Show all your computations and justify/explain your answers. Electronic devices are NOT allowed.

In the following $\mathbb{Z}$ and $\mathbb{Z}_n$ are considered with their usual addition.

(1) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 6 & 1 & 2 & 7 & 5 & 8 \end{pmatrix}$.
   
   (a) (3 pts) Write $\sigma$ as a product of disjoint cycles.
   
   (b) (3 pts) Determine whether $\sigma$ is even or odd.

(2) (4 pts) Let $\sigma = (2,6,1)(4,3,5)$ and $\tau = (2,3,4)(6,1,5)$ be two elements in $S_6$. Find a permutation $\mu \in S_6$ so that $\sigma = \mu \tau \mu^{-1}$.

(3) (8 pts) List all the abelian groups of order 540 up to isomorphism.

(4) (a) (3 pts) Find the order of $(1,3)$ in $\mathbb{Z}_6 \oplus \mathbb{Z}_{10}$.
   
   (b) (6 pts) Find a homomorphism from $\mathbb{Z}_6 \oplus \mathbb{Z}_{10}$ onto $\mathbb{Z}_{30}$.

(5) Let $H = \langle (1,3) \rangle$ in $\mathbb{Z}_9 \oplus \mathbb{Z}_6$.
   
   (a) (6 pts) Find all the left cosets of $H$ in $\mathbb{Z}_9 \oplus \mathbb{Z}_6$.
   
   (b) (3 pts) Find the order of $(0,2) + H$ in $\mathbb{Z}_9 \oplus \mathbb{Z}_6/H$.

(6) (6 pts) Let $G$ be an abelian group of order 309. Show that $G$ is isomorphic to $\mathbb{Z}_{309}$.

(7) Let $H = \langle (2,3) \rangle$ in $\mathbb{Z} \oplus \mathbb{Z}$.
   
   (a) (5 pts) Find all the left cosets of $H$ in $\mathbb{Z} \oplus \mathbb{Z}$.
   
   (b) (5 pts) Show that $\mathbb{Z} \oplus \mathbb{Z}/H$ is a cyclic group.

(8) (5 pts) Let $G$ be a finite abelian group. Assume that $G$ has at least three elements of order 3. Show that $|G|$ is divisible by 9.

(9) (5 pts) Let $G$ be a group and let $H$ be a normal subgroup of $G$. Assume that $(G : H) = n$. Show that $g^n \in H$ for all $g \in G$. 