Homework due Friday, January 18, 3:00 pm.

(1) Let $X$ be a non-empty set. Let $S = \{f : X \to X | f \text{ is a bijection}\}$. Show that $S$ together with composition of functions forms a group.

(2) Let $G = \mathcal{P}(\{1, 2\})$, the power set of $\{1, 2\}$. For any two sets $A, B \in G$ define $A \ast B = A \Delta B$. It was discussed in class that $(G, \ast)$ is a group. List all the subgroups of $G$.

(3) Let $(G, \ast)$ be as in problem 2 (above). Show that $(G, \ast)$ is not isomorphic to $(\mathbb{Z}_4, +_4)$. (Hint: suppose there exists an isomorphic from $G$ onto $\mathbb{Z}_4$, can the element $1 \in \mathbb{Z}_4$ be in the image?)

(4) Let $(G, \ast)$ be a group and let $H$ be a subgroup of $G$. Define the relation $g_1 \sim g_2 \iff g_1^{-1} \ast g_2 \in H$.

(a) Show that $\sim$ is an equivalence relation.
(b) Find the class of $e$, the identity element.

(5) Let $a$ be a positive real number. Note that for all $x, y \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ and a unique $0 \leq r < a$ so that $x + y = r + ka$.

Denote $[0, a)$, the half open interval, by $\mathbb{R}_a$ and define the following “addition” on $\mathbb{R}_a$.

$x +_a y = r,$
where $x + y = r + ka$ and $r \in [0, a)$.

(a) Show that $(\mathbb{R}_a, +_a)$ is a group.
(b) Show that $(\mathbb{R}_1, +_1)$ is isomorphic to $(\mathbb{R}_a, +_a)$ for any $a > 0$. (Therefore, $(\mathbb{R}_a, +_a)$ and $(\mathbb{R}_b, +_b)$ are isomorphic for any $a, b > 0$).
(c) (Bonus Problem) Prove or disprove: $(\mathbb{R}_1, +_1)$ is isomorphic to $(\mathbb{R}, +)$.

(6) Exercise 4 page 45: 3, 12, 19