Math 103A Homework 4 Solutions

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Problem 1. Let $a, b$ be two positive integers and define
\[ H := \{am + bn : m, n \in \mathbb{Z}\}. \]
a) Show that $H$ is a subgroup of $\mathbb{Z}$ with the usual addition.
b) Show that $H = d\mathbb{Z}$ where $d = \gcd(a, b)$.

Solution. a) First note that $0 \in H$ which we can see by letting $m = n = 0$. Next, given $(am_1 + bn_1), (am_2 + bn_2) \in H$, we have that
\[ (am_1 + bn_1) + (am_2 + bn_2) = a(m_1 + m_2) + b(n_1 + n_2) \in H \]
so we have that $H$ is closed under multiplication. Lastly, given $am + bn \in H$, we have that $a(-m) + b(-n) \in H$ is so that
\[ (am + bn) + (a(-m) + b(-n)) = 0 = (a(-m) + b(-n)) + (am + bn) \]
and hence we have that $H$ is a subgroup of $\mathbb{Z}$.
b) Given any $am + bn \in H$, we have that $d = \gcd(a, b)$ is so that $d|a$ and $d|b$ and hence there exists $c_1, c_2 \in \mathbb{Z}$ so that $am + bn = dc_1m + dc_2n = (c_1m + c_2n) \in d\mathbb{Z}$. Next, since $d = \gcd(a, b)$, there exists $m', n' \in \mathbb{Z}$ so that $am' + bn' = d$, and hence given any $dc \in d\mathbb{Z}$ we have that
\[ dc = (am' + bn')c = a(m'c) + b(n'c) \in H \]
and hence we have that $H = d\mathbb{Z}$.

Problem 2 (29 pg. 45). Show that if $G$ is a finite group with identity $e$ and with an even number of elements, then there is an $e \neq a \in G$ so that $a * a = e$

Solution. Seeking a contradiction, suppose that for all $a \in G$ so that $a \neq e$ we have that $a * a \neq e$. Then, we have that $a \neq a^{-1}$ for all $a \in G$ so that $a \neq e$. Hence we have an even number of elements in the group which are not the identity element and since there is a unique identity element, we necessarily have that the group has odd order; a contradiction.

Problem 3 (11 pg. 55). Determine whether the set of $n \times n$ matrices with determinant $-1$ is a subgroup of $GL(n, \mathbb{R})$.

Solution. Let $S_{-1}$ be the set of $n \times n$ matrices with determinant $-1$. Then, given $A, B \in S_{-1}$ we have that $\det(A) = \det(B) = -1$, but $\det(AB) = \det(A) \det(B) = (-1)(-1) = 1$ and hence $AB \notin S_{-1}$. Also, the identity matrix $I_n \notin S_{-1}$. Thus $S_{-1}$ is not a subgroup.

Problem 4 (12 pg. 55). Determine whether the set of $n \times n$ matrices with determinant $-1$ or $1$ is a subgroup of $GL(n, \mathbb{R})$.

Solution. Let $S_{\pm 1}$ be the set of $n \times n$ matrices with determinant $-1$ or $1$. First $I_n \in S_{\pm 1}$. Then, given $A, B \in S_{\pm 1}$, we have that $\det(AB) = \det(A) \det(B) = \pm 1$ and hence $AB \in S_{\pm 1}$. Lastly, given $A \in S_{\pm 1}$ we have that $A^{-1}$ exists and $\det(A^{-1}) = \frac{1}{\det(A)} = \pm 1$ and hence $A^{-1} \in S_{\pm 1}$. Thus $S_{\pm 1}$ forms a subgroup of $GL(n, \mathbb{R})$.

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Problem 5 (13 pg. 55). Determine whether the set of \( n \times n \) matrices with \( (A)^T A = I_n \) is a subgroup of \( GL(n, \mathbb{R}) \).

**Solution.** Let \( O_n := \{ A \in GL(n, \mathbb{R}) : A^{-1} = A^T \} \) be the set of \( n \times n \) real orthogonal matrices. First, we have that \( I_n \in O_n \). Next, given \( A, B \in O_n \), we have that
\[
(AB)^T (AB) = B^T A^T AB = B^T B = I_n.
\]
Finally, given \( A \in O_n \), we have that \( A = (A^T)^T = (A^{-1})^T \) and hence we have that
\[
(A^{-1})^T A^{-1} = AA^{-1} = I_n
\]
and thus \( A^{-1} \in O_n \) so that \( O_n \) is a subgroup of \( GL(n, \mathbb{R}) \).

Problem 6 (31 pg. 55). Find the order of the cyclic subgroup generated by the following element in \( GL(n, \mathbb{R}) \).
\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

**Solution.** A calculation yields that:
\[
A^2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} = I_4
\]
and hence \( \text{ord}(A) = 2 \). One can also solve this problem using adjacency matrices from graph theory or by noting that the above matrix is a permutation matrix which corresponds to the permutation \( \sigma = (3,4,1,2) \) in one line notation or \( \sigma = (13)(24) \) in cycle notation.

Problem 7 (41 pg. 55). Let \( (G, *) \) and \( (G', \ast') \) be groups and \( \phi : G \to G' \) a group isomorphism. Let \( H \subseteq G \) be a subgroup of \( G \), then show that \( \phi(H) := \{ \phi(h) : h \in H \} \) is a subgroup of \( G' \).

**Solution.** As we showed previously, if \( e' \) is the identity element of \( G' \), then we have that \( \phi(e) = e' \) and hence \( e' \in \phi(H) \). Next, given \( \phi(h_1), \phi(h_2) \in \phi(H) \), we have that
\[
\phi(h_1) \ast' \phi(h_2) = \phi(h_1 \ast h_2) \in \phi(H)
\]
since we have that \( h_1, h_2 \in H \) and hence \( h_1 \ast h_2 \in H \). Finally, consider \( \phi(h) \in H \) and now we claim that \( \phi(h)^{-1} = \phi(h^{-1}) \) and hence since \( h^{-1} \in H \), we necessarily have that \( \phi(h)^{-1} \in \phi(H) \). We have that
\[
\phi(h) \ast' \phi(h^{-1}) = \phi(h \ast h^{-1}) = \phi(e) = e'
\]
and thus we have that \( \phi(h)^{-1} = \phi(h^{-1}) \). As a result \( \phi(H) := \{ \phi(h) : h \in H \} \) is a subgroup of \( G' \).

Problem 8 (48 pg. 55). Let \( G \) be a abelian group with identity element \( e \). Prove that \( H_n := \{ x : x^n = e \} \) is a subgroup of \( G \) for \( n \geq 1 \).

**Solution.** First, observe that \( e \in H_n \). Next, given \( a, b \in H_n \) and using the abelian nature of the group \( G \), we have that
\[
(a \ast b)^n = a^n \ast b^n = e \ast e = e.
\]
Finally, considering \( a \in H_n \), we again use the fact that the group is abelian to get that
\[
(a^{-1})^n = (a^{-1})^n \ast e = (a^{-1})^n \ast a^n = (a^{-1} \ast a)^n = e^n = e
\]
and hence we have that \( H_n := \{ x : x^n = e \} \) is a subgroup of \( G \) for \( n \geq 1 \).