_{\tau}\|\lambda - \lambda'\|_{\tau}. \tag{78}$$

Recall that $U(\tau)$ is a cone on the polyhedron $P(\tau)$.

Lemma 8.1. There exists a constant $\hat{L}_{\tau}$, depending on $L_{\tau}$, with the following property. For every $\lambda, \lambda' \in P(\tau)$ we have $\left|\frac{1}{\ell_M(\lambda)} - \frac{1}{\ell_M(\lambda')}\right| \leq \hat{L}_{\tau}$. \[\square\]

Proof. First note that there exists some $\ell_{M,\tau} > 1$ so that $1/\ell_{M,\tau} \leq \ell_M(\lambda) \leq \ell_{M,\tau}$ for all $\lambda \in P(\tau)$. The claim thus follows from (78).

For any $T > 0$, let $C_M(\tau, T) = \{\lambda \in U(\tau) : \ell_M(\lambda) \leq T\}$. To simplify the notation we will write $C_M(\tau)$ for $C_M(\tau, 1)$. Let $S_M(\tau) = \{\lambda \in U(\tau) : \ell_M(\lambda) = 1\}$. Then

$$C_M(\tau, T) = TC_M(\tau) = [0, T]S_M(\tau).$$

Proof of Theorem 1.1 Let $M \in \mathcal{M}(S)$. Let $\tau_1, \ldots, \tau_c$ be finitely many maximal train tracks with the following properties.

- $\mathcal{M}L(S) = \bigcup_{i=1}^c U(\tau_i)$, and
- $\ell_M : U(\tau_i) \to \mathbb{R}$ is $L_i$-Lipschitz for all $1 \leq i \leq c$.

Let $L = \max L_i$; increasing $L$ if necessary we will also assume that the conclusion of Lemma 8.1 holds with $L$.

\[\square\]

\[\text{As we remarked in the introduction, the point here is that we are counting the number of point in one Mod}_g\text{-orbit.}\]
Let us fix some $1 \leq i \leq c$ and write $\tau = \tau_i$; when there is no confusion we drop $\tau$ from the notation for the norm and normalization in $U(\tau)$. We will first consider the contribution coming from $U(\tau)$ and then will combine contributions of different $\tau_i$ for $1 \leq i \leq c$.

In the following we will use the following upper bound estimate for the number of integral point in a Euclidean region: the number of lattice points in a Euclidean region is $\ll$ the volume of the 1-neighborhood of the region.

Let $\gamma_0$ be a rational (multi) geodesic. For every $T > 0$ define
\begin{equation}
N_\tau(\gamma_0, T) = \# \{ g \gamma_0 \in U(\tau) : \ell_M(g \gamma_0) \leq T \}.
\end{equation}

Fix some $\delta > 0$; this will be optimized later and will be chosen to be of size $T^{-\kappa}$. Define
\begin{equation}
P_{\geq \delta}(\tau) := \{ (b_i) \in P(\tau) : b_i \geq 2 \delta \text{ for all } i \}.
\end{equation}

Cover $P(\tau)$ with cubes of size $\delta$ with disjoint interior. Let $\{ U_j : j \in J_{\delta} \}$ be the subcollection of these cubes so that $U_j \cap P_{\geq \delta}(\tau) \neq \emptyset$.

For every $j$, let $\lambda_j \in U_j$ be the center of $U_j$. The number of $U_j$’s required to cover $P(\tau)$ is $\ll \delta^{-N_{16}}$ for some $N_{16}$ depending on $\tau$.

There is some $\kappa_{30}$ depending only on the dimension with the following property. If $\delta \geq T^{-\kappa_{30}}$ then the number of integral points $\gamma \in U(\tau)$ with $\| \gamma \| \leq \ell_M, \tau T$ and
\begin{equation}
\bar{\gamma} = \gamma / \| \gamma \| \in P(\tau) - P_{\geq \delta}(\tau)
\end{equation}
is $\ll \delta T^h$.

For each $j$, let $U_{j,-}$ denote the cube which has the same center $\lambda_j$ as $U_j$, but has size $\delta - \delta^{N_{17}}$ where $N_{17} = N_{16} + 1$.

Then, if $\delta^{N_{17}} \geq T^{-\kappa_{30}}$, the number of integral points $\gamma \in U(\tau)$ with $\| \gamma \| \leq \ell_M, \tau T$ and
\begin{equation}
\bar{\gamma} \in \bigcup_j U_j - U_{j,-}
\end{equation}
is $\ll \delta^{-N_{16}} T^h \ll \delta T^h$.

Altogether, we have: if $\delta^{N_{17}} \geq T^{-\kappa_{30}}$, then
\begin{equation}
\# \{ \gamma \in \text{Mod}_g, \gamma_0 \cap U(\tau) : \ell_M(\gamma) \leq T, \bar{\gamma} \text{ satisfies } (81) \text{ or } (82) \} \ll \delta T^h.
\end{equation}

We now find an estimate for
\begin{equation}
\# \{ \gamma \in \text{Mod}_g, \gamma_0 \cap C_M(\tau, T) : \bar{\gamma} \in \cup U_{j,-} \}.
\end{equation}

Put $U_{j,-,+} = \{ \frac{\lambda}{\ell_M(\lambda_j) - L_3} : \lambda \in U_{j,-} \}$ and $U_{j,-,-} = \{ \frac{\lambda}{\ell_M(\lambda_j) + L_3} : \lambda \in U_{j,-} \}$. Then it follows from (78) that
\begin{equation}
[0, 1] U_{j,-,-} \subset \{ \lambda \in C_M(1, \tau) : \bar{\lambda} \in U_{j,-} \} \subset [0, 1] U_{j,-,+}
\end{equation}
Therefore, applying Theorem 7.1 with $U = U_{j,-,\pm}$, we get that

$$\frac{v(\gamma_0)\mu_{Th}(U_{j,-,\pm})}{h(\ell_M(\lambda_j)+18\delta)^h} T^h + O_{r,\gamma_0}(T^{h-\kappa_24}) \leq \#\{\gamma \in \text{Mod}_g : \gamma \in C_M(\tau, T), \gamma \in U_{j,-,\pm} \} \leq \frac{v(\gamma_0)\mu_{Th}(U_{j,-,\pm})}{h(\ell_M(\lambda_j)-18\delta)^h} T^h + O_{r,\gamma_0}(T^{h-\kappa_24});$$

this estimate implies that

$$(84) \quad \#\{\gamma \in \text{Mod}_g : \gamma \in C_M(\tau, T), \gamma \in U_{j,-,\pm} \} = \frac{v(\gamma_0)\mu_{Th}(U_{j,-,\pm})}{h(\ell_M(\lambda_j))^h} T^h + O_{r,\gamma_0}(\delta \mu_{Th}(U_{j,-,\pm})T^h + T^{h-\kappa_24}).$$

Let us put $S_M(\tau,j) = \{\lambda \in S_M(\tau) : \lambda \in U_{j,-,\pm}\}$. Then by Lemma 8.1 we have

$$\mu_{Th}([0,1]S_M(\tau,j)) = \int_{U_{j,-,\pm}} \frac{1}{h(\ell_M(\lambda_j))^k} d\mu_{Th} = \frac{\mu_{Th}(U_{j,-,\pm})}{h(\ell_M(\lambda_j))^h} + O(\delta) \mu_{Th}(U_{j,-,\pm}).$$

This observation together with (84) gives that

$$(85) \quad \#\{\gamma \in \text{Mod}_g : \gamma \in C_M(\tau, T), \gamma \in U_{j,-,\pm} \} = v(\gamma_0)\mu_{Th}([0,1]S_M(\tau,j))T^h + O_{r,\gamma_0}(\delta \mu_{Th}(U_{j,-,\pm})T^h + T^{h-\kappa_24}).$$

Recall also that $\ell_M^{\pm 1}$ is bounded on $P(\tau)$; we have $\sum \mu_{Th}([0,1]S_M(\tau,j)) = \mu_{Th}([0,1]S_M(\tau)) + O(\delta^*).$ Hence, summing (85) over all $j$’s we get

$$(86) \quad \#\{\gamma \in \text{Mod}_g : \gamma \in C_M(\tau, T), \gamma \in \bigcup_{j \in \mathbb{Z}} U_{j,-,\pm} \} = v(\gamma_0)\mu_{Th}([0,1]S_M(\tau))T^h + O_{r,\gamma_0}(\delta^* T^h + \delta^{-}\text{N_{16}T^{h-\kappa_24}}).$$

Now choose $\delta = T^*$ so that $\delta^* T^h + \delta^{-}\text{N_{16}T^{h-\kappa_24}} = T^{h-\kappa_31}$. Then we get from (86) and (83) that

$$(87) \quad \#\{\gamma \in \text{Mod}_g : \gamma \in C_M(\tau, T) \} = v(\gamma_0)\mu_{Th}([0,1]S_M(\tau))T^h + O(T^{h-\kappa_31}).$$

This conclude the contribution arising from a single train track chart $U(\tau)$.

Recall now that the regions in $U(\tau_1)$ which are carried by other $U(\tau'_{i'})$ are finite sided polyhedra, see Lemma 5.1. We may thus find disjoint finite sided polyhedra $\mathcal{U}_i \subset P(\tau_1)$ to the $\cup \mathbb{R}^+, \mathcal{U}_i = \mathcal{ML}(S)$. Repeating the above argument for each $\mathcal{U}_i$, the theorem follows from the estimate in (87).

We conclude with the following which are of independent interest. Let $\Gamma \subset \text{Mod}_g$ be a finite index subgroup and let $\tau$ be a maximal train track. Define

$$\mathcal{N}_{\Gamma,\tau}(\gamma_0, T) := \{\gamma \in \Gamma : \gamma \cap U(\tau) : \|\gamma\|_{\tau} \leq T\}.$$

**Theorem 8.2.** There exists some $\kappa_{32} = \kappa_{32}^{\Gamma}$ so that the following holds. For every rational multi curve $\gamma_0 \in U(\tau)$, there exists some constant $c_{\Gamma,\tau}(\gamma_0)$ so that

$$\#\mathcal{N}_{\Gamma,\tau}(\gamma_0, T) = c_{\Gamma,\tau}(\gamma_0) T^{69-6} + O_{\gamma_0,\tau,\Gamma}(T^{69-6-\kappa_{32}})$$
Proof. The argument is similar to our argument in the proof of Theorem \[1.2\]. Recall that we normalized the Masur-Veech measure to be a probability measure on \( Q_1(1, \ldots, 1) \). Let \( \mu_\Gamma \) denote the lift of the Masur-Veech measure to \( Q_1^1 \mathcal{T}(1, \ldots, 1)/\Gamma \), then \( \mu_\Gamma(Q_1^1 \mathcal{T}(1, \ldots, 1)/\Gamma) = [\text{Mod}_g : \Gamma] \).

Similar to (62), define \( v_\Gamma(\gamma_0) \) to be the measure of the lift of \( W^u(q_0) \) to \( Q_1^1 \mathcal{T}(1, \ldots, 1) \) where \( \mathcal{I}(q_0) = \gamma_0 \).

Now, by virtue of Theorem \[7.1\], we have
\[
\# \{ \gamma \in \Gamma. \gamma_0 \cap U(\tau) : \| \gamma \|_\tau \leq T \} = v_\Gamma(\gamma_0)/[\text{Mod}_g : \Gamma] \text{ and } v_\Gamma(\gamma_0) \text{ is as above.}
\]

The exponent \( \kappa_{32} \) depends on the exponential mixing rate for the Teichmüller geodesic flow on \( (Q_1^1 \mathcal{T}(1, \ldots, 1)/\Gamma, \mu_\Gamma) \). \( \square \)

Let \( \Gamma \subset \text{Mod}_g \) be a finite index subgroup. Given a rational multi-geodesics \( \gamma_0 \) on \( M \) define
\[
s_{M, \Gamma}(\gamma_0, T) := \# \{ \gamma \in \Gamma. \gamma_0 : \ell_M(\gamma) \leq T \}
\]
We also have the following generalization of Theorem \[1.1\].

**Theorem 8.3.** There exists some \( \kappa_{33} = \kappa_{33}(\Gamma) > 0 \), dependence on \( \Gamma \) is related to the exponential mixing rate for the Teichmüller geodesic flow on \( Q^1_1 \mathcal{T}(1, \ldots, 1)/\Gamma \), and some \( c = c(\gamma_0, M, \Gamma) \) so that the following holds.
\[
s_{M, \Gamma}(\gamma_0, T) = cT^{6g-6} + O_{\gamma_0, M, \Gamma}(T^{6g-6-\kappa_{33}})
\]

**Proof.** Similar to the discussion in the proof of Theorem \[8.2\], the proof of Theorem \[1.1\] applies mutatis mutandis to \( s_{M, \Gamma}(\gamma_0, T) \). \( \square \)

## Index

- \( A(g, x) \) the Kontsevich-Zorich cocycle, 5
- \( \alpha \) multiplicities of zeros, 3
- \( B_r(q) \) and \( B^*_r(q) \), 27
- \( B_r(x) \) ball of radius \( r \) around \( x \), 9
- \( B_r(\bar{x}) \), 10
- \( B^u_r(x), B^s_r(x), B^c_r(x) \), 9
- \( C_M(\tau) \) the set of \( \lambda \in U(\tau) \) with \( \ell_M(\lambda) \leq 1 \), 39
- \( C_M(\tau, T) \) the set of \( \lambda \in U(\tau) \) with \( \ell_M(\lambda) \leq T \), 39
- \( C^1(\varphi) \) the \( C^1 \)-norm of \( \varphi \), 10
- \( D_{\text{cusp}}(\epsilon) \) the \( \epsilon \)-thin part of \( W^u(q) \), 32
- \( D_{\bar{x}} \), 34
- \( D_0 \), 33
- \( E^u(x) \), 9
- Ext the extremal length, 7

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E_t(x, K), 18

Φ the period map, 4
φ^u, φ^s, φ^c, φ^cs, φ^cu, φ^s, φ^cs, 9

γ a rational multicurve, 2
g a mapping class, 2
g the genus of S, 1

Ω(S) the moduli space of Abelian differentials, 4
Ω_1(S) the moduli space of area one Abelian differentials, 4
H(α) a stratum of Abelian differentials, 4
H(α) a stratum of area one Abelian differentials, 4
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h the topological entropy of Teichmüller geodesic flow, 4
H^u_t, int, 19
H^u_t(x, θ), 12
H^u_t(x), 19
H^u_t(x, K), 17

J(q^{1/2}) imaginary foliation of q, 27
i(, ) algebraic intersection pairing, 6

K(ε) the ε-thick part, 32

ℓ_M(γ) the hyperbolic length of γ, 2
λ a lamination, 23
ℓ_M, τ, 39
L_ε the Lipschitz constant of ℓ_M in U(τ), 39
λ^* and λ, 27

Mod_g the mapping class group of S, 1
M affine invariant manifold, 5
M(S) the moduli space of S, 1
ML(S) the space of measured laminations on S, 2
M orienting double cover of M, 4
μ affine SL(2, R)-invariant measure, 5
μ^u, μ^s conditional measures of μ along W^u(x), W^s(x), 9
μ^{Th} the Thurston measure, 2

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N(q_0, t, U, ε), 34
∥∥_x the modified Hodge norm at x, 8
∥∥_{AGY, x} the AGY norm at x, 8
∥∥_{H, x} the Hodge norm at x, 5
∥∥_x and ∥∥ sum of the weights of λ ∈ U(τ), 27
n_{q_0} the Mirzakhani function, 2
\( N\)\(_{nc}(t, \psi, \phi) \), 18
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\( O_{r}(\gamma_0, T, U, e) \), 31
\( O_{r}(\gamma_0, e, U) \), 30
\( P(\tau) \) polyhedron of laminations whose weights add up to one, 27
\( \bar{P}_1 \) the normalized Hubbard-Masur map, 25
\( \pi \) the covering map, 4
\( P \) the Hubbard-Masur map, 25
\( P_{\geq \delta}(\tau) \) points in \( P(\tau) \) where each coordinate is at least 2\( \delta \), 40
\( p_x(e) \), 8
\( p \) the natural map from \( H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R}) \), 8
\( Q_1(S) \) the moduli space of area one quadratic differentials, 3
\( Q(S) \) the moduli space of quadratic differentials, 3
\( Q(\alpha) \) a stratum of quadratic differentials, 4
\( Q_1(\alpha) \) a stratum of area one quadratic differentials, 3
\( Q^1 T(\alpha) \) the universal cover of \( Q_1(\alpha) \), 4
\( \mathfrak{R}(q^{1/2}) \) real foliation of \( q \), 27
\( S \) a compact surface of genus \( g \), 1
\( S(E, r, L) \) the set of Borel functions supported in \( E \) which may be approximated by smooth functions, 11
\( S_W(E, r, L) \), 11
\( S_M(\tau) \) the set of \( \lambda \in U(\tau) \) with \( \ell_M(\lambda) = 1 \), 39
\(*\) the Hodge star operator, 6
\( S(E, r) \), 12
\( S(x, r) \), 12
\( s_M(\gamma_0, L) \) number of simple closed curves, 2
\( s_{M,1}(\gamma_0, L) \), 12
\( T(S) \) the Teichmüller space of \( S \), 1
\( \tau \) a train track, 23
\( U \) a polyhedron in \( U(\tau) \) of dimension \( \dim U(\tau) - 1 \), 27
\( U(\tau) \) a train track chart in \( ML(S) \), 24
\( u(x) \) the Margulis function for the cusp, 12
\( \mathbf{v}(x) \) the direction of the geodesic flow at \( x \), 9
\( \mathbf{v}(\gamma_0) \) volume of \( W^u(\pi \circ \bar{P}_1^{-1}([\eta], \gamma_0)) \) for a rational multigeodesic \( \gamma_0 \), 33
\( \mathcal{V}(\tau) \), 24
\( W^{cs}(x) \) center-stable foliation in \( Q_1(\alpha) \), 9
\( W^{cu}(x) \) center-unstable foliation in \( Q_1(\alpha) \), 9
$W^s(x)$ stable foliation in $Q_1(\alpha)$, 9
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