Homework due Thursday, April 9, 9:00 pm, on Gradescope.

(1) Let \( A \) be a ring and \( x \) and indeterminate. Prove that
(a) \( \text{Nil}(A[x]) = \text{Nil}(A)[x] \).
(b) \( A[x]^\times = \{ \sum a_i x^i \in A[x] : a_0 \in A^\times, a_i \in \text{Nil}(A), i > 0 \} \)
(c) \( J(A[x]) = \text{Nil}(A[x]) = \text{Nil}(A)[x] \).
(This is parts of Exercise 2 of chapter 1 in the book.)
Hint: part (1): one inclusion is clear, for the other inclusion use the fact that \( p[x] \) is a prime ideal in \( A[x] \) for any prime ideal \( p \triangleleft A \).
Part (2): For one inclusion show and use the fact that if \( B \) is a ring, \( u \in B^\times \) and \( n \in \text{Nil}(B) \), then \( n + u \in B^\times \). For the other inclusion prove and use the fact that if \( B \) is an integral domain \( B[x]^\times = B^\times \) —apply this to \( A/p \) for prime ideals \( p \).
Part (3): One inclusion is clear, for the other use part (2).

(2) Complete the proof of McCoy’s Theorem: Let \( a, b_1, \ldots, b_n \triangleleft A \). Suppose
\[
\mathfrak{a} \subset \bigcup_{i=1}^{n} \mathfrak{b}_i \quad \text{and} \quad \mathfrak{a} \not\subset \bigcup_{i \neq j} \mathfrak{b}_i \quad \text{for every} \quad j.
\]
Then there exists some \( k \in \mathbb{N} \) so that \( \mathfrak{a}^k \subset \bigcap_i \mathfrak{b}_i \).

(3) Prove the following statements:
(a) \( (\mathfrak{a} : \mathfrak{b} : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{bc}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b}) \).
(b) \( \sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt[\mathfrak{a} + \sqrt{\mathfrak{b}}] \).
(c) If \( \mathfrak{p} \) is a prime ideal, \( \sqrt[\mathfrak{p}] = \mathfrak{p} \).
(d) \( \sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{a} \).

(4) Prove the following statements:
(a) Any non-empty closed subset of \( \text{Spec}(A) \) intersects \( \text{Max}(A) \) non-trivially.
(b) \( \{ \mathfrak{p} \in \text{Spec}(A) : \{ \mathfrak{p} \} \text{ is closed} \} = \text{Max}(A) \).
(c) If \( A \) is an integral domain, \( \{0\} \) is dense in \( \text{Spec}(A) \).

(5) Let \( \mathfrak{a} \triangleleft A \) and let \( \pi : A \to A/\mathfrak{a} \) be the natural map. Then \( \pi^* \) induces a bijection from \( \text{Spec}(A/\mathfrak{a}) \) to \( V(\mathfrak{a}) \).

(6) Let \( A \) be a local ring, \( M \) and \( N \) finitely generated \( A \)-modules. Prove that if \( M \otimes N = 0 \), then either \( M = 0 \) or \( N = 0 \). (Exercise 3, chapter 2 in the book.)