Problem 1. Let $G$ be a group. Let $H \leq G$ and $K \leq G$ be two subgroups of $G$. Show that $H \cup K$ is a subgroup if and only if either $H \subseteq K$ or $K \subseteq H$.

Solution. In the backwards direction where either $H \subseteq K$ or $K \subseteq H$, we have that $H \cup K$ is equal to $K$ or $H$ respectively, so clearly we have that $H \cup K$ is a subgroup. In the forward direction, suppose that $H \cup K$ is a subgroup and that neither $H \subseteq K$ nor $K \subseteq H$ holds. That is, there exists $k \in K$ so that $k \notin H$ and an $h \in H$ so that $h \notin K$. Now, we consider $hk \in H \cup K$ since $h,k \in H \cup K$ and subgroups are closed under multiplication. Now, we claim that $hk \notin H$. Suppose it was, then left multiplication by $h^{-1} \in H$ yields that $h^{-1} \cdot hk = k \in H$, a contradiction. Hence $hk \notin H$. Thus, we necessarily have that $hk \in K$. Now, right multiplication by $k^{-1} \in K$ yields that $h \in K$; a contradiction.

Problem 2. Let $G$ be a group and let $a$ be one fixed element of $G$. Show that $H_a := \{x \in G : xa = ax\}$ is a subgroup of $G$.

Solution. First, we have that $e \in H_a$ since $ea = ae$. Next, if we have that $x,y \in H_a$ then
\[(xy)a = x(ya) = x(ay) = (xa)y = a(xy)\]
so that $xy \in H_a$. Finally, if $x \in H_a$, then
\[x^{-1}a = x^{-1}ax^{-1} = x^{-1}(ax)x^{-1} = x^{-1}(xa)x^{-1} = ax^{-1}\]
and hence $x^{-1} \in H_a$ so that $H_a$ is a subgroup.

Problem 3. Generalizing Exercise 51, let $S$ be any subset of a group $G$.

a) Show that $H_S := \{g \in G : gs = sg \text{ for all } s \in S\}$ is a subgroup of $G$.

b) In reference to part a), the subgroup $H_G$ is center of $G$. Prove that $H_G$ is abelian.

Solution. a) First, we have that $e \in H_S$ since $es = se$ for all $s \in S$. Next, if we have that $x,y \in H_S$ then
\[(xy)s = x(ys) = x(sy) = (xs)y = s(xy)\]
for all $s \in S$ so that $xy \in H_S$. Finally, if $x \in H_S$, then
\[x^{-1}s = x^{-1}sx^{-1} = x^{-1}(sx)x^{-1} = x^{-1}(xs)x^{-1} = sx^{-1}\]
for all $s \in S$ and hence $x^{-1} \in H_S$ so that $H_S$ is a subgroup.

b) Fix $a,b \in H_G$ and then we have that $b \in H_G \subseteq G$, so by definition of $a \in H_G$, we have that $ab = ba$ and thus $H_G$ is abelian.
Problem 4. Find the number of generators of a cyclic group having order 12.

Solution. In order to find a generator of $\mathbb{Z}_{12}$ we need an $i$ so that $1 \leq i < 12$ where $\gcd(i, 12) = 1$. We thus have that 1, 5, 7, 11 are generators of $\mathbb{Z}_{12}$ for which any cyclic group of order 12 is isomorphic to. We thus have 4 generators. There is more general theory (i.e. not guess and check) in terms of the number of generators of the cyclic group of order $n$ involving the Euler $\phi$-function and noting that $12 = 2^2 \cdot 3$.

Problem 5. Find the number of automorphisms of the group $\mathbb{Z}_8$.

Solution. Since we necessarily need to send the generator 1 $\in \mathbb{Z}_8$ to one of the corresponding generators and this then completely determines the rest of the isomorphism by Exercise 44, there are four such automorphisms; one for each of the generators of the group $\mathbb{Z}_8$.

Problem 6. Find the number of automorphisms of the group $\mathbb{Z}$.

Solution. We have exactly two automorphism of the group $\mathbb{Z}$; which is the identity map and the map where $\phi(1) = -1$ and hence $\phi(n) = -n$. Seeking a contradiction, suppose we had another such automorphism $\phi : \mathbb{Z} \to \mathbb{Z}$. Then $\phi(1) = x$ where we have that $|x| > 1$ and so that there exists $k \in \mathbb{Z}$ so that $\phi(k) = 1$. First, let’s consider the case where $x > 1$. Now, $\phi(1) = x = \phi(k) + \cdots + \phi(k) = \phi(kx)$. The injectivity of $\phi$ then yields that $kx = 1$; a contradiction since $k \in \mathbb{Z}$ and $|x| > 1$. The case where $x < -1$ is similar; we simply consider $\phi(-1)$. Thus $\phi_1 : \mathbb{Z} \to \mathbb{Z}$ where $\phi_1 : n \mapsto n$ and $\phi_2 : \mathbb{Z} \to \mathbb{Z}$ where $\phi_2 : n \mapsto -n$ are the only automorphisms on $\mathbb{Z}$.

Problem 7. Find the number of elements in cyclic subgroup of $\mathbb{Z}_{42}$ generated by 30.

Solution. By analyzing the prime factorizations of 30 and 42, we see that $\text{lcm}(30, 42) = 5 \cdot 2 \cdot 3 \cdot 7 = 7(30)$ and hence we see that ord$(30)) = 7$ and hence there are 7 elements in the cyclic subgroup of $\mathbb{Z}_{42}$ generated by 30 which are $\{0, 30, 2(30), \ldots , 6(30)\}$.

Problem 8. Find all order of subgroups of $\mathbb{Z}_{12}$.

Solution. We have that $\mathbb{Z}_{12}$ is cyclic and hence all of its subgroups are cyclic. Thus, it suffices to determine all possible orders of the elements in $\mathbb{Z}_{12}$. Lastly, since ord$(a) = \text{ord}(a^{-1})$, it suffices to determine the order of elements 1, 2, ..., 6. We have that ord$(1) = 12$, ord$(2) = 6$, ord$(3) = 4$, ord$(4) = 3$, ord$(5) = 12$, ord$(6) = 2$. Hence, we have that the only orders of the subgroups of $\mathbb{Z}_{12}$ are 1, 2, 3, 4, 6, 12.

Problem 9. Let $G$ be a group and suppose that $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

Solution. Following the hint, fixing some $x \in G$, we consider the element $(xax^{-1})^2 \in G$ and observe that
\[(xax^{-1})^2 = (xax^{-1})(xax^{-1}) = xa^2x^{-1} = xx^{-1} = e\]
and since $a$ is the unique element of order 2, we have that $xa^{-1} = e$ or $xax^{-1} = a$. In the first case, this yields that $xa = x$ and hence $a = e$; a contradiction. Hence, we necessarily have that second case where $xax^{-1} = a \iff xa = ax$ which is what we wanted to show.

Another way to do this problem is to note that the map $\phi_x : G \to G$ given by $\phi(a) = xax^{-1}$ is a group isomorphism (and hence automorphism) for all $x \in G$. (I am leaving this for you guys to check; it is not hard to show.) Therefore, we have that ord$(a) = \text{ord}(\phi_x(a)) = \text{ord}(xax^{-1}) = 2$ and thus $a = xax^{-1}$ since $a$ is the only element of order 2 in the group.