Problem 1. Consider $\mathbb{Z}_3$ with the addition and multiplication mod 3 as usual. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Define

$$(a, b) + (a', b') := (a + a', b + b')$$

$$(a, b) \cdot (a', b') := (aa' - bb', ab' + a'b)$$

(a) Show that $(R, +, \cdot)$ is a commutative ring.
(b) Show that $(1, 0)$ is the identity element for the multiplication.
(c) Show that the equation $x^2 = -1$ has exactly two solutions in $R$.

Solution. a) We first note that $(R, +)$ is the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$ we saw in Math 103A as it is the direct product of two abelian groups for which the binary operation is applied component-wise. The associativity of multiplication follows as

$$(a_1, a_2) \cdot [(b_1, b_2) \cdot (c_1, c_2)] = (a_1, a_2) \cdot (b_1c_1 - b_2c_2, b_1c_2 + b_2c_1)$$

$$= (a_1b_1c_1 - a_1b_2c_2 - a_2b_1c_2 - a_2b_2c_1, a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 - a_2b_2c_2).$$

and since we have that

$$[(a_1, a_2) \cdot (b_1, b_2)] \cdot (c_1, c_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) \cdot (c_1, c_2)$$

$$= (a_1b_1c_1 - a_1b_2c_2 - a_2b_1c_2 - a_2b_2c_1, a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 - a_2b_2c_2).$$

Putting these together, for all $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$,

$$(a_1, a_2) \cdot [(b_1, b_2) \cdot (c_1, c_2)] = [(a_1, a_2) \cdot (b_1, b_2)] \cdot (c_1, c_2).$$

Next, we observe that $\cdot : R \times R \to R$ is commutative as for all $(a_1, a_2), (b_1, b_2) \in R$,

$$(a_1, a_2) \times (b_1, b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) = (b_1a_1 - b_2a_2, b_1a_2 + b_2a_1) = (b_1, b_2) \cdot (a_1, a_2).$$
Finally, we check the left distribution law (note that the left distribution law implies the right distribution law when $\cdot$ is commutative). We have

$$(a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2) = (a_1b_1 + a_1c_1 - a_2b_2 - a_2c_2, a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1)$$

and also that

$$(a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) + (a_1c_1 - a_2c_2, a_1c_2 + a_2c_1)$$

and we finish by noting that

$$(a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) + (a_1c_1 - a_2c_2, a_1c_2 + a_2c_1) = (a_1b_1 + a_1c_1 - a_2b_2 - a_2c_2, a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1).$$

Thus $(R, +, \cdot)$ is a commutative ring.

b) Since we are a commutative ring it suffices to note that for all $(a_1, a_2) \in R$ that

$$(a_1, a_2) \cdot (1, 0) = (a_1(1) - a_2(0), a_1(0) + a_2(1)) = (a_1, a_2).$$

c) We first note that $-(1, 0) = (-1, 0) = (2, 0)$ in $R$ and that $x = (x_1, x_2) \in R$ is so that

$$x^2 = (x_1, x_2) \cdot (x_1, x_2) = (x_1^2 - x_2^2, 2(x_1x_2)).$$

Thus $x^2 = -1$ implies that $x_1^2 - x_2^2 = 2$ and $x_1x_2 = 0$. As $\mathbb{Z}_3$ is an integral domain, we therefore have either $x_1 = 0$ or $x_2 = 0$. In the case where they are both zero, they do not satisfy the condition in the first coordinate. When $x_1 = 0$, we could have $x_2 = 1$ or $x_2 = 2$ as these both satisfy the condition in the first coordinate. However, when $x_2 = 0$, we need $x_1^2 = 2$ which has no solutions in $\mathbb{Z}_3$. Thus, $(0, 1)$ and $(0, 2)$ are the only solutions.

**Problem 2.** Let $R$ be a ring so that $x^2 = x$ for all $x \in R$.

(a) Compute $(x + x)(x + x)$ (by multiplying) for every $x \in R$.

(b) Conclude from part (a) and $(x + x)^2 = x + x$ that $x + x = 0$ for every $x \in R$.

(c) Show that $R$ is a commutative ring. (Hint: Note that $x + y = (x + y)^2$ for all $x, y \in R$. Compute $(x + y)^2$ also by multiplying $(x + y)(x + y)$. Use part (b) to conclude.)

**Solution.**

(a) For any $x \in R$, by distributivity, $(x + x)(x + x) = x(x + x) + x(x + x) = x^2 + x^2 + x^2 + x^2$. By our assumption, $x^2 = x$, so $(x + x)(x + x) = x + x + x + x$.

b) We have that,

$$x + x + x + x = (x + x)(x + x) = (x + x)^2 = x + x.$$

The equality of the first and last expressions implies that

$$x + x + (x + x) + -(x + x) = 0,$$
which implies that $x + x = 0$, as desired.

c) We conclude that $R$ is a commutative ring, by showing that $xy = yx$ for any $x, y \in R$. Let $x, y \in R$. Note that by distributivity,

$$(x + y)(x + y) = x(x + y) + y(x + y) = x^2 + xy + yx + y^2 = x + xy + yx + y.$$  

Note that $(x + y)^2 = x + y$ by assumption and thus $x + xy + yx + y = x + y$. Then, we have that $xy + yx = 0$. However, by part b) we have that $xy + xy = 0$ as well, and thus $xy + yx = xy + xy$, which implies that $xy = yx$, as desired.

**Problem 3.** Find characteristic of the rings $\mathbb{Z}_5 \times \mathbb{Z}_6$ and $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9$ (additions and multiplication are the usual addition and multiplication).

**Solution.** We claim that the characteristic of $\mathbb{Z}_5 \times \mathbb{Z}_6$ is 30. As $(1, 1)$ is the multiplicative identity, by Theorem 19.15, it suffices to show that 30 is the smallest positive integer such that $30 \cdot (1, 1) = (0, 0)$. First, $30 \cdot (1, 1) = (30, 30) = (0, 0)$ since $5 \mid 30$ and $6 \mid 30$. Now note that if a positive integer $n$ satisfies that $n \cdot (1, 1) = (0, 0)$, then we must have that $n \cdot 1 = 0$ in $\mathbb{Z}_5$ and $n \cdot 6 = 0$ in $\mathbb{Z}_6$. Thus, we must have that $5 \mid n$ and $6 \mid n$, which implies that the least common multiple 30 must divide $n$. Thus, 30 is indeed the minimum positive integer such that $n \cdot (1, 1) = (0, 0)$, making it the characteristic of $\mathbb{Z}_5 \times \mathbb{Z}_6$.

We claim that the characteristic of $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9$ is 72. Again as $(1, 1, 1)$ is the multiplicative identity, it suffices to show that 72 is the smallest positive integer such that $72 \cdot (1, 1, 1) = (0, 0, 0)$. First, $72 \cdot (1, 1, 1) = (0, 0, 0)$ since $6 \mid 72$, $8 \mid 72$, and $9 \mid 72$. Furthermore, if a positive integer $n$ has $n \cdot (1, 1, 1) = (0, 0, 0)$, we must have that $6 \mid n$, $8 \mid n$, and $9 \mid n$ and thus the least common multiple 72 must divide $n$. Thus, 72 is the minimum positive integer such that $n \cdot (1, 1, 1) = (0, 0, 0)$, making it the characteristic of $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9$.

**Problem 4.** Let $R$ be ring without zero divisor. Let $a, b \in R$ and $a \neq 0$. Show that the equation $ax = b$ can have at most one solution. Give an example where no solution exist.

**Solution.** Seeking a contradiction, suppose at least two such solutions exists. Then, there exists $x_1, x_2 \in R$ so that $ax_1 = b$ and $ax_2 = b$ for $x_1 \neq x_2$. This then implies

$$0 = b - b = (ax_1) - (ax_2) = a(x_1 - x_2)$$

which is a contradiction as $0 \neq a$ and $R$ does not have any zero divisors. An example for which there are no solutions is the ring $R = \mathbb{Z}$ and the equation $2x = 1$.

**Problem 5.** Let $R$ be a ring with unity. Assume $a \in R$ is a unit, show that $a$ is not a zero divisor.

**Solution.** Seeking a contradiction, suppose that $a \in R$ is a unit and a zero divisor. Then, there exists nonzero $c \in R$ so that $ac = 0$ and $a \in R$ so that $ab = 1 = ba$. Then we get a contradiction as

$$c = 1 \cdot c = (ba) \cdot c = b(ac) = b(0) = 0.$$
Problem 6 (§19.2). Solve the equation $3x = 2$ in $\mathbb{Z}_7$ and $\mathbb{Z}_{23}$.

Solution. In $\mathbb{Z}_7$, we note that $5 \cdot 3 = 1$ and hence,

$$x = (1)x = (5 \cdot 3)x = 5 \cdot (3x) = 5(2) = 3.$$ 

Note that $\mathbb{Z}_7$ is actually a field and $3^{-1} = 5$.

In $\mathbb{Z}_{23}$, we note that $3 \cdot 8 = 24 = 1$ and hence,

$$x = (1)x = (8 \cdot 3)x = 8 \cdot (3x) = 8(2) = 16.$$ 

Note that there are other ways to solve this by brute force case analysis.

Problem 7 (§19.10). Find the characteristic of the ring $\mathbb{Z}_6 \times \mathbb{Z}_{15}$.

Solution. We need to find the minimum $n \geq 1$ so that $n(1, 1) = (0, 0)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$. Observe that $\text{lcm}(6, 15) = 30$ and that $30(1, 1) = (0, 0)$ and this is minimal by considering the second coordinate and the fact that $15(1, 1) = (3, 0)$. Thus, the characteristic of the ring $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ is 30. Note that this is not prime and $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ is NOT an integral domain.

Problem 8 (§19.12). Let $R$ be a commutative ring with unity of characteristic 3. Compute and simplify $(a + b)^9$ for $a, b \in R$.

Solution. The Binomial Theorem, which applies here as $R$ is a commutative ring, yields

$$(a + b)^9 = \sum_{i=0}^{9} \binom{9}{i} a^i b^{9-i}.$$ 

It is therefore our task to determine which of the terms $\binom{9}{i}$ are divisible by three. Recall

$$\binom{9}{i} = \frac{9!}{i!(9-i)!} = \binom{9}{9-i}$$ 

and hence we only need to check $i = 0, 1, 2, 3, 4, 5$. It turns out these are all zero modulo 3 besides the term corresponding to $i = 0$ (and hence $i = 9$) which yields

$$(a + b)^9 = a^9 + b^9.$$ 

Another way to solve this is to note that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3$$

and that

$$(a + b)^9 = ((a + b)^3)^3 = (a^3 + b^3)^3 = (a^3)^3 + (b^3)^3 = a^9 + b^9$$

where the second to last equality follows by applying Equation (1) with $a^3$ and $b^3$.
Problem 9 (§19.14). Show that the matrix
\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\]
is a divisor of zero in $M_2(\mathbb{Z})$.

Solution. We first note that
\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
x + 2y \\
2x + 4y
\end{bmatrix},
\]
which is 0 if $x = -2y$. Thus, consider $x = -2, y = 1$, and note that
\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 \\
1
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix},
\]
so we can add a duplicate of this column to get that
\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & -2 \\
1 & 1
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
which shows that this matrix is a divisor of 0 in $M_2(\mathbb{Z})$. 

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