ISOLATIONS OF GEODESIC PLANES IN THE FRAME BUNDLE OF A HYPERBOLIC 3-MANIFOLD

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Abstract. We present a quantitative isolation property of the lifts of properly immersed geodesic planes in the frame bundle of a geometrically finite hyperbolic 3-manifold. Our estimates are polynomials in the tight areas and Bowen-Margulis-Sullivan densities of geodesic planes, with degree given by the modified critical exponents.

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1. Introduction

Let $\mathbb{H}^3$ denote the hyperbolic 3-space, and let $G := \text{PSL}_2(\mathbb{C}) \simeq \text{Isom}^+(\mathbb{H}^3)$. Any complete hyperbolic 3-manifold can be presented as a quotient $M = \Gamma \backslash \mathbb{H}^3$ where $\Gamma$ is a torsion-free discrete subgroup of $G$. A geodesic plane in $M$ is the image of a totally geodesic immersion of the hyperbolic plane $\mathbb{H}^2$ in $M$. Set $X := \Gamma \backslash G$. Via the identification of $X$ with the oriented frame bundle $FM$, a geodesic plane in $M$ arises as the image of a unique $\text{PSL}_2(\mathbb{R})$-orbit under the base point projection map

$$\pi : X \simeq FM \to M.$$  

Moreover a properly immersed geodesic plane in $M$ corresponds to a closed $\text{PSL}_2(\mathbb{R})$-orbit in $X$.

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Setting $H := \text{PSL}_2(\mathbb{R})$, the main goal of this paper is to obtain a quantitative isolation result for closed $H$-orbits in $X$ when $\Gamma$ is a geometrically finite group. Fix a left invariant Riemannian metric on $G$, which projects to the hyperbolic metric on $\mathbb{H}^3$. This induces the distance $d$ on $X$ so that the canonical projection $G \rightarrow X$ is a local isometry. We use this Riemannian structure on $G$ to define the volume of a closed $H$-orbit in $X$. For a closed subset $S \subset X$ and $\varepsilon > 0$, $B(S, \varepsilon)$ denotes the $\varepsilon$-neighborhood of $S$.

**The case when $M$ is compact.** We first state the result for compact hyperbolic 3-manifolds. In this case, Ratner [21] and Shah [27] independently showed that every $H$-orbit is either compact or dense in $X$. Moreover, there are only countably many compact $H$-orbits in $X$. Mozes and Shah [19] proved that an infinite sequence of compact $H$-orbits becomes equidistributed in $X$. Our questions concern the following quantitative isolation property: for given compact $H$-orbits $Y$ and $Z$ in $X$,

1. How close can $Y$ approach $Z$?
2. Given $\varepsilon > 0$, what portion of $Y$ enters into the $\varepsilon$-neighborhood of $Z$?

It turns out that volumes of compact orbits are the only complexity which measures their quantitative isolation property. The following theorem was proved by Margulis in an unpublished note:

**Theorem 1.1** (Margulis). Let $\Gamma$ be a cocompact lattice in $G$. For every $1/3 \leq s < 1$, the following hold for any compact $H$-orbits $Y \neq Z$ in $X$:

1. \[ d(Y, Z) \gg \alpha_s^{-4/s} \cdot \text{Vol}(Y)^{-1/s} \cdot \text{Vol}(Z)^{-1/s} \]
   where $\alpha_s = (1 - s)^{1/(1-s)}$.
2. For all $0 < \varepsilon < 1$,
   \[ m_Y(Y \cap B(Z, \varepsilon)) \ll \alpha_s^4 \cdot \varepsilon^s \cdot \text{Vol}(Z) \]
   where $m_Y$ denotes the $H$-invariant probability measure on $Y$.

In both statements, the implied constants depend only on $\Gamma$.

**Remark 1.2.**

1. By recent works ([15], [2]), there may be infinitely many compact $H$-orbits only when $\Gamma$ is an arithmetic lattice.
2. Theorem 1.1 for some exponent $s$ is proved in [9, Lemma 10.3]. The proof in [9] is based on the effective ergodic theorem which relies on the arithmeticity of $\Gamma$ via uniform spectral gap on compact $H$-orbits; the exponent $s$ obtained in their approach however is much smaller than 1.
3. Margulis’ proof does not rely on the arithmeticity of $\Gamma$ and is based on the construction of a certain function on $Y$ which measures the distance $d(y, Z)$ for $y \in Y$ (cf. (1.13)). A similar function appeared first in the work of Eskin, Mozes and Margulis in the study of a quantitative version of the Oppenheim conjecture [11], and later in several other works (e.g., [10], [4], and [12]).
General geometrically finite case. We now consider a general hyperbolic 3-manifold \( M = \Gamma \setminus \mathbb{H}^3 \). Denote by \( \Lambda \subset \partial \mathbb{H}^3 \) the limit set of \( \Gamma \) and by \( \text{core} \ M \) the convex core of \( M \), i.e.,

\[
\text{core} \ M = \Gamma \setminus \text{hull} \ \Lambda \subset M
\]

where \( \text{hull} \ \Lambda \subset \mathbb{H}^3 \) denotes the convex hull of \( \Lambda \). In the rest of the introduction, we assume that \( M \) is geometrically finite, that is, the unit neighborhood of \( \text{core} \ M \) has finite volume.

Let \( Y \subset X \) be a closed \( H \)-orbit and \( S_Y = \Delta_Y \setminus \mathbb{H}^2 \) be the associated hyperbolic surface, where \( \Delta_Y < H \) is the stabilizer in \( H \) of a point in \( Y \). We assume that \( Y \) is non-elementary, that is, \( \Delta_Y \) is not virtually cyclic; otherwise, we cannot expect an isolation phenomenon for \( Y \), as there is a continuous family of parallel elementary closed \( H \)-orbits in general when \( M \) is of infinite volume. It is known that \( S_Y \) is always geometrically finite [20].

Let \( 0 < \delta(Y) \leq 1 \) denote the critical exponent of \( S_Y \), i.e., the abscissa of the convergence of the series \( \sum_{\gamma \in \Delta_Y} e^{-sd(o,\gamma(o))} \) for some \( o \in \mathbb{H}^2 \). We define the following modified critical exponent of \( Y \):

\[
\delta_Y := \begin{cases} 
\delta(Y) & \text{if } S_Y \text{ has no cusp} \\
2\delta(Y) - 1 & \text{otherwise};
\end{cases}
\]

note that \( 0 < \delta_Y \leq \delta(Y) \leq 1 \), and \( \delta_Y = 1 \) if and only if \( S_Y \) has finite area.

In generalizing Theorem 1.1(1), we first observe that the distance \( d(Y, Z) \) between two closed \( H \)-orbits \( Y, Z \) may be zero, e.g., if they both have cusps going into the same cuspidal end of \( X \). To remedy this issue, we use the thick-thin decomposition of \( \text{core} \ M \). For \( p \in M \), we denote by \( \text{inj} \ p \) the injectivity radius at \( p \). For all \( \varepsilon > 0 \), the \( \varepsilon \)-thick part

\[
(\text{core} \ M)_\varepsilon := \{ p \in \text{core} \ M : \text{inj} \ p \geq \varepsilon \}
\]

is compact, and for all sufficiently small \( \varepsilon > 0 \), the \( \varepsilon \)-thin part given by \( \text{core} \ M - (\text{core} \ M)_\varepsilon \) is contained in finitely many disjoint horoballs. Let \( X_0 \subset X \) denote the renormalized frame bundle \( RFM \) (see (2.1)). Using the fact that the projection of \( X_0 \) is contained in \( \text{core} \ M \) under \( \pi \), we define the \( \varepsilon \)-thick part of \( X_0 \) as follows:

\[
X_\varepsilon := \{ x \in X_0 : \pi(x) \in (\text{core} \ M)_\varepsilon \}.
\]

The following theorem extends Theorem 1.1 to all geometrically finite hyperbolic manifolds:

**Theorem 1.5.** Let \( M \) be a geometrically finite hyperbolic 3-manifold. Let \( Y \neq Z \) be non-elementary closed \( H \)-orbits in \( X \), and denote by \( m_Y \) the probability Bowen-Margulis-Sullivan measure on \( Y \). For every \( \delta_Y \leq s < \delta_Y \) the following hold.

1. For all \( 0 < \varepsilon \ll 1 \), we have

\[
d(Y \cap X_\varepsilon, Z) \geq \alpha_{Y,s}^{-s} \cdot \left( \frac{\text{area}_\varepsilon}{\text{area}_Z} \right)^{1/s}
\]
where

- $v_{Y,\epsilon} = \min_{y \in Y \cap X} m_Y(B_Y(y, \epsilon))$ where $B_Y(y, \epsilon)$ is the $\epsilon$-ball around $y$ in the induced metric on $Y$.
- $\text{area}_t Z$ denotes the tight area of $S_Z$ relative to $M$.
- $\alpha_{Y,s} := \left(\frac{s_Y}{\delta_Y - s}\right)^{1/(\delta_Y - s)}$ where $s_Y$ is the shadow constant of $Y$.

(2) For all $0 < \epsilon \ll 1$,

$$m_Y(Y \cap B(Z, \epsilon)) \ll \alpha_{Y,s}^* \cdot \epsilon^s \cdot \text{area}_t Z.$$ 

In both statements, the implied constants and $\ast$ depend only on $\Gamma$.

When $X$ has finite volume, we have $\delta_Y = 1$ and $m_Y$ is $H$-invariant so that $v_{Y,\epsilon} \asymp \epsilon^3 \text{Vol}(Y)^{-1}$. Moreover, the tight area $\text{area}_t Z$ and the shadow constant $s_Y$ are simply the usual area of $S_Z$ and a fixed constant (in fact, 2) respectively. Therefore Theorem 1.5 recovers Theorem 1.1.

We now give definitions of the tight area $\text{area}_t Z$ and the shadow constant $s_Y$ for a general geometrically finite case; these are new geometric invariants introduced in this paper.

**Definition 1.7** (Tight area of $S$). For a properly immersed geodesic plane $S$ of $M$, the tight-area of $S$ relative to $M$ is given by

$$\text{area}_t(S) := \text{area}(S \cap \mathcal{N}(\text{core } M))$$

where $\mathcal{N}(\text{core } M) = \{p \in M : d(p, q) \leq \text{inj}(q) \text{ for some } q \in \text{core } M\}$ is the tight neighborhood of core $M$.

We show $\text{area}_t(S)$ is finite in Theorem 3.3, by proving that $S \cap \mathcal{N}(\text{core } M)$ is contained in the union of a bounded neighborhood of $\text{core}(S)$ and finitely many cusp-like regions (see Fig.1). We remark that the area of the intersection $S \cap B(\text{core } M, 1)$ is not finite in general.
Definition 1.8 (Shadow constant of $Y$). For a closed $H$-orbit $Y$ in $X$, let $\Lambda_Y \subset \partial \mathbb{H}^2$ denote the limit set of $\Delta_Y$, $\{\nu_p : p \in \mathbb{H}^2\}$ the Patterson-Sullivan density for $\Delta_Y$, and $B_p(\xi, \varepsilon)$ the $\varepsilon$-neighborhood of $\xi \in \partial \mathbb{H}^2$ with respect to the Gromov metric at $p$. The shadow constant of $Y$ is defined as follows:

$$(1.9) \quad s_Y := \sup_{\xi \in \Lambda_Y, p \in [\xi, \Lambda_Y], 0 < \varepsilon \leq 1/2} \nu_p(B_p(\xi, \varepsilon))^{1/\delta_Y} \cdot \nu_p(B_p(\xi, 1/2))^{1/\delta_Y},$$

where $[\xi, \Lambda_Y]$ is the union of all geodesics connecting $\xi$ to a point in $\Lambda_Y$.

We show that $s_Y < \infty$ in Theorem 4.8.

Remark 1.10. (1) If $Y$ is convex cocompact, then for all $0 < \varepsilon < 1$, $v_{Y, \varepsilon} \asymp \varepsilon^{1+2\delta_Y}$ with the implied constant depending on $Y$. When $Y$ has a cusp, Sullivan’s shadow lemma (cf. Proposition 4.11) implies that $\lim_{\varepsilon \to 0} \frac{\log v_{Y, \varepsilon}}{\log \varepsilon}$ does not exist.

(2) We give a proof of a more general version of Theorem 1.5(1) where $Z$ is allowed to be equal to $Y$ (see Corollary 10.5 for a precise statement).

A hyperbolic 3-manifold $M$ is called convex cocompact acylindrical if $\text{core } M$ is a compact manifold with no essential discs or cylinders which are not boundary parallel. For such a manifold, there exists a uniform positive lower bound for $\delta(Y) = \delta_Y$ for all non-elementary closed $H$-orbits $Y$ [17]; therefore the dependence of $\delta_Y$ can be removed in Theorem 1.5 if one is content with taking some $s$ which works uniformly for all such orbits.

Examples of $X$ with infinitely many closed $H$-orbits are provided by the following theorem which can be deduced from ([17], [18], [3]):

Theorem 1.11. Let $M_0$ be an arithmetic hyperbolic 3-manifold with a properly immersed geodesic plane. Any geometrically finite acylindrical hyperbolic 3-manifold $M$ which covers $M_0$ contains infinitely many non-elementary properly immersed geodesic planes.

It is easy to construct examples of $M$ satisfying the hypothesis of this theorem. For instance, if $M_0$ is an arithmetic hyperbolic 3-manifold with a properly imbedded compact geodesic plane $P$, $M_0$ is covered by a geometrically finite acylindrical manifold $M$ whose convex core has boundary isometric to $P$.

Finally, we mention the following application of Theorem 1.5 in view of recent interests in related counting problems [7].

Corollary 1.12. Let $\text{Vol}(M) < \infty$, and let $\mathcal{N}(T)$ denote the number of properly immersed totally geodesic planes $P$ in $M$ of area at most $T$. Then for any $1/2 < s < 1$, we have

$$\mathcal{N}(T) \ll_s T^{(6/s) - 1} \quad \text{for all } T > 1$$

where the implied constant depends only on $s$. 
We remark that when $\text{Vol}(M) < \infty$, the heuristics suggest $s = \dim G/H = 3$ in Theorem 1.5 and hence $N(T) \ll T$ in Corollary 1.12. Indeed, when $\Gamma = \text{PSL}_2(\mathbb{Z}[i])$, the asymptotic $N(T) \sim c \cdot T$, as suggested in [24], has been obtained by Jung [13] based on subtle number theoretic arguments.

**Discussion on proofs.** We discuss some of the main ingredients of the proof of Theorem 1.5. First consider the case when $X = \Gamma \backslash G$ is compact (the account below deviates slightly from Margulis’ original argument). Let $\varepsilon_X$ be the minimum injectivity radius of points in $X$. The Lie algebra of $G$ decomposes as $\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$. Hence, for each $y \in Y$, the set

$$I_Z(y) := \{v \in i\mathfrak{sl}_2(\mathbb{R}) : 0 < \|v\| < \varepsilon_X, \ y \exp(v) \in Z\}$$

keeps track of all points of $Z \cap B(y, \varepsilon_X)$ in the direction transversal to $H$ (see Fig. 2).

Therefore, the following function $f_s : Y \to [2, \infty)$ ($0 < s < 1$) encodes the information on the distance $d(y, Z)$:

$$f_s(y) = \begin{cases} \sum_{v \in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \neq \emptyset, \\ \varepsilon_X^{-s} & \text{otherwise} \end{cases}.$$

A function of this type is referred to as a *Margulis function* in literature.

The proof of Theorem 1.1 is based on the following fact: the average of $f_s$ is controlled by the volume of $Z$, i.e.,

$$m_Y(f_s) \ll_s \text{Vol}(Z).$$

We prove the estimate in (1.14) using the following super-harmonicity type inequality: for any $1/3 \leq s < 1$, there exist $t = t_s > 0$ and $b = b_s > 1$ such that for all $y \in Y$,

$$A_t f_s(y) \leq \frac{1}{2} f_s(y) + b \text{Vol}(Z).$$
where \((A_t f_s)(y) = \int_0^1 f_s(yu_r a_t dr, u_r = (1 \ 0), and a_t = \left( e^{t/2} \ 0 \ 0 \ e^{-t/2} \right) \).

The proof of (1.15) is based on the inequality (11.1), which is essentially a lemma in linear algebra. We refer to the Appendix (section 11), where a complete proof of Theorem 1.1 is given.

For a general geometrically finite hyperbolic manifold, many changes are required, and several technical difficulties arise. In general, there is no positive lower bound for the injectivity radius on \(X\), and the shadow constant of \(Y\) appears in the linear algebra lemma (Lemma 5.6). These facts force us to incorporate the height of \(y\) as well as the shadow constant of \(Y\) in the definition of the Margulis function (see Def. 9.1). The correct substitutes for the volume measures on \(Y\) and \(Z\) turn out to be the Bowen-Margulis-Sullivan probability measure \(m_Y\) and the tight area of \(Z\) respectively.

It is more common in the existing literature on the subject to define the operator \(A_t\) using averages over large spheres in \(H^2\). Our operator \(A_t\) however is defined using averages over expanding horocyclic pieces; this choice is more amenable to the change of variables and iteration arguments for Patterson-Sullivan measures. Indeed, for a locally bounded Borel function \(f\) on \(Y \cap X_0\) and for any \(y \in Y \cap X_0\),

\[
(A_t f)(y) = \frac{1}{\mu_y([-1, 1])} \int_{-1}^1 f(yu_r a_t) d\mu_y(r)
\]

where \(\mu_y\) is the Patterson-Sullivan measure on \(yU\) (see (4.2)).

When \(X\) is compact and hence \(m_Y\) is \(H\)-invariant, (1.14) follows by simply integrating (1.15) with respect to \(m_Y\). In general, we resort to Lemma 7.3 the proof of which is based on an iterated version of (1.15) for \(A_{nt_0}, n \in \mathbb{N}\), for some \(t_0 > 0\) as well as on the fact that the Bowen-Margulis-Sullivan measure \(m_Y\) is \(a_{t_0}\)-ergodic.

In fact, the main technical result of this paper can be summarized as follows:

**Proposition 1.16.** Let \(\Gamma\) be a geometrically finite group of \(G\). Let \(Y \neq Z\) be non-elementary closed \(H\)-orbits in \(X = \Gamma \setminus G\), and set \(Y_0 := Y \cap X_0\). For any \(\frac{s_Y}{3} \leq s < s_Y\), there exist \(t_s > 0\) and a locally bounded Borel function \(F_s : Y_0 \to (0, \infty)\) with the following properties:

1. For all \(y \in Y_0\),

\[
d(y, Z)^{-s} \leq s_Y^s F_s(y).
\]

2. For all \(y \in Y_0\) and \(n \geq 1\),

\[
(A_{nt_s} F_s)(y) \leq \frac{1}{2^n} F_s(y) + \alpha_{Y,s}^t Z.
\]

3. There exists \(1 < \sigma \ll s_Y^s\) such that for all \(y \in Y_0\) and for all \(h \in H\) with \(\|h\| \geq 2\) and \(yh \in Y_0\),

\[
\sigma^{-1} F_s(y) \leq F_s(yh) \leq \sigma F_s(y).
\]
Finally we mention that the reason that we can take the exponent \( s \) arbitrarily close to \( \delta_Y \) lies in the two ingredients of our proof: firstly, the linear algebra lemma (Lemma 5.6) is obtained for all \( \delta_Y/3 \leq s < \delta_Y \) and secondly, for any \( y \in Y \cap X_0 \), we can find \( |r| < 1 \) so that \( yur \in X_0 \) and the height of \( yur \) can be lowered to be \( O(1) \) by the geodesic flow of time comparable to the logarithmic height of \( y \); see Lemma 8.4 for the precise statement.

**Organization.** We end this introduction with an outline of the paper. In §2, we fix some notation and conventions to be used throughout the paper. In §3, we show the finiteness of the tight area of a properly immersed geodesic plane. In §4, we show the finiteness of the shadow constant of a closed \( H \)-orbit. In §5, we prove a lemma from linear algebra; this lemma is a key ingredient to prove a local version of our main inequality. §6 is devoted to the study of the height function in \( X_0 \). In §7, the definition of the Markov operator and a basic property of this operator are discussed. In §8, we prove the return lemma, and use it to obtain a uniform control on the number of sheets of \( Z \) in a neighborhood of \( y \). In §9, we construct the desired Margulis function and prove the main inequalities. In §10, we give a proof of Theorem 1.5. In the Appendix (§11), we provide a proof of Theorem 1.1.

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### 2. Notation and preliminaries

In this section, we review some definitions and introduce notation which will be used throughout the paper.

We set \( G = \text{PSL}_2(\mathbb{C}) \simeq \text{Isom}^+(\mathbb{H}^3) \), and \( H = \text{PSL}_2(\mathbb{R}) \). We fix \( \mathbb{H}^2 \subset \mathbb{H}^3 \) so that \( \{ g \in G : g(\mathbb{H}^2) = \mathbb{H}^2 \} = H \). Let \( A \) denote the following one-parameter subgroup of \( G \):

\[
A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

Set \( K_0 = \text{PSU}(2) \) and \( M_0 \) the centralizer of \( A \) in \( K_0 \). We fix a point \( o \in \mathbb{H}^2 \subset \mathbb{H}^3 \) and a unit tangent vector \( v_o \in T_o(\mathbb{H}^3) \) so that their stabilizer subgroups are \( K_0 \) and \( M_0 \) respectively. The isometric action of \( G \) on \( \mathbb{H}^3 \) induces identifications \( G/K_0 = \mathbb{H}^3 \), \( G/M_0 = T^1 \mathbb{H}^3 \), and \( G = F \mathbb{H}^3 \) where \( T^1 \mathbb{H}^3 \) and \( F \mathbb{H}^3 \) denote, respectively, the unit tangent bundle and the oriented frame bundle over \( \mathbb{H}^3 \). Note also that \( H \cap K_0 = \text{PSO}(2) \) and that \( H(o) = \mathbb{H}^2 \).

The right translation action of \( A \) on \( G \) induces the geodesic/frame flow on \( T^1 \mathbb{H}^3 \) and \( F \mathbb{H}^3 \), respectively. Let \( v_o^\pm \in \partial \mathbb{H}^3 \) denote the forward and backward end points of the geodesic given by \( v_o \). For \( g \in G \), we define

\[
g^\pm := g(v_o^\pm) \in \partial \mathbb{H}^3.
\]
Let $\Gamma < G$ be a discrete torsion-free subgroup. We set
\[ M := \Gamma \setminus \mathbb{H}^3 \quad \text{and} \quad X := \Gamma \setminus G \simeq FM. \]
We denote by $\pi : X \rightarrow M$ the base point projection map. Denote by $\Lambda = \Lambda(\Gamma)$ the limit set of $\Gamma$. The convex core of $M$ is given by core $M = \Gamma \setminus \text{hull}\Lambda$. Let $X_0$ denote the renormalized frame bundle $RFM$, i.e.,
\[
(2.1) \quad X_0 = \{ [g] \in X : g^\pm \in \Lambda \},
\]
that is, $X_0$ is the union of $A$-orbits whose projections to $M$ stay inside core $M$. We remark that $X_0$ does not surject onto core $M$ in general.

In the whole paper, we assume that $\Gamma$ is geometrically finite, that is, the unit neighborhood of core $M$ has finite volume. This is equivalent to the condition that $\Lambda$ is the union of the radial limit points and bounded parabolic limit points: $\Lambda = \Lambda_{rad} \cup \Lambda_{bp}$ (cf. [5], [16]). A point $\xi \in \Lambda$ is called \textit{radial} if a geodesic ray toward to $\xi$ accumulates on $M$, \textit{parabolic} if it is fixed by a parabolic element of $\Gamma$, and \textit{bounded parabolic} if it is parabolic and $\text{Stab}_\Gamma(\xi)$ acts co-compactly on $\Lambda - \{ \xi \}$. In particular, for $\Gamma$ geometrically finite, the set of parabolic limit points $\Lambda_p$ is equal to $\Lambda_{bp}$. For $\xi \in \Lambda_p$, the rank of the free abelian subgroup $\text{Stab}_\Gamma(\xi)$ is referred to as the rank of $\xi$.

A geometrically finite group $\Gamma$ is called \textit{convex cocompact} if core $M$ is compact, or equivalently, if $\Lambda = \Lambda_{rad}$.

We denote by $N$ the expanding horospherical subgroup of $G$ for the action of $A$:
\[ N = \left\{ u_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{C} \right\}. \]
For $\xi \in \Lambda_p$, a horoball $\tilde{h}_\xi \subset G$ based at $\xi$ is of the form
\[
(2.2) \quad \tilde{h}_\xi(T) = g^N A_{(-\infty, -T]} K_0 \quad \text{for some} \ T \geq 1
\]
where $g \in G$ is such that $g^- = \xi$ and $A_{(-\infty, -T]} = \{ a_t : -\infty < t \leq -T \}$. Its image $\tilde{h}_\xi(o)$ in $\mathbb{H}^3$ is called a horoball in $\mathbb{H}^3$ based at $\xi$. By a horoball $h_\xi$ in $X$ and in $M$, we mean their respective images of horoballs $\tilde{h}_\xi$ and $\tilde{h}_\xi(o)$ in $X$ and $M$ under the corresponding projection maps.

**Thick-thin decomposition of $X_0$.** We fix a Riemannian metric $d$ on $G$ which induces the hyperbolic metric on $\mathbb{H}^3$. By abuse of notation, we use $d$ to denote the distance function on $X$ induced by $d$, as well as on $M$. For a subset $S \subset \spadesuit$ and $\varepsilon > 0$, $B_\spadesuit(S, \varepsilon)$ denotes the set $\{ x \in \spadesuit : d(x, S) \leq \varepsilon \}$. When $\spadesuit$ is a subgroup of $G$ and $S = \{ \epsilon \}$, we simply write $B_\spadesuit(\epsilon)$ instead of $B_\spadesuit(S, \epsilon)$. When there is no room for confusion for the ambient space $\spadesuit$, we omit the subscript $\spadesuit$.

For $p \in M$, we denote by $\text{inj} p$ the injectivity radius at $p \in M$, that is: the supremum $r > 0$ such that the projection map $\mathbb{H}^3 \rightarrow M = \tilde{\Gamma} \setminus \mathbb{H}^3$ is injective on the ball $B_{\mathbb{H}^3}(\tilde{p}, r)$ where $\tilde{p} \in \mathbb{H}^3$ is such that $p = [\tilde{p}]$. For $S \subset M$ and $\varepsilon > 0$, we call the subsets $\{ p \in S : \text{inj}(p) \geq \varepsilon \}$ and $\{ p \in S : \text{inj}(p) < \varepsilon \}$ the $\varepsilon$-thick part and the $\varepsilon$-thin part of $S$ respectively.
As $M$ is geometrically finite, core $M$ is contained in a union of its $\varepsilon$-thick part (core $M)_{\varepsilon}$ and finitely many disjoint horoballs for all small $\varepsilon > 0$ (cf. [16]). If $p = [g]u_{s}a_{-t}o$ is contained in a horoball $h_{\xi} = gNA(\infty, -T)(o)$, then $\text{inj}(p) \asymp e^{-t}$ for all $t \gg T$.

Let $\varepsilon_{M} > 0$ be the supremum of $\varepsilon$ with respect to which such a decomposition of core $M$ holds. We call the $\varepsilon_{M}$-thick part of core $M$ the compact core of $M$, and denote by $M_{\text{cpt}}$.

For $x = [g]u_{s}a_{-t}o \in X$, we denote by $\text{inj}(x)$ the injectivity radius of $\pi(x) \in M$. For $\varepsilon > 0$, we set

$$X_{\varepsilon} := \{x \in X_{0} : \text{inj}(x) \geq \varepsilon\}.$$

We set $\varepsilon_{X} = \varepsilon_{M}$; note that $X_{0} - X_{\varepsilon_{X}}$ is either empty or is contained in a union of horoballs in $X$.

**Convention.** By an absolute constant, we mean a constant which depends at most on $G$ and $\Gamma$. We will use the notation $A \asymp B$ when the ratio between the two lies in $[C^{-1}, C]$ for some absolute constant $C \geq 1$. We write $A \ll B$ (resp. $A \asymp B$, $A \ll C B$) to mean that $A \leq C B^{L}$ (resp. $C^{-1}B^{L} \leq A \leq C B^{L}$, $A \leq C : B$) for some absolute constants $C > 0$ and $L > 0$; we remark that when $\Gamma$ is a lattice, the exponent $L$ depends only on $G$.

3. **Tight area of a properly immersed geodesic plane**

In this section, we show that the tight area of a properly immersed geodesic plane of $M$ is finite.

For a closed subset $Q \subset M$, we define the **tight neighborhood** of $Q$ by

$$N(Q) := \{p \in M : d(p, q) \leq \text{inj}(q) \text{ for some } q \in Q\}.$$

We are mainly interested in the tight neighborhood of core $M$. If $M$ is convex cocompact, $N(\text{core } M)$ is compact. In order to describe the shape of $N(\text{core } M)$ in the presence of cusps, fix a set $\xi_{1}, \cdots, \xi_{\ell}$ of $\Gamma$-representatives of $\Lambda_{p}$. Then core $M$ is contained in the union of $M_{\text{cpt}}$ and a disjoint union $\bigcup h_{\xi_{i}}$ of horoballs based at $\xi_{i}$’s.

Consider the upper half-space model $\mathbb{H}^{3} = \{(x_{1}, x_{2}, y) : y > 0\} = \mathbb{R}^{2} \times \mathbb{R}_{>0}$, and let $\infty \in \Lambda_{p}$. Let $p : \mathbb{H}^{3} \rightarrow M$ denote the canonical projection map. As $\infty$ is a bounded parabolic fixed point, there exists a rectangle, say, $I \subset \mathbb{R}^{2}$ and $r > 0$ (depending on $\infty$) such that

1. $p(I \times \{y > r\}) \supset N(h_{\infty} \cap \text{core } M)$ and
2. $p(I \times \{r\}) \subset B(M_{\text{cpt}}, R)$

where $R$ depends only on $M$. We call this set $\mathcal{C}_{\infty} := I \times \{y \geq r\}$ a chimney for $\infty$ (cf. Fig 3).

Note that

$$N(\text{core } M) \subset B(M_{\text{cpt}}, R) \cup \left( \bigcup_{1 \leq i \leq \ell} p(\mathcal{C}_{\xi_{i}}) \right)$$

where $\mathcal{C}_{\xi_{i}}$ is a chimney for $\xi_{i}$.
Definition 3.2. For a properly immersed geodesic plane $S$ of $M$, we define the tight-area of $S$ relative to $M$ as follows:

$$\text{area}_t(S) := \text{area}(S \cap N(\text{core } M)).$$

Theorem 3.3. For a properly immersed non-elementary geodesic plane $S$ of $M$, we have

$$1 \ll \text{area}_t(S) < \infty$$

where the implied multiplicative constant depends only on $M$.

Proof. Since no horoball can contain a complete geodesic, it follows that $S$ intersects the compact core $M_{\text{cpt}}$. Therefore,

$$\text{area}_t S \geq \varepsilon_X^2,$$

as $\varepsilon_X$ is the minimum of injectivity radius of points in $M_{\text{cpt}}$ (see section 2). This implies the lower bound.

We now turn to the proof of the upper bound. We use the notation in (3.1). Fix a geodesic plane $P \subset \mathbb{H}^3$ which covers $S$ and let $\Delta = \text{Stab}_\Gamma(P)$. Fix a Dirichlet domain $D$ in $P$ for the action of $\Delta$. As $\Delta \backslash P$ is geometrically finite, $D \cap \text{hull}(\Delta)$ has finite area, and the set $D - \text{hull}(\Delta)$ is a disjoint union of finitely many flares. Fixing a flare $F \subset D - \text{hull}(\Delta)$, it suffices to show that $\{x \in F : p(x) \in N(\text{core } M)\}$ has finite area. As $S$ is properly immersed, the set $\{x \in F : d(p(x), M_{\text{cpt}}) \leq R\}$ is bounded. Therefore, fixing a chimney $\mathcal{C}_{\xi_i}$ as above, it suffices to show that the set $\{x \in F : p(x) \in \mathcal{C}_{\xi_i}\} = F \cap \Gamma \mathcal{C}_{\xi_i}$ has finite area.

Without loss of generality, we may assume $\xi_i = \infty$. We will denote by $\partial F$ the intersection of the closure of $F$ and $\partial P$, and let $F_\varepsilon \subset F$ denote the $\varepsilon$-neighborhood of $\partial F$ in the Euclidean metric in the unit disc model of $P$ (cf. Fig. 4).

Fix $\varepsilon_0 > 0$ so that

$$F_{\varepsilon_0} \cap \{x \in D : d(p(x), M_{\text{cpt}}) < R\} = \emptyset;$$

such $\varepsilon_0$ exists, as $S$ is a proper immersion. Writing $\mathcal{C}_{\infty} = I \times \{y \geq r\}$ as above, let $H_{\infty} := \mathbb{R}^2 \times \{y > r\}$, and set $\Gamma_{\infty} := \text{Stab}_\Gamma(\infty)$. 

**Figure 3. Chimney**
We claim that
\[(3.5) \quad \#\{\gamma H_\infty : F_{\varepsilon_0/2} \cap \gamma \mathcal{C}_\infty \neq \emptyset\} < \infty.\]

Suppose not. Since \(\Gamma H_\infty\) is closed in the space of all horoballs in \(\mathbb{H}^3\), there exists a sequence of distinct \(\gamma_i(\infty) \in \Gamma(\infty)\) such that \(F_{\varepsilon_0/2} \cap \gamma_i \mathcal{C}_\infty \neq \emptyset\) and the size of the horoballs \(\gamma_i H_\infty\) goes to 0 in the Euclidean metric in the ball model of \(\mathbb{H}^3\). Note that if \(\infty\) has rank 2, then \(\Gamma_\infty(I \times \{r\}) = \mathbb{R}^2 \times \{r\}\) and that if \(\infty\) has rank 1, then \(\Gamma_\infty(I \times \{r\})\) contains a region between two parallel horocycles in \(\mathbb{R}^2\). Since \(\bar{\gamma}_i \mathcal{C}_\infty\) is not a limit point for \(\Delta\), it follows that \(\bar{\gamma}_i \mathcal{C}_\infty\) is contained in the \(R\)-neighborhood of \(M_{\text{opt}}\), proving the claim.

By Claim 3.5, it is now enough to show that, fixing a horoball \(\gamma H_\infty\), the intersection \(F_{\varepsilon_0} \cap \gamma \mathcal{C}_\infty\) has finite area. Suppose that \(F_{\varepsilon_0} \cap \gamma \mathcal{C}_\infty\) is unbounded in \(P\); otherwise the claim is clear. Without loss of generality, we may assume \(\gamma = e\), by replacing \(P\) by \(\gamma^{-1}P\) if necessary. If \(\infty \notin \partial P\), then \(F_{\varepsilon_0} \cap \mathcal{C}_\infty\), being contained in \(P \cap H_\infty\), is a bounded subset of \(P\); contradiction. Therefore, \(\infty \in \partial P\). Then, as \(F_{\varepsilon_0} \cap \mathcal{C}_\infty\subset F_{\varepsilon_0} \cap H_\infty\) is unbounded, we have \(\infty \in \partial F\). Since \(F\) is a flare, it follows that \(\infty\) is not a limit point for \(\Delta\). This implies that the rank of \(\infty\) in \(\Lambda_p\) is 1 [20, Lem. 6.2]. Therefore \(\mathcal{C}_\infty\) is contained in a subset of the form \(T \times \{y \geq r\}\) where \(T\) is a strip between two parallel lines \(L_1, L_2\) in \(\mathbb{R}^2\). Since \(\infty\) is not a limit point for \(\Delta\), the vertical plane \(P\) is not parallel to \(L_i\). Therefore the intersection \(F_{\varepsilon_0} \cap \mathcal{C}_\infty\), being a subset of \(P \cap (T \times \{y \geq r\})\), is contained in a cusp-like region, isometric to \(\{(x, y) \in \mathbb{H}^2 : y \geq r\}\). This finishes the proof. \(\square\)

The proof of the above theorem demonstrates that the portion of \(S\), especially of the flares of \(S\), staying in the tight neighborhood of core \(M\) can go to infinity only in cusp-like shapes, by visiting the chimneys of horoballs.

**Figure 4. Flare \(F\) and \(F_\varepsilon\)**


of core $M$ (Fig. 1). This is not true any more if we replace the tight neighborhood of core $M$ by the unit neighborhood of core $M$. More precisely if $\Lambda$ contains a parabolic limit point of rank one which is not stabilized by any element of $\pi_1(S)$, then some region of $S$ with infinite area can stay inside the unit neighborhood of core $M$. This situation may be compared to the presence of divergent geodesics in finite area setting.

4. Shadow constants

In this section, fixing a closed non-elementary $H$-orbit $Y$ in $X$, we recall the definition of Patterson-Sullivan measures $\mu_y$ on horocycles in $Y$, and relate its density with the shadow constant $s_Y$, which we show is a finite number.

Set $\Delta_Y := \text{Stab}_H(y_0)$ to be the stabilizer of a point $y_0 \in Y$; note that despite the notation, $\Delta_Y$ is uniquely determined up to a conjugation by an element of $H$. As $\Gamma$ is geometrically finite, the subgroup $\Delta_Y$ is a geometrically finite subgroup of $H$. We denote by $\Lambda_Y \subset \partial \mathbb{H}^2$ the limit set of $\Delta_Y$. Let $0 < \delta(Y) \leq 1$ denote the critical exponent of $\Delta_Y$, or equivalently, the Hausdorff dimension of $\Lambda_Y$.

We denote by $\{\nu_p = \nu_{Y,p} : p \in \mathbb{H}^2\}$ the Patterson-Sullivan density for $\Delta_Y$, normalized so that $|\nu_0| = 1$. This means that $\nu_p$'s are Borel measures on $\Lambda_Y$ satisfying that for all $\gamma \in \Delta_Y$, $p, q \in \mathbb{H}^2$, $\xi \in \Lambda_Y$,

$$\frac{d\gamma \nu_p}{d\nu_p}(\xi) = e^{-\delta(Y)\beta_\xi(\gamma^{-1}(p), p)} \quad \text{and} \quad \frac{d\nu_q}{d\nu_p}(\xi) = e^{-\delta(Y)\beta_\xi(q,p)}$$

where $\beta_\xi(\cdot, \cdot)$ denotes the Busemann function. As $\Delta_Y$ is geometrically finite, there exists a unique Patterson-Sullivan density up to a constant multiple.

**PS-measures on $U$-orbits.** Set

$$U := \left\{ u_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R} \right\} = \mathbb{R} \cap H$$

which is the expanding horocylic subgroup of $H$. Using the parametrization $s \mapsto u_s$, we may identify $U$ with $\mathbb{R}$. Note that for all $s, t \in \mathbb{R}$,

$$a_{-t}u_s a_t = u_{e^t s}.$$

For any $h \in H$, the restriction of the visual map $g \mapsto g^+$ is a diffeomorphism between $hU$ and $\partial \mathbb{H}^2 - \{h^-\}$. Using this diffeomorphism, we can define a measure $\mu_{hU}$ on $hU$:

$$d\mu_{hU}(hu_r) = e^{\delta(Y)\beta_{(hu_r)+}(p, hu_r(p))} d\nu_p(hu_r)^+;$$

this is independent of the choice of $p \in \mathbb{H}^2$. We simply write $d\mu_h(r)$ for $d\mu_{hU}(hu_r)$. Note that these measures depend on the $U$-orbits but not on the individual points. By the $\Delta_Y$-invariance and the conformal property of the PS-density, we have

$$d\mu_h(\mathcal{O}) = d\mu_{\gamma h}(\mathcal{O})$$
for any $\gamma \in \Delta_Y$ and for any bounded Borel set $O \subset \mathbb{R}$; therefore $\mu_y(O)$ is well-defined for $y \in \Delta_Y \setminus H$.

For any $y \in \Delta_Y \setminus H$ and any $s \in \mathbb{R}$, we have:

$$\mu_y([-e^s, e^s]) = e^{\delta(Y)s} \mu_{ya_{-s}}([-1, 1]).$$

Set

$$Y_0 := \{ [h] \in \Delta_Y \setminus H : h^\pm \in \Lambda_Y \}$$

where $h^\pm = \lim_{t \to \pm \infty} ha_t(o)$.

**Shadow constant.** As in the introduction, we define the modified critical exponent of $Y$:

$$\delta_Y = \begin{cases} 
\delta(Y) & \text{if } Y \text{ is convex cocompact} \\
2\delta(Y) - 1 & \text{otherwise}
\end{cases}$$

If $Y$ has a cusp, then $\delta(Y) > 1/2$, and hence $0 < \delta_Y \leq \delta(Y) \leq 1$.

We define

$$1 \leq p_Y := \sup_{y \in Y_0, 0 \leq \varepsilon \leq 2} \frac{\mu_y([-\varepsilon, \varepsilon])^{1/\delta_Y}}{\varepsilon \cdot \mu_y([-1, 1])^{1/\delta_Y}}.$$  

Recall the shadow constant $s_Y = \sup_{0 < \varepsilon \leq 1/2} s_Y(\varepsilon)$ in (1.8) where

$$s_Y(\varepsilon)^{\delta_Y} := \sup_{\xi \in \Lambda_Y, p \in [\Lambda_Y, \xi]} \frac{\nu_p(B_p(\xi, \varepsilon))}{\varepsilon^{\delta_Y} \nu_p(B_p(\xi, 1/2))}.$$  

The main goal of this section is devoted to the proof of the following theorem using a uniform version of Sullivan’s shadow lemma.

**Theorem 4.8.** We have

$$s_Y \asymp p_Y < \infty.$$  

In principle, this definition of $s_Y$ involves making a choice of $\Delta_Y = \text{Stab}_H(y_0)$, i.e., the choice of $y_0 \in Y$, as $\Lambda_Y$ is the limit set of $\Delta_Y$. However we observe the following:

**Lemma 4.9.** The constant $s_Y$ is independent of the choice of $y_0 \in Y$.

**Proof.** Let $y = y_0 h^{-1} \in Y$ for $h \in H$. Define $s_Y'$ similar to $s_Y$ using $\Delta'_Y = \text{Stab}_H(y) = h\Delta_Y h^{-1}$. If $\xi \in \Lambda_Y$, then

$$\frac{d((h\gamma h^{-1})_s p^\prime)}{d\nu_p^\prime}(h\xi) = \frac{d((h\gamma)_s \nu_{h^{-1}p})}{dh_s \nu_{h^{-1}p}}(h\xi) = \frac{d\gamma_s \nu_{h^{-1}p}}{dh \nu_{h^{-1}p}}(\xi).$$

Since the limit set of $\Delta'_Y$ is given by $h\Lambda_Y$, this implies that the family $\{p^\prime = h_s \nu_{h^{-1}p} : p \in \mathbb{H}^2\}$ is the Patterson-Sullivan density for $\Delta'_Y$. Now for any $0 < \varepsilon \leq 1$ and $\xi \in \Lambda_Y$, we have

$$\nu_{hp}(B_{hp}(h\xi, \varepsilon)) = h_s \nu_p(B_{hp}(h\xi, \varepsilon)) = \nu_p(h^{-1} B_{hp}(h\xi, \varepsilon)) = \nu_p(B_p(h\xi, \varepsilon)).$$

It follows that $s_Y = s'_Y$. \qed
Shadow lemma. Consider the associated hyperbolic plane and its convex core:

\[ S_Y := \Delta_Y \setminus \mathbb{H}^2 \quad \text{and} \quad \text{core}(S_Y) := \Delta_Y \setminus \text{hull}(\Lambda_Y). \]

We denote by \( C_Y \) the compact core of \( S_Y \), defined as the minimal connected surface whose complement in \( \text{core}(S_Y) \) is a union of disjoint cusps. If \( S_Y \) is convex cocompact, then \( C_Y = S_Y \). Let

\[ d_Y := \text{diam}(C_Y). \]

We can write \( \text{core}(S_Y) \) as the disjoint union of the compact core \( C_0 := C_Y \) and finitely many cusps, say, \( C_1, \ldots, C_m \). Fix a Dirichlet domain \( \mathcal{F}_Y \subset \mathbb{H}^2 \) for \( \Delta_Y \) containing the base point \( o \). For each \( C_i \), \( 0 \leq i \leq m \), choose the lift \( \tilde{C}_i \subset \mathcal{F}_Y \cap \text{hull}(\Lambda_Y) \) so that \( \Delta_Y \setminus \Delta_Y C_i = C_i \). In particular, \( \partial \tilde{C}_0 \) intersects \( \tilde{C}_i \) as an interval for \( i \geq 1 \). Let \( \xi_i \in \Lambda_Y \) be the base point of the horodisc \( \tilde{C}_i \), i.e., \( \xi_i = \partial \tilde{C}_i \cap \partial \mathbb{H}^2 \). Let \( F_{\xi_i} \subset \partial \mathbb{H}^2 - \{ \xi_i \} \) be a minimal interval so that \( \Lambda_Y - \{ \xi_i \} \subset \text{Stab}_{\Delta_Y}(\xi_i)F_{\xi_i} \).

For \( p \in \mathbb{H}^2 \), let \( d_p \) denote the Gromov distance on \( \partial \mathbb{H}^2 \): for \( \xi \neq \eta \in \partial \mathbb{H}^2 \),

\[ d_p(\xi, \eta) = e^{-\beta_2(p,q)+\beta_0(p,q))/2} \]

where \( q \) is any point on the geodesic connecting \( \xi \) and \( \eta \). The diameter of \((\partial \mathbb{H}^2, d_p)\) is equal to 1.

For any \( h \in H \), we have \( d_p(\xi, \eta) = d_{h(p)}(h(\xi), h(\eta)) \). For \( \xi \in \partial \mathbb{H}^2 \), and \( r > 0 \), set

\[ B_p(\xi, r) = \{ \eta \in \partial \mathbb{H}^2 : d_p(\eta, \xi) \leq r \}. \]

Also, denote by \( V(p, \xi, r) \) the set of all \( \eta \in \partial \mathbb{H}^2 \) such that the distance between \( p \) and the orthogonal projection of \( \eta \) onto the geodesic \( [p, \xi] \) is at least \( r \). There exists some \( b > 0 \) so that

\[ V(p, \xi, r + b) \subset B_p(\xi, e^{-r}) \subset V(p, \xi, r - b) \quad \text{for all} \quad r \geq b. \]

The following is a uniform version of Sullivan’s shadow lemma [28]. The proof of this proposition can be found in [25, Thm. 3.4]; since the dependence on the multiplicative constant is important to us, we give a sketch of the proof while making the dependence of constants explicit.

Proposition 4.11. There exists a constant \( c \asymp e^{-d_Y} \) such that for all \( \xi \in \Lambda_Y \), \( p \in \tilde{C}_0 \), and \( t > 0 \),

\[ c^{-1} \cdot \nu_p(F_{\xi_i}) \beta_Y e^{-\delta(Y)t + (1-\delta(Y))d(\xi_i, \Delta_Y(p))} \leq \nu_p(V(p, \xi, t)) \]

\[ \leq c \cdot \nu_p(F_{\xi_i}) e^{-\delta(Y)t + (1-\delta(Y))d(\xi_i, \Delta_Y(p))} \]

where

- \( \{ \xi_i \} \) is the unit speed geodesic ray \([p, \xi]\) so that \( d(p, \xi_i) = t \);
- \( F_{\xi_i} = \partial \mathbb{H}^2 \) if \( \xi_i \in \tilde{C}_0 \), and \( F_{\xi_i} \) if \( \xi_i \in \tilde{C}_i \) for \( 1 \leq i \leq m \);
- \( \beta_Y := \inf_{\eta \in \Lambda_Y, q \in \tilde{C}_0} \nu_q(B_q(\eta, e^{-d_Y})) \).
Proof. Let \( p, \xi \in \Lambda_Y \) and \( \xi_t \) be as in the statement. By the \( \delta(Y) \)-conformality of the PS density, we have

\[
\nu_p(V(p, \xi, t)) = e^{-\delta(Y)t} \nu_{\xi_t}(V(p, \xi, t)).
\]

Therefore it suffices to show

\[
\nu_{\xi_t}(V(p, \xi, t)) \asymp \nu_p(F_{\xi_t}) \cdot e^{(1-\delta(Y))d(\xi_t, \Delta_Y(p))}
\]

while making the dependence of the implied constant explicit.

**Claim A.** If \( \xi_t \in \Delta_Y \mathcal{C}_0 \), then

\[
e^{-\delta(Y)d_Y} \cdot \inf_{\eta \in \Lambda_Y} \nu_p(B(\eta, e^{-d_Y})) \ll \nu_{\xi_t}(V(p, \xi, t)) \ll e^{\delta(Y)d_Y} |\nu_p|
\]

where the implied constants are absolute.

As \( \xi_t \in \Delta_Y \mathcal{C}_0 \), there exists \( \gamma \in \Delta_Y \) such that \( d(\xi_t, \gamma p) \leq d_Y \). Hence

\[
e^{-\delta(Y)d_Y} \nu_{\xi_t}(V(p, \xi, t)) \leq \nu_{\gamma p}(V(p, \xi, t)) = \nu_p(V(\gamma^{-1}p, \gamma^{-1}\xi, t)) \leq e^{\delta(Y)d_Y} \nu_{\xi_t}(V(p, \xi, t)).
\]

The upper bound in (4.12) follows from the first inequality, while the lower bound follows from the second inequality; indeed

\[
V(\gamma^{-1}p, \gamma^{-1}\xi, t) = V(\gamma^{-1}\xi_t, \gamma^{-1}\xi, 0)
\]

and the latter contains \( B_p(\gamma^{-1}\xi, e^{-d_Y}) \), since \( d(p, \gamma^{-1}\xi_t) \leq d_Y \).

**Claim B.** Let \( \xi \) be a parabolic limit point in \( \Lambda_Y \). Assume that for some \( i \geq 1 \), \( \xi_t \in \mathcal{C}_i \) for all large \( t \).

We claim:

\[
\nu_{\xi_t}(V(p, \xi, t)) \asymp e^{\delta(Y)d_Y} \cdot \nu_p(F_{\xi_t}) \cdot e^{(1-\delta(Y))(d(\xi_t, \Delta_Y(p)) + d_Y)}
\]

and

\[
\nu_{\xi_t}(\partial \mathbb{H}^2 - V(p, \xi, t)) \asymp e^{\delta(Y)d_Y} \cdot \nu_p(F_{\xi_t}) \cdot e^{(1-\delta(Y))(d(\xi_t, \Delta_Y(p)) + d_Y)}.
\]

Let \( s_t \geq 0 \) be such that \( \xi_{s_t} \in \partial \mathcal{C}_i \). Then for all \( t \geq s_t \),

\[
|d(\xi_t, \Delta_Y(p)) - (t - s_t)| \leq d_Y.
\]

Hence for (4.13), it suffices to show

\[
\nu_{\xi_t}(V(p, \xi, t)) \asymp e^{(1-\delta(Y))(t-s_t)} \nu_p(F_{\xi_t}).
\]

Note that if we set \( \Gamma_\xi = \text{Stab}_{\Delta_Y}(\xi) \),

\[
\nu_{\xi_t}(V(p, \xi, t)) \asymp \sum_{\gamma \in \Gamma_\xi, \gamma F_{\xi_t} \cap V(p, t, t) \neq \emptyset} \nu_{\xi_t}(\gamma F_{\xi_t}).
\]

If \( \gamma F_{\xi_t} \cap V(p, \xi_t, t) \neq \emptyset \), then \( d(p, \gamma p) \geq 2t - k \) for some absolute \( k \), and hence

\[
\nu_{\xi_t}(V(p, \xi, t)) \asymp \sum_{\gamma \in \Gamma_\xi, d(p, \gamma p) \geq 2t} \nu_{\xi_t}(\gamma F_{\xi_t}).
\]
Let $F_\xi^*$ denote the image of $F_\xi$ on the horocycle based at $\xi$ passing through $p$ via the visual map. We use the fact that if $d(p, \gamma p) \geq 2t$, then for all $\eta \in F_\xi$,

$$|\beta_\eta(\gamma^{-1} \xi_\eta, \xi_\eta) - d(p, \gamma p) + 2t| \ll \text{diam} F_\xi^* \leq d_Y$$

(cf. proof of [25, Lemma 2.9]). Since $\nu_\xi(\gamma F_\xi) = \int_{F_\xi} e^{-\delta(Y)} \beta_\eta(\xi_\eta, \xi_\eta) d\nu_\xi(\eta)$, $\nu_\xi(F_\xi) \asymp e^{-\delta(Y)t} \nu_p(F_\xi)$, we deduce, with multiplicative constant $\asymp e^{\delta(Y)d_Y}$,

$$\sum_{\gamma \in \Gamma_\xi, d(p, \gamma p) \geq 2t} \nu_\xi(\gamma F_\xi) \asymp \sum_{\gamma \in \Gamma_\xi, d(p, \gamma p) \geq 2t} e^{2\delta(Y)t - \delta(Y)d(p, \gamma p)} \nu_\xi(F_\xi)$$

$$\times \nu_p(F_\xi) e^{\delta(Y)t} \sum_{\gamma \in \Gamma_\xi, d(p, \gamma p) \geq 2t} e^{-\delta(Y)d(p, \gamma p)}$$

$$\times \nu_p(F_\xi) e^{(1-\delta(Y))t}$$

using $a_n := \#\{\gamma \in \Gamma_\xi : n < d(p, \gamma p) \leq n + 1\} \asymp n^{\pm 2}$ in the last estimate. This proves (4.13).

The estimate (4.15) follows similarly now using

$$\nu_\xi(\partial \mathbb{H}^2 - V(p, \xi, t)) \asymp \sum_{\gamma \in \Gamma_\xi, d(p, \gamma p) \leq 2t} \nu_\xi(\gamma F)$$

and $\sum_{n=0}^{2t} a_n e^{-\delta(Y)n} \asymp e^{(1-2\delta(Y))t}$.

The next two remaining cases are deduced from Claims A and B.

**Claim C.** When $\xi$ be a parabolic limit point, (4.13) holds with multiplicative constant $\asymp e^{\delta d_Y}$ (see the proof of [25, Prop. 3.6]).

**Claim D.** If $\xi_i \in \Delta_Y \bar{C}_i$ for some $i$, then (4.13) holds with multiplicative constant $\asymp e^{\delta d_Y}$ (see the proof of [25, Lem. 3.8]).

**Proposition 4.16.** Fix $p = p_Y \in \bar{C}_0$. There exists $R_Y \asymp e^{\delta d_Y}$ such that for all $y \in Y_0$, we have

$$R_Y^{-1} \beta_Y e^{(1-\delta(Y))d(C_Y, \pi(y))} |\mu| \leq \mu_y([-1, 1]) \leq R_Y e^{(1-\delta(Y))d(C_Y, \pi(y))} |\mu|$$

where $\pi$ denotes the base point projection $\Delta_Y \setminus H = T^1(S_Y) \to S_Y$.

**Proof.** The following argument is a slight modification of the proof of [26, Prop. 5.1]. Since the map $y \mapsto \mu_y([-1, 1])$ is continuous on $Y_0$ and $\{[h] \in Y_0 : h^- \text{ is a radial limit point of } \Lambda_Y\}$ is dense in $Y_0$, it suffices to prove the claim for $y = [h]$, assuming that $h^-$ is a radial limit point for $\Delta_Y$.

Recall that $\mu_y([-1, 1]) = e^{\delta(Y)t} \mu_y([-e^{-t}, e^{-t}])$ for all $t \in \mathbb{R}$. Let $t \geq 0$ be the minimal number so that $\pi(ya_{-t}) \in C_Y$; this exists as $h^-$ is a radial limit point. Then

$$d(\pi(y), C_Y) \leq d(\pi(y), \pi(ya_{-t})) \leq d_Y + d(\pi(y), C_Y).$$

Set $\xi_t = ha_{-t}(o)$. Then

$$\mu_{ha_{-t}[-e^{-t}, e^{-t}]} \asymp \nu_\xi(V(\xi_t, h^+, t))$$
(cf. [25, Lemma 4.4]).

Since \( ya_{-\ell} \in C_Y \), \( F_{\xi_t} = \partial \mathbb{H}^2 \). So \( \nu_{\xi_t}(F_{\xi_t}) = |\nu_{\xi_t}| \sim |\nu_p| \) up to a multiplicative constant \( e^{*d_Y} \). Therefore, for some implied constant \( \asymp e^{*d_Y} \), we have

\[
\beta_Y e^{-\delta(Y)t + (1 - \delta(Y))d(\pi(y), \pi(ya_{-\ell}))} |\nu_p| \ll \nu_{\xi_t}(V(\xi_t, h^+, t)) \ll e^{-\delta(Y)t + (1 - \delta(Y))d(\pi(y), \pi(ya_{-\ell}))} |\nu_p|.
\]

This estimate and (4.17), therefore, imply that

\[
\beta_Y e^{(1 - \delta(Y))d(\pi(y), C_Y)} |\nu_p| \ll \mu_y([-1, 1]) \ll e^{(1 - \delta(Y))d(\pi(y), C_Y)} |\nu_p|
\]

with the implied constant \( \asymp e^{*d_Y} \), proving the claim. \( \Box \)

We use the following result, essentially obtained by Schapira-Maucourant ([28], [26]):

**Corollary 4.18.** Fix \( \rho > 0 \). Then for all \( 0 < \varepsilon \leq \rho \),

\[
R_Y^{-2} \cdot \beta_Y \leq \sup_{y \in Y_0} \frac{\mu_y([-\varepsilon, \varepsilon])}{\varepsilon^{\delta_Y} \mu_y([-1, 1])} \leq \max\{1, \rho^2\} \cdot R_Y^{-2} \cdot \beta_Y^{-1} < \infty,
\]

where \( R_Y \) is as in Proposition 4.16.

**Proof.** By (4.3), we have \( \mu_y([-\varepsilon, \varepsilon]) = \varepsilon^{\delta(Y)} \mu_{ya_{-\log \varepsilon}}([-1, 1]) \). Hence the case when \( Y \) is convex cocompact follows from Proposition 4.16.

Now suppose that \( Y \) has a cusp. Let \( y \in Y_0 \). Using the triangle inequality, we get that \( d(\pi(ya_{-\log \varepsilon}), C_Y) - d(\pi(y), C_Y) \leq |\log \varepsilon| \). Therefore, by Proposition 4.16, we have

\[
\frac{\mu_{ya_{-\log \varepsilon}}([-1, 1])}{\mu_y([-1, 1])} \leq R_Y^2 \beta_Y^{-1} \cdot e^{(1 - \delta(Y))d(\pi(ya_{-\log \varepsilon}), C_Y) - d(\pi(y), C_Y)} \\
\leq \begin{cases} 
R_Y^2 \beta_Y^{-1} \cdot \varepsilon^{\delta(Y) - 1} & \text{if } 0 \leq \varepsilon < 1 \\
R_Y^2 \beta_Y^{-1} \cdot \varepsilon^{1 - \delta(Y)} & \text{if } \varepsilon \geq 1
\end{cases}
\]

In consequence, we have

\[
\frac{\mu_y([-\varepsilon, \varepsilon])}{\varepsilon^{2(1 - \delta(Y)) - 2} \mu_y([-1, 1])} \leq \begin{cases} 
R_Y^2 \beta_Y^{-1} & \text{if } 0 \leq \varepsilon < 1 \\
R_Y^2 \beta_Y^{-1} \cdot \rho^2 & \text{if } \rho \geq 1 \text{ and } 1 \leq \varepsilon \leq \rho
\end{cases}
\]

which establishes the upper bound.

By choosing \( y \in Y_0 \) such that \( d(\pi(ya_{-\log \varepsilon}), C_Y) - d(\pi(y), C_Y) = |\log \varepsilon| \), we get the lower bound. \( \Box \)

Theorem 4.8 follows from the following:

**Proposition 4.19.** We have

1. for any \( 0 < \varepsilon \leq 1/2, 0 < s_Y(\varepsilon) < \infty \).
2. \( s_Y \asymp p_Y \ll e^{*d_Y/\delta_Y} \beta_Y^{-1} \).
Proof. Let \( h \in H \) and set \( y = |h| \). Fix \( 0 < r \leq 2 \). Recall
\[
\mu_y([-r, r]) = \int e^{-\delta(Y)\beta_{h,r}^+(h(o),hu_r(o))} d\nu_{h(o)}(hu_r^+).
\]
Since \( |\beta_{h,r}^+(h(o),hu_r(o))| \leq d(o, u_r(o)) \), we
\[
e^{-\delta(Y)\beta_{h,r}^+(h(o),hu_r(o))} \ll 1
\]
with the implied constant independent of all \( 0 < r \leq 2 \).
Since \( d_0(u^+_r, e^+) = d_{h(o)}((hu_r)^+, h^+) \), we have
\[
\nu_{h(o)}(B_{h(o)}(h^+, \frac{c}{\sqrt{1+r^2}})) \ll \mu_y([-r, r]) \ll \nu_{h(o)}(B_{h(o)}(h^+, \frac{cr}{\sqrt{1+r^2}}))
\]
for some \( c > 1 \) independent of \( r \) and \( h \).
This implies that
\[
\mu_y([-\varepsilon/c', \varepsilon/c']) \ll \nu_{h(o)}(B_{h(o)}(h^+, \varepsilon)) \ll \mu_y([-c'\varepsilon, c'\varepsilon])
\]
as well as
\[
\frac{\mu_y([-\varepsilon/c', \varepsilon/c'])}{\varepsilon^\delta \mu_y([-c'/2, c'/2])} \ll \frac{\nu_{h(o)}(B_{h(o)}(h^+, \varepsilon))}{\varepsilon^\delta \nu_{h(o)}(B_{h(o)}(h^+, 1/2))} \ll \frac{\mu_y([-c'\varepsilon, c'\varepsilon])}{\varepsilon^\delta \mu_y([-1/(2c'), 1/(2c')])}
\]
where \( c' > 1 \) is independent of \( 0 < \varepsilon < 1/2 \) and \( h \in H \). The claims thus follow from Corollary 4.18.

5. Linear algebra lemma

The goal of this section is to prove the linear algebra lemma (Lemma 5.6) and its slight variant (Lemma 5.13).

In this section, it is more convenient to identify \( G \) as SO(Q)\(^o\) for the quadratic form
\[
Q(x_1, x_2, x_3, x_4) = 2x_1x_4 + x_2^2 + x_3^2.
\]
As \( Q \) has signature \((3, 1)\), PSL\(_2(\mathbb{C}) \simeq SO(Q)\(^o\)\) as real Lie groups. We consider the standard representation of \( G \) on the space \( \mathbb{R}^4 \) of row vectors and denote the Euclidean norm on \( \mathbb{R}^4 \) by \( \| \cdot \| \). We have
\[
H = \text{Stab}_G(e_3) \simeq SO(2, 1)\(^o\) \quad \text{and}
\]
\[
A = \{ a_t = \text{diag}(e^t, 1, 1, e^{-t}) : t \in \mathbb{R} \} < H.
\]
Set
\[
V := \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_4.
\]
Then the restriction of the standard representation of \( G \) to \( H \) induces a representation of \( H \) on \( V \), which is isomorphic to the adjoint representation of \( H \) on its Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \); in particular, it is irreducible.

Note that for each \( t > 0 \), \( \mathbb{R}e_2 = \{ v \in V : \text{va}_t = v \} \), \( \mathbb{R}e_1 \) is the subspace of all vectors with eigenvalues \( > 1 \), and \( \mathbb{R}e_4 \) is the subspace of all vectors with eigenvalues \( < 1 \).
Let $p : V \to \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $p^+ : V \to \mathbb{R}e_1$ denote the natural projections. Writing $v = v_1e_1 + v_2e_2 + v_4e_4$, a direct computation yields that for any $r \in \mathbb{R}$,

\begin{equation}
(5.1)\quad p(vu_r) = (v_1 + v_2r + \frac{v_4^2}{2})e_1 + (v_2 + v_4r)e_2
\end{equation}

and

\begin{equation}
(5.2)\quad p^+(vu_r) = (v_1 + v_2r + \frac{v_4^2}{2})e_1.
\end{equation}

For a unit vector $v \in V$ and $\varepsilon > 0$, define

\begin{align*}
D(v, \varepsilon) &= \{ r \in [-1, 1] : \|p(vu_r)\| \leq \varepsilon \}; \\
D^+(v, \varepsilon) &= \{ r \in [-1, 1] : \|p^+(vu_r)\| \leq \varepsilon \}.
\end{align*}

Lemma 5.2. For all $0 < \varepsilon < 1/2$ and a unit vector $v \in V$, we have

\begin{equation}
\ell(D(v, \varepsilon)) \ll \varepsilon \quad \text{and} \quad \ell(D^+(v, \varepsilon)) \ll \varepsilon^{1/2}
\end{equation}

where $\ell$ denotes the Lebesgue measure on $\mathbb{R}$.

Proof. Writing $v = v_1e_1 + v_2e_2 + v_4e_4$, we have

\begin{equation}
\ell(D(v, \varepsilon)) \leq \ell\{ r \in [-1, 1] : |v_1 + v_2r + \frac{v_4^2}{2}| \leq \varepsilon \text{ and } |v_2 + v_4r| \leq \varepsilon \}.
\end{equation}

If $|v_4| \geq 0.01$, then

\begin{equation}
\ell(D(v, \varepsilon)) \leq \ell\{ r \in [-1, 1] : |v_2 + v_4r| \leq \varepsilon \} \leq 200\varepsilon.
\end{equation}

If $|v_4| < 0.01$ but $0.1 \leq |v_2| \leq 1$, then for $r \in [-1, 1]$, we have $|v_2 + v_4r| \geq 0.09$, and hence for all $\varepsilon < 1/2$,

\begin{equation}
\ell(D(v, \varepsilon)) \leq \ell\{ r \in [-1, 1] : |v_2 + v_4r| \leq \varepsilon \} = 0.
\end{equation}

Now consider the case when $|v_4| \leq 0.01$ and $|v_2| \leq 0.1$. Then, since $\|v\| = 1$, we get that $|v_1| \geq 0.7$. Hence for all $r \in [-1, 1]$, $|v_1 + v_2r + v_4r^2/2| > 0.5$ In consequence, for all $\varepsilon < 1/2$,

\begin{equation}
\ell(D(v, \varepsilon)) \leq \ell\{ r \in [-1, 1] : |v_1 + v_2r + v_4r^2/2| \leq \varepsilon \} = 0,
\end{equation}

proving the estimate on $D(v, \varepsilon)$. To estimate $D^+(v, \varepsilon)$, observe that $p^+(vu_r) = (v_1 + v_2r + \frac{v_4^2}{2})e_1$ is a polynomial map of degree at most 2. Moreover, since $\|v\| = 1$, we have

\begin{equation}
\max\{ |v_1|, |v_2|, |v_4| \} \gg 1.
\end{equation}

Therefore, $\sup_{r \in [-1, 1]} \|p^+(vu_r)\| \gg 1$. The claim about $D^+(v, \varepsilon)$ now follows using Lagrange’s interpolation, see [6] for a more general statement. \qed

For the rest of this section, we fix a closed non-elementary $H$-orbit $Y$.

Lemma 5.3. There exists an absolute constant $b_0 > 0$ for which the following holds: for any $y \in Y_0$ and $0 < \varepsilon < 1$, we have

\begin{equation}
(5.4)\quad \sup_{v \in V, \|v\|=1} \mu_y(D(v, \varepsilon)) \leq b_0 \|p_Y\|^{\delta_Y} \varepsilon^{\delta_Y} \mu_y([-1, 1]),
\end{equation}

and

\begin{equation}
(5.5)\quad \sup_{v \in V, \|v\|=1} \mu_y(D^+(v, \varepsilon)) \leq b_0 \|p_Y\|^{\delta_Y} \varepsilon^{\delta_Y/2} \mu_y([-1, 1]).
\end{equation}
where \( p_Y \) is given as in (4.6).

**Proof.** By (5.1), the set \( D(v, \varepsilon) \) (resp. \( D^+(v, \varepsilon) \)) consists of at most 3 (resp. 2) intervals. By Lemma 5.2, \( D(v, \varepsilon) \) (resp. \( D^+(v, \varepsilon) \)) may be covered by \( \ll 1 \) many intervals of length \( \varepsilon \) (resp. \( \varepsilon^{1/2} \)). Therefore (5.4) (resp. (5.5)) follows from the definition of \( p_Y \). \( \square \)

We use Lemma 5.3 to prove the following lemma which will be crucial in the sequel.

**Lemma 5.6** (Linear algebra lemma). For any \( \delta_Y \leq s < \delta_Y, 1 \leq \rho \leq 2, \) and \( t > 0 \), we have

\[
(5.7) \quad \sup_{y \in Y_0, v \in V, \|v\|=1} \frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \leq b_0 \frac{p_Y \delta_Y e^{-(\delta_Y-s)t/4}}{(\delta_Y-s)}
\]

where \( b_0 \geq 2 \) is an absolute constant.

**Proof.** Since \( \mu_y[-\rho, \rho] = p^{\gamma}(\mu_{ya_{-\log \rho}}[-1,1]) \) and \( Y_0 \) is \( A \)-invariant, it suffices to prove the claim for \( \rho = 1 \). Fix \( 0 < s < \delta_Y \) and \( t > 0 \). We observe that for all \( r \in \mathbb{R} \),

\[
(5.8) \quad \|vu_r a_t\| \geq \|p(vu_r)\| \quad \text{and} \quad \|vu_r a_t\| \geq e^t \|p^+(vu_r)\|.
\]

For simplicity, set \( \beta_y := \frac{1}{\mu_y([-1,1])} \). The inequality (5.4) and the first estimate in (5.8) imply that for any \( 0 < \varepsilon \leq 1 \) and any unit vector \( v \in V \), we have

\[
\beta_y \int_{r \in D(v, \varepsilon) - D(v, \varepsilon/2)} \|vu_r a_t\|^{-s} d\mu_y(r) \leq b_0 p_Y \delta_Y e^{\delta_Y} \cdot (\varepsilon/2)^{-s} \leq 2b_0 p_Y \delta_Y e^{\delta_Y-s}.
\]

We write \( D(v, \varepsilon) = \bigcup_{k=0}^{\infty} D(v, \varepsilon/2^k) - D(v, \varepsilon/2^{k+1}) \). Now applying the above estimate for each \( \varepsilon/2^k \) and summing up the geometric series, we get that for any \( 0 < \varepsilon < 1 \),

\[
(5.9) \quad \beta_y \int_{r \in D(v, \varepsilon)} \|vu_r a_t\|^{-s} d\mu_y(r) \leq \frac{2b_0 p_Y \delta_Y e^{\delta_Y-s}}{1 - 2s^{\delta_Y}}.
\]

Moreover, using (5.5) and the first estimate in (5.8) again, for any \( \kappa > 0 \), we have

\[
(5.10) \quad \beta_y \int_{r \in D^+(v, \kappa) - D(v, \varepsilon)} \|vu_r a_t\|^{-s} d\mu_y(r) \leq 2b_0 p_Y \delta_Y \kappa^{\delta_Y/2} e^{-s}.
\]

Finally, the definition of \( D^+(v, \kappa) \) and the second estimate in (5.8) imply

\[
(5.11) \quad \beta_y \int_{r \in [-1,1] - D^+(v, \kappa)} \|vu_r a_t\|^{-s} d\mu_y(r) \leq \kappa^{-s} e^{-st}.
\]
Combining (5.9), (5.10), and (5.11) and using the inequality \[ \frac{1}{1-2^{-\delta_Y-s}} \leq \frac{2}{\delta_Y-s}, \] we deduce that for any \( 0 < \varepsilon, \kappa < 1, \)

\[ \beta_y \int_{-1}^{1} \|v u_t a_t\|^{-s} d\mu_y(r) \leq \frac{2b_0 y^{\delta_Y}}{\delta_Y-s} \left( e^{\delta_Y-s} + \kappa^{\delta_Y/2} e^{-\kappa} + \kappa^{-s} e^{-st} \right). \]

Let \( \varepsilon = e^{-t/4} \) and \( \kappa = \varepsilon^{2} \). As \( \delta_Y/3 \leq s < \delta_Y \), we have \( e^{-s/2} \leq e^{(s-\delta_Y)/4} \). This yields:

\[ \beta_y \int_{-1}^{1} \|v u_t a_t\|^{-s} d\mu_y(r) \leq \frac{6b_0 y^{\delta_Y}}{\delta_Y-s} e^{-(\delta_Y-s)t/4}, \]

as we claimed. \( \square \)

We will extend the upper bound in Lemma 5.6 to all unit vectors \( v \in e_1 G \), based on the fact that the vectors in \( e_1 G \) are projectively away from the \( H \)-invariant point corresponding to \( \mathbb{R} e_3 \).

**Lemma 5.12.** There exists an absolute constant \( b_1 > 1 \) such that for any vector \( v \in e_1 G \subset \mathbb{R}^4 \),

\[ \|v\| \leq b_1 \|v_1\| \]

where \( v_1 \) is the projection of \( v \in \mathbb{R}^4 \) to \( V = \mathbb{R} e_1 \oplus \mathbb{R} e_2 \oplus \mathbb{R} e_4 \).

**Proof.** Since \( Q(e_1) = 0 \) and \( G = \text{SO}(Q)^{\circ} \), we have \( Q(e_1 g) = 0 \) for every \( g \in G \). Since \( Q(e_3) = 1 \), the set \( \{\|v\|^{-1} v : v \in e_1 G\} \) is a compact subset of the unit sphere in \( \mathbb{R}^4 \) not containing \( e_3 \). Therefore there exists an absolute constant \( 0 < \eta < 1 \) such that if we write \( v = v_1 + re_3 \in e_1 G \), then \( |r| \leq \eta \|v\| \). Therefore \( \|v_1\|^2 = \|v\|^2 - r^2 \geq (1 - \eta^2)\|v\|^2 \). Hence it suffices to set \( b_1 = (1 - \eta^2)^{-1/2} \). \( \square \)

**Lemma 5.13** (Linear algebra lemma II). For any \( \frac{\delta_Y}{3} \leq s < \delta_Y, 1 \leq \rho \leq 2, \) and \( t > 0 \), we have

\[ \sup_{y \in Y_0, v \in e_1 G, \|v\|=1} \frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \frac{1}{\|v u_t a_t\|^{-s}} d\mu_y(r) \leq b_0 b_1 \frac{y^{\delta_Y} e^{-(\delta_Y-s)t/4}}{\delta_Y-s} \]

where \( b_0 \geq 2 \) and \( b_1 > 1 \) are absolute constants as in Lemmas 5.6 and 5.12 respectively.

**Proof.** Let \( v \in e_1 G \) be a unit vector, and write \( v = v_0 + v_1 \) where \( v_0 \in \mathbb{R} e_3 \) and \( v_1 \in V \). Since \( e_3 \) is \( H \)-invariant, we have \( vh = v_0 + v_1 h \in \mathbb{R} e_3 \oplus V \) for
all \( h \in H \). Therefore,

\[
\frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \frac{1}{\|v_1 u_v a_t\|^s} \, d\mu_y(r) \leq \frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \frac{1}{\|v_1 u_v a_t\|^s} \, d\mu_y(r)
\]

\[
\leq \frac{b_0 p_Y \delta_Y e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \|v_1\|^{-s} \quad \text{by Lemma 5.6}
\]

\[
\leq \frac{b_0 b_1 p_Y \delta_Y e^{-(\delta_Y - s)t/4}}{(\delta_Y - s)} \|v\|^{-s} \quad \text{by Lemma 5.12.}
\]

\[
\square
\]

6. Height function \( \omega \)

In this section we define the height function \( \omega : X_0 \to (0, \infty) \) and show that \( \omega(x) \) is comparable to the reciprocal of the injectivity radius at \( x \).

For this purpose, we continue to realize \( G \) as \( \SO(Q)^0 \) acting on \( \mathbb{R}^4 \) by the standard representation, as in Section 5. Observe that \( \Q(e_1) = 0 \) and the stabilizer of \( e_1 \) in \( G \) is equal to \( M_0 N \).

Fixing a set of \( \Gamma \)-representatives \( \xi_1, \ldots, \xi_\ell \) in \( \Lambda_{bp} \), choose elements \( g_i \in G \) so that \( g_i = \xi_i \) and \( \|e_1 g_i^{-1}\| = 1 \), and set

\[
(6.1) \quad v_i := e_1 g_i^{-1} \in e_1 G.
\]

Note that

\[
\text{Stab}_G(\xi_i) = g_i AM_0 N g_i^{-1} \quad \text{and} \quad \text{Stab}_G(v_i) = g_i M_0 N g_i^{-1}.
\]

By Witt’s theorem, we have that for each \( i \),

\[
\{v \in \mathbb{R}^4 - \{0\} : Q(v) = 0 \} = v_i G \simeq g_i M_0 N g_i^{-1} \backslash \mathbb{R}^4.
\]

**Lemma 6.2.** For each \( 1 \leq i \leq \ell \), the orbit \( v_i \Gamma \) is a closed (and hence discrete) subset of \( \mathbb{R}^4 \).

**Proof.** The condition \( \xi_i \in \Lambda_{bp} \) implies that \( \Gamma \backslash \Gamma g_i M_0 N \) is a closed subset of \( X \). Equivalently, \( \Gamma g_i M_0 N \) as well as \( \Gamma g_i M_0 N g_i^{-1} \) is closed in \( G \). Therefore, its inverse \( g_i M_0 N g_i^{-1} \Gamma \) is a closed subset of \( G \). In consequence, \( v_i \Gamma \subset \mathbb{R}^4 \) is a closed subset of \( v_i G = \{v \in \mathbb{R}^4 - \{0\} : Q(v) = 0 \} \).

It remains to show that \( v_i G \) does not accumulate on 0. Suppose on the contrary that there exists an infinite sequence \( v_i \gamma_\ell \) converging to 0 for some \( \gamma_\ell \in \Gamma \). Using the Iwasawa decomposition \( G = g_i N A K_0 \), we may write \( \gamma_\ell = g_i n_\ell a_\ell k_\ell \) with \( n_\ell \in N, t_\ell \in \mathbb{R} \) and \( k_\ell \in K_0 \). Since

\[
v_i \gamma_\ell = e^{t_\ell}(e_1 k_\ell),
\]

the assumption that \( v_i \gamma_\ell \to 0 \) implies that \( t_\ell \to -\infty \).

On the other hand, as \( \xi_i \in \Lambda_{bp} \), \( \text{Stab}_\Gamma(\xi_i) = \Gamma \cap g_i AM_0 N g_i^{-1} \) contains a parabolic element, say, \( \gamma' \neq e \). Note that \( n_0 := g_i^{-1} \gamma' g_i \) is then an element
of $N$, as any parabolic element of $AM_0 N$ belongs to $N$ in the group $G \simeq \text{PSL}_2(\mathbb{C})$. Now observe that, as $N$ is abelian,
\[
\gamma_\ell^{-1} \gamma_j \gamma_\ell = k_\ell^{-1} a_{-t_\ell} (n_\ell^{-1} g_\ell^{-1} \gamma_j' g_\ell n_\ell) a_{t_\ell} k_\ell = k_\ell^{-1} (a_{-t_\ell} n_0 a_{t_\ell}) k_\ell.
\]

Since $t_\ell \to -\infty$, the sequence $a_{-t_\ell} n_0 a_{t_\ell}$ converges to $e$. Since $\{k_\ell^{-1}\}$ is a bounded sequence, it follows that, up to passing to a subsequence, $\gamma_\ell^{-1} \gamma_j \gamma_\ell$ is an infinite sequence converging to $e$, contradicting the discreteness of $\Gamma$. □

**Definition 6.3** (Height function). Define the height function $\omega : X_0 \to [2, \infty)$ by

\[
\omega(x) := \max_{1 \leq i \leq \ell} \omega_i(x)
\]

where

\[
\omega_i(x) := \max_{\gamma \in \Gamma} \left\{ 2, \| v_i \gamma g \|^{-1} \right\} \quad \text{for any } g \in G \text{ with } x = [g];
\]

this is well-defined by Lemma 6.2.

By the definition of $\varepsilon_X$, $X_0$ is contained in the union of $X_{\varepsilon_X} \cup \bigcup_{j=1}^\ell \tilde{h}_j$ where $\tilde{h}_j$ is a horoball based at $\xi_j$. Fix $T_j > 0$ so that $\tilde{h}_j = [g_j]N\mathcal{A}_{(-\infty,-T_j]} K_0$.

Set $\tilde{h}_j := g_j N\mathcal{A}_{(-\infty,-T_j]} K_0$.

The following is an immediate consequence of the think-thin decomposition of $M$:

**Lemma 6.4.** If $\tilde{h}_j \cap \gamma \tilde{h}_i \neq \emptyset$ for some $1 \leq i, j \leq \ell$ and $\gamma \in \Gamma$, then $i = j, \gamma \in \text{Stab}_G(\xi_i) = \text{Stab} \tilde{h}_i$, and hence $\tilde{h}_j = \gamma \tilde{h}_i$.

**Lemma 6.5.** For all $1 \leq i, j \leq \ell$ and $\gamma \in \Gamma$ such that $\tilde{h}_j \neq \gamma \tilde{h}_i$,

\[
\inf_{g \in \tilde{h}_i} \| v_j \gamma g \| \geq \eta_0
\]

where $\eta_0 := \min_{1 \leq m \leq \ell} e^{-T_m}$.

**Proof.** Let $g \in \tilde{h}_i$ and $\gamma \in \Gamma$. Using $G = g_j N\mathcal{A} K_0$, write $\gamma g = g_j u a_s k \in g_j N\mathcal{A} K_0$. Then $\| v_j \gamma g \| = e^s$. Hence if $\| v_j \gamma g \| < \eta_0$, then $s \leq -T_j$. So $\gamma g \in \tilde{h}_j$. Therefore $\tilde{h}_j \cap \gamma \tilde{h}_i \neq \emptyset$. By Lemma 6.4, $\tilde{h}_j = \gamma \tilde{h}_i$. □

**Proposition 6.7.** There is an absolute constant $\alpha \geq 2$ such that for all $x \in X_0$,

\[
\frac{1}{2\alpha} \cdot \text{inj}(x) \leq \omega(x)^{-1} \leq \frac{\alpha}{2} \cdot \text{inj}(x).
\]

**Proof.** Fixing $1 \leq j \leq \ell$, it suffices to show the claim for all $x \in X_0 \cap \tilde{h}_j$.

Let $g \in g_i u a_{-t} k \in \tilde{h}_i$ be so that $x = [g]$, where $u a_{-t} k \in N\mathcal{A}_{(-\infty,-T_j]} K_0$.

Note that

\[
\omega_i(x)^{-1} = \| v_i g \| = \| e_1 g_\ell^{-1} (g_i u a_{-t}) k \| = \| e_1 u a_{-t} k \| = e^{-t}.
\]

In view of the definition of $\omega$ and $\omega_i$, this together with Lemma 6.5 implies that

\[
\omega(x) = \omega_i(x) = e^t.
\]

Since $\text{inj}(x) \asymp e^{-t}$, this finishes proof. □
In this section we define a Markov operator $A_t$ and prove Proposition 7.5 which relates the average $m_Y(F)$ of a locally bounded, log-continuous, Borel function $F$ on $Y_0$ with a super-harmonic type inequality for $A_t F$. This proposition will serve as a main tool in our approach to prove Theorem 1.5.

Fix a closed non-elementary $H$-orbit $Y$ in $X$.

**Bowen-Margulis-Sullivan measure $m_Y$.** We denote by $m_Y$ the Bowen-Margulis-Sullivan probability measure on $\Delta_Y \setminus H = T^1(S_Y)$, which is the unique probability measure of maximal entropy (that is $\delta(Y)$) for the geodesic flow. We will also use the same notation $m_Y$ to denote the push-forward of the measure to $Y$ via the map $\text{Stab}_H(y_0) \setminus H \to Y$ given by $[h] \to y_0 h$. Considered as a measure on $Y$, $m_Y$ is well-defined, independent of the choice of $y_0 \in Y$.

Recall the definition of $Y_0$ in (4.4); note that $Y_0 = \text{supp} m_Y$. In the following, all of our Borel functions are assumed to be defined everywhere in their domains. By a locally bounded function, we mean a function which is bounded on every compact subset.

**Definition 7.1 (Markov Operator).** Let $t \in \mathbb{R}$ and $\rho > 0$. For a locally bounded Borel function $\psi : Y_0 \to \mathbb{R}$, we define

$$ (A_{t,\rho} \psi)(y) := \frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \psi(yu_ta_t) d\mu_y(r). $$

We set $A_t := A_{t,1}$.

Note that $A_{t,\rho} \psi$ is a locally bounded Borel function on $Y_0$. Although $\lim_{n \to \infty} A_{nt} \psi = m_Y(\psi)$ for any $\psi \in C_c(Y_0)$ and any $t > 0$ [20], the Margulis function $F$ we will be constructing is not a continuous function on $Y_0$, and hence we cannot use such an equidistribution statement to control $m_Y(F)$. We will use the following lemma instead:

**Lemma 7.3.** Let $F : Y_0 \to [2, \infty)$ be a locally bounded Borel function. Assume that there exist some $t > 0$ and $D > 0$ such that

$$ \limsup_{n \to \infty} A_{nt} F(y) \leq D \quad \text{for all } y \in Y_0. $$

Then

$$ m_Y(F) \leq 8D. $$

**Proof.** For every $k \geq 2$, let $F_k : Y_0 \to [2, \infty)$ be given by

$$ F_k(y) := \min\{F(y), k\}. $$

As $F_k$ is bounded, it belongs to $L^1(Y_0, m_Y)$. Since the action of $A$ is mixing for $m_Y$ by the work of Babillot [1], we have $m_Y$ is $a_t$-ergodic for each $t \neq 0$. Hence, by the Birkhoff ergodic theorem, for $m_Y$-a.e. $y \in Y_0$, we have

$$ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_k(ya_nt) = \int F_k \, dm_Y. $$
Therefore, using Egorov’s theorem, for every $\varepsilon > 0$, there exist $N_\varepsilon > 1$ and a measurable subset $Y_\varepsilon' \subset Y_0$ with $m_Y(Y_\varepsilon') > 1 - \varepsilon^2$ such that for every $y \in Y_\varepsilon'$ and all $N > N_\varepsilon$, we have

$$\frac{1}{N} \sum_{n=1}^{N} F_k(ya_n) > \frac{1}{2} \int F_k \, dm_Y.$$ 

Now by the maximal ergodic theorem [23, Thm. 17], if $\varepsilon$ is small enough, there exists a measurable subset $Y_\varepsilon \subset Y_\varepsilon'$ with $m_Y(Y_\varepsilon) > 1 - \varepsilon$ so that for all $y \in Y_\varepsilon$, we have

$$\mu_y \{ r \in [-1, 1] : yur \in Y_\varepsilon' \} > \frac{1}{2} \mu_y([-1, 1]).$$

 Altogether, if $y \in Y_\varepsilon$ and $N > N_\varepsilon$, we have

$$\frac{1}{N} \sum_{n=1}^{N} A_{nt}F_k(y) = \frac{1}{\mu([-1, 1])} \int_{-1}^{1} \frac{1}{N} \sum_{n=1}^{N} F_k(yu_r a_n) d\mu_y(r) > \frac{1}{4} \int F_k \, dm_Y.$$ 

Fix $y \in Y_\varepsilon$. By the hypothesis (7.4), there exists $n_0 = n_0(y)$ such that for all $n \geq n_0$, we have

$$A_{nt}F_k(y) \leq A_{nt}F(y) \leq 2D.$$ 

Therefore, we deduce that for all sufficiently large $N \gg 1$,

$$\frac{1}{4} \int F_k \, dm_Y \leq \frac{1}{N} \left( \sum_{n=1}^{n_0} A_{nt}F_k(y) + \sum_{n=n_0+1}^{N} A_{nt}F_k(y) \right) \leq \frac{k_n}{N} + \frac{2D(N-n_0)}{N}. $$

By sending $N \to \infty$, we get that for all $k > 2$,

$$\int F_k \, dm_Y \leq 8D.$$ 

Since $\{F_k : k = 3, 4, \ldots\}$ is an increasing sequence of positive functions converging to $F$ point-wise, the monotone convergence theorem implies

$$\int F \, dm_Y = \lim_{k \to \infty} \int F_k \, dm_Y \leq 8D$$

as we claimed. \qed

We remark that in [11], the Markov operator $A_t$ was defined using the integral over the translates $SO(2)a_t$, whereas we use the integral over the translates $U_{[-\rho, \rho]}a_t$ of a horocyclic piece. The proof of the following proposition, which is an analogue of [11, §5.3], is the main reason for our digression from their definition, as the handling of the PS-measure on $U$ is more manageable than that of the PS-measure on $SO(2)$ in performing change of variables.

**Proposition 7.5.** Let $F : Y_0 \to [2, \infty)$ be a locally bounded Borel function satisfying the following properties:

(a) There exists $\sigma \geq 2$ such that for all $h \in B_H(2)$ and $y \in Y_0$,

$$\sigma^{-1}F(y) \leq F(yh) \leq \sigma F(y).$$
(b) There exist \( t \geq 2 \) and \( D_0 > 0 \) such that for all \( y \in Y_0 \) and \( 1 \leq \rho \leq 2 \),

\[
A_{t, \rho} F(y) \leq \frac{1}{8\sigma p_Y^{\delta_Y}} \cdot F(y) + D_0
\]

where \( p_Y \) is as in (4.6).

Then

\[
m_Y(F) \leq 64D_0p_Y^{\delta_Y}.
\]

In view of Lemma 7.3, Proposition 7.5 is an immediate consequence of the following:

**Proposition 7.6.** Let \( F \) be as in Proposition 7.5. Then for all \( y \in Y_0 \) and \( n \geq 1 \), we have

(7.7)

\[
A_{t, \rho} F(y) \leq \frac{1}{2n} F(y) + 8D_0p_Y^{\delta_Y}.
\]

**Proof.** The main step of the proof is the following estimate.

**Claim:** For any \( 1 \leq \rho \leq 2 \), \( y \in Y_0 \) and \( n \in \mathbb{N} \), we have

(7.8)

\[
A_{(n+1)t, \rho} F(y) \leq \frac{1}{2} A_{nt, \rho + e^{-nt}} F(y) + \hat{D}
\]

where \( \hat{D} := 4D_0p_Y^{\delta_Y} \).

Let us first assume this claim and prove the proposition. We observe

- \( \sum_{j \geq 1} e^{-jt} \leq 1/2 \) (as \( t \geq 2 \)),
- \( (8\sigma p_Y^{\delta_Y})^{-1} \leq 1/2 \), and
- \( D_0 \leq \hat{D} \).

Using the assumption (b) of Proposition 7.5 with \( \rho = 1 + \sum_{j \geq 1} e^{-jt} \), we deduce that for any \( n \geq 2 \),

\[
A_{nt} F(y) \leq \frac{1}{2n} A_{t, 1 + \sum_{j=1}^{\infty} e^{-jt}} F(y) + \hat{D} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \right)
\]

\[
\leq \frac{1}{2n} \left( (8\sigma p_Y^{\delta_Y})^{-1} F(y) + D_0 \right) + \hat{D} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \right)
\]

(7.9)

\[
\leq \frac{1}{2n} F(y) + 2\hat{D}
\]

which establishes the proposition.

We now prove the claim (7.8). For \( y \in Y_0 \) and \( \rho > 0 \), set

\[
b_y(\rho) := \mu_y([-\rho, \rho]) \quad \text{and} \quad b_y = b_y(1).
\]

To ease the notation, we prove (7.8) with \( \rho = 1 \); the proof in general is similar. By assumption (a) and (b) of Proposition 7.5, we have

(7.10)

\[
A_t F(y) \leq c_0 F(y) + D_0 \leq \left( \frac{c_0 \sigma}{b_y} \int_{-1}^{1} F(yu_r) d\mu_y(r) \right) + D_0
\]

where \( c_0 = (8\sigma p_Y^{\delta_Y})^{-1} \).

Set \( \rho_n := e^{-nt} \). Let \( \{ [r_j - \rho_n, r_j + \rho_n] : j \in J \} \) be a covering of \( [-1, 1] \cap \text{supp}(\mu_y) \)
with $r_j \in [-1, 1] \cap \text{supp}(\mu_y)$ and with multiplicity bounded by 2. For each $j \in J$, let $z_j := yu_{r_j}$. Then

$$
\sum_j b_{z_j}(\rho_n) = \sum_j \mu_y(z_ju_{r_j-\rho_n,r_j+\rho_n}) \leq 2b_y(2).
$$

Moreover, we get

$$
A_{n+1}F(y) = \frac{1}{b_y} \int_{-1}^1 F(yu_t a_{n+1}t) d\mu_y(r)
\leq \frac{1}{b_y} \sum_j b_{z_j}(\rho_n) \int_{-\rho_n}^{\rho_n} F(z_ja_{nt}u_t a_t) d\mu_{z_j}(r)
= \frac{1}{b_y} \sum_j b_{z_j}(\rho_n) b_{z_j} a_{nt} \int_{-\rho_n}^{\rho_n} F(z_ja_{nt}u_t a_t) d\mu_{z_j}(r). \tag{7.12}
$$

We now make the change of variables $s = re^{nt}$. In view of (7.12), we have

$$
A_{n+1}F(y) \leq \frac{1}{b_y} \sum_j b_{z_j}(\rho_n) c_0 \sigma \int_{-1}^1 F(z_ja_{nt}u_t a_t) d\mu_{z_j}(s) + \frac{1}{b_y} \sum_j b_{z_j}(\rho_n) D_0. \tag{7.13}
$$

By (7.11), we have \( \frac{1}{b_y} \sum_j b_{z_j}(\rho_n) D_0 \leq \hat{D}. \)

Therefore, reversing the change of variable, i.e., now letting $r = e^{-nt}s$, we get from (7.13) the following:

$$
A_{n+1}F(y) \leq \frac{1}{b_y} \sum_j c_0 \sigma \int_{-\rho_n}^{\rho_n} F(z_j u_t a_{nt})d\mu_{z_j}(r) + \hat{D}
\leq \frac{2c_0 \sigma}{b_y} \int_{-(1+\rho_n)}^{1+\rho_n} F(yu_t a_{nt})d\mu_y(r) + \hat{D}
= \frac{2c_0 \sigma b_y (1 + \rho_n)}{b_y} A_{nt,1+\rho_n} F(y) + \hat{D}.
$$

Since

$$
\sup_{y \in Y_0} \frac{2c_0 \sigma b_y (2)}{b_y} = (4p^\delta Y)^{-1} \sup_{y \in Y_0} \frac{b_y (2)}{b_y} \leq \frac{1}{2},
$$

we get

$$
A_{n+1}F(y) \leq \frac{1}{2} A_{nt,1+\rho_n} F(y) + \hat{D}.
$$

The proof is complete.
8. Return lemma and number of nearby sheets

We fix closed non-elementary $H$-orbits $Y$ and $Z$ in $X$. Since $Z$ is closed, a fixed ball around $y \in Y_0$ intersects only finitely many sheets of $Z$ (Fig. 2). The aim of this section is to show that the number of sheets of $Z$ in $B(y, \text{inj}(y))$ is controlled by the tight area of $S_Z$ with a multiplicative constant depending on $p_Y$ and $\delta_Y$.

The main ingredient is a return lemma which says that for any $y \in Y_0$, there exists some point in $\{yu_r \in Y_0 : r \in [-1, 1]\}$ whose minimum return time to a fixed compact subset under the geodesic flow is comparable to $\log(\omega(y))$ (see Lemma 8.4).

**Return lemma.** We use the notation of section 6.

Since the exponential map $\text{Lie}(G) \rightarrow G$ defines a local diffeomorphism, we have an absolute constant $c_1 \geq 2$ such that for all $x \in X$, and $w \in \text{Lie}(G)$ with $\|w\| \leq \max(1, \varepsilon_X)$,

\[(8.1) \quad c_1^{-1}\|w\| \leq d(x, x \exp(w)) \leq c_1\|w\|\]

where the norm $\|\cdot\|$ on $\text{Lie}(G)$ is defined using an inner product with respect to which $\mathfrak{s\mathfrak{l}}_2(\mathbb{R})$ and $i\mathfrak{s\mathfrak{l}}_2(\mathbb{R})$ are orthogonal to each other.

We choose an absolute constant $d_X \geq 24$ so that

\[X_{\varepsilon_X} \subset \{x \in X_0 : \omega(x) \leq d_X\}.

Let $D_1 := D_1(Y)$ be given by

\[(8.2) \quad D_1 = c_1\alpha\left(\frac{\delta_0}{\eta_0} + d_X\right)\]

where $\kappa$ is defined by $\hat{b}_0p_Y^{\delta_Y}\kappa^{\delta_Y}/2 = 1/2$, $0 < \eta_0 < 1$ is as in (6.6), $\alpha \geq 1$ is as in (6.8), and $c_1$ is as in (8.1). Define

\[(8.3) \quad K_Y = \{y \in Y_0 : \omega(y) \leq D_1/(c_1\alpha)\}.

Note that $X_{\varepsilon_X} \cap Y_0 \subset K_Y$.

The choices of the above parameters are motivated by our applications in the following lemmas. Indeed the choice of $\kappa$ is used in (8.5). The multiplicative parameter $c_1\alpha$, which features in the definitions of $D_1$ and $K_Y$, is tailored so that we may utilize Lemma 8.9 in the proof of Lemma 8.12.

**Lemma 8.4** (Return lemma). For every $y \in Y_0$, there exists some $|r| \leq 1$ so that $yu_r a_{-t} \in K_Y$ where $t = \log(\eta_0\omega(y)/6)$.

**Proof.** Let $y \in Y_0 - K_Y$. There exist $1 \leq j \leq \ell$ and $g \in \tilde{h}_j$ so that $y = [g]$. So $g$ is of the form $g_jua_{-t}k$ where $u \in N$, $t > T_j$ and $k \in K_0$. Set $v := v_jg$. Note that

\[\|v\|^{-1} = e^t = \omega(y).

Let $\kappa > 0$ be as used in (8.2). Then (5.5) implies that

\[(8.5) \quad \mu_g(D^+(\|\frac{v}{\|v\|}, \kappa)) \leq \frac{1}{2}\mu_g([-1, 1]).\]
Then there exists \( r \in \text{supp}(\mu_y) \cap \left( [-1, 1] - D^+(\frac{\nu}{\|v\|}, \kappa) \right) \). This means that \( yu_r \in Y_0 \) and that \( \|p^+(vu_r)\| > \kappa\|v\| \). Set \( t := \log(\eta_0\omega(y)/6) \). Then
\[
\kappa\|v\| \cdot \frac{\eta_0\omega(y)}{6} = \kappa\|v\|e^t \leq \|p^+(vu_r)_{at}\| \leq \|vu_r_{at}\| \leq 2\|v\| \cdot \frac{\eta_0\omega(y)}{6}.
\]
Hence, using the fact that \( \omega(y) = \|v\|^{-1} \),
\[
\frac{\kappa\eta_0}{6} \leq \|vu_r_{at}\| = \|v_j gu_r_{at}\| \leq \frac{\eta_0}{3}.
\]
In view of the above upper bound, Lemma 6.5 now implies that
\[
\omega(yu_r_{at}) = \|v_j gu_r_{at}\|^{-1}.
\]
Therefore,
\[
\omega(yu_r_{at}) \leq \frac{6}{\kappa\eta_0} \leq D_1/(c_1\alpha)
\]
proving the claim. \( \square \)

**Number of nearby sheets.** Recalling that \( \mathfrak{s}\mathfrak{l}_2(\mathbb{C}) = \mathfrak{s}\mathfrak{l}_2(\mathbb{R}) \oplus i\mathfrak{s}\mathfrak{l}_2(\mathbb{R}) \), we set \( V = i\mathfrak{s}\mathfrak{l}_2(\mathbb{R}) \) and consider the action of \( H \) on \( V \) via the adjoint representation; so \( vh = h^{-1}vh \) for \( v \in V \) and \( h \in H \). We use the relation \( g(\exp v)h = gh \exp(\nu h) \) which is valid for all \( g \in G, v \in V, h \in H \).

If \( D \geq \alpha/2 \) for \( \alpha \) as in (6.7), then \( D^{-1}\omega(y)^{-1} \leq \frac{1}{4} \text{inj}(y) \).

**Definition 8.6.** For \( y \in Y_0 \) and \( D \geq \alpha/2 \), we define
\[
I_Z(y,D) = \{v \in V - \{0\} : \|v\| < D^{-1}\omega(y)^{-1}, y \exp(v) \in Z\}.
\]
Since \( V \) is the orthogonal complement to \( \text{Lie}(H) \), the set \( I_Z(y,D) \) can be understood as the number of sheets of \( Z \) in the ball around \( y \) of radius \( D^{-1}\omega(y)^{-1} \).

It turns out that \( \#I_Z(y,D) \) can be controlled in terms of the tight area of \( S_Z \), uniformly over all \( y \in Y_0 \) for an appropriate \( D > 1 \).

**Notation 8.8.** We set
\[
\tau_Z := \text{area}_t(S_Z).
\]

Theorem 3.3 shows that \( 1 \ll \tau_Z < \infty \) where the implied constant depends only on \( M \).

We begin with the following lemma:

**Lemma 8.9.** With \( c_1 \geq 2 \) and \( \alpha \geq 2 \) given respectively in (8.1) and (6.7), we have that for all \( y \in Y_0 \),
\[
\#I_Z(y,c_1\alpha) \ll \omega(y)^3\tau_Z.
\]

**Proof.** Let \( c_1 \geq 1 \) and \( \alpha \) be absolute constants given in (8.1) and (6.7) respectively. It follows that for any \( y \in Y_0 \) and \( v \in I_Z(y,\alpha) \),
\[
d(y, y \exp(v)) \leq c_1\|v\| \leq c_1(c_1\alpha)^{-1} \cdot \omega(y)^{-1} < \frac{1}{2} \cdot \text{inj}(y).
\]
It follows that for each \( v \in I_Z(y, c_1 \alpha) \), \( \text{inj}(y \exp v) \geq \text{inj}(y)/2 \). Hence the balls \( B_Z(y \exp v, \text{inj}(y)/2) \), \( v \in I_Z(y, c_1 \alpha) \) are disjoint from each other, and hence

\[
\#I_Z(y, \alpha) \cdot \text{Vol}(B_H(e, \text{inj}(y)/2)) = \text{Vol}\left( \bigcup B_Z(y \exp v, \text{inj}(y)/2) : v \in I_Z(y, \alpha) \right).
\]

On the other hand, if we set \( \rho_y := \min\{1, \text{inj}(y)/2\} \), then

\[
\pi\left( \bigcup B_Z(y \exp v, \rho_y) : v \in I_Z(y, c_1 \alpha) \right) \subset S_Z \cap N(\text{core}(M)).
\]

Therefore

\[
\#I_Z(y, c_1 \alpha) \leq \text{Vol}(B_H(e, \rho_y))^{-1} \cdot \tau_Z \ll \rho_y^{-3/2} \ll \omega(y)^3 \tau_Z.
\]

\[\square\]

**Lemma 8.12** (Number of sheets). For \( D_1 = D_1(Y) \ll p_Y^2 \) as in (8.2), we have

\[
\sup_{y \in Y_0} \#I_Z(y, D_1) \leq c_0 \rho_Y^6 \cdot \tau_Z
\]

where \( c_0 \geq 2 \) is an absolute constant.

**Proof.** Let \( K_Y \) be as in (8.3):

\[
K_Y = \{ y \in Y_0 : \omega(y) \leq (c_1 \alpha)^{-1} D_1 \}.
\]

If \( y \in K_Y \), then, by Lemma 8.9,

\[
\#I_Z(y, D_1) \leq \#I_Z(y, c_1 \alpha) \ll D_1^3 \tau_Z \ll p_Y^6 \tau_Z.
\]

Now suppose that \( y \in Y_0 - K_Y \). By Lemma 8.4, there exist \( |r| < 1 \) and \( t = \log(\eta_0 \cdot \omega(y)/6) \), where \( 0 < \eta_0 \leq 1 \) is as in (6.6), such that

\[
yu_r a_t \in K_Y.
\]

We claim that if \( v \in I_Z(y, D_1) \), then \( v(u_r a_t) \in I_Z(yu_r a_t, c_1 \alpha) \). Firstly, note that, plugging \( t = \log(\eta_0 \cdot \omega(y)/6) \) and using \( 0 < \eta \leq 1 \),

\[
\|v(u_r a_t)\| \leq 3e^t\|v\| = \frac{3\eta_0 \omega(y)\|v\|}{6} < \omega(y) \cdot \|v\|.
\]

Hence for \( v \in I_Z(y, D_1) \), as \( \omega(y)\|v\| < D_1^{-1} \),

\[
\|v(u_r a_t)\| < \omega(y) \cdot \|v\| \leq D_1^{-1} \leq (c_1 \alpha)^{-1} \omega(yu_r a_t)^{-1}.
\]

where we used the fact that \( (c_1 \alpha)^{-1} D_1 > \omega(yu_r a_t) \).

Since \( y(\exp v)u_r a_t = (yu_r a_t) \exp(v(u_r a_t)) \in Z \), this implies that \( yu_r a_t \in I_Z(c_1 \alpha) \). Therefore the map \( v \mapsto v(u_r a_t) \) is an injective map from \( I_Z(y, D_1) \) into \( I_Z(yu_r a_t, c_1 \alpha) \). Consequently,

\[
\#I_Z(y, D_1) \leq \#I_Z(yu_r a_t, c_1 \alpha) \ll p_Y^6 \cdot \tau_Z.
\]

This finishes the proof. \[\square\]
9. Margulis function: construction and estimate

Throughout this section, we fix closed non-elementary $H$-orbits $Y, Z$ in $X$ and let

$$\frac{\delta_Y}{3} \leq s < \delta_Y.$$ 

In this section, we define a family of Margulis functions $F_{s, \lambda} = F_{s, \lambda, Y, Z}$, $\lambda > 1$ and show that the hypothesis of Proposition 7.5 is satisfied for a certain choice of $\lambda$, which we will denote by $\lambda_s$. As a consequence, we will get an estimate on $m_Y(F_{s, \lambda_s})$ in Theorem 9.18.

We set

$$I_Z(y) := \{v \in V - \{0\} : \|v\| < D_1^{-1}\omega(y)^{-1}, y \exp(v) \in Z\}$$

for $D_1 > 1$ as given in Lemma 8.12.

**Definition 9.1 (Margulis function).**

1. Define $f_s := f_{s, Y, Z} : Y_0 \to (0, \infty)$ by

$$f_s(y) := \begin{cases} 
\sum_{v \in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \neq \emptyset \\
\omega(y)^s & \text{otherwise.}
\end{cases}$$

2. For $\lambda \geq 1$, define $F_{s, \lambda} = F_{s, \lambda, Y, Z} : Y_0 \to (0, \infty)$ as follows:

$$F_{s, \lambda}(y) = f_s(y) + \lambda \omega(y)^s.$$ 

Note that for all $y \in Y_0$

$$\omega(y)^s \leq f_s(y) < \infty.$$ 

Since $Y$ and $Z$ are closed orbits, both $f_s$ and $F_{s, \lambda}$ are locally bounded. Moreover, they are also Borel functions. Indeed, $\omega^s$ is continuous on $Y_0$, and $f_s$ is continuous on the open subset $\{y \in Y_0 : I_Z(y) \neq \emptyset\}$ as well as on its complement.

In this section, we specify choices of parameters $t_s$ and $\lambda_s$ so that the average $A_{t_s} F_{s, \lambda_s}$ satisfies the hypothesis of Proposition 7.5 with controlled size of the additive term (Lemma 9.14).

**Notation 9.4 (Parameters).**

1. For $0 < c < 1$, define $t(c, s) > 0$ by

$$\frac{b_0 b_1 p_{Y}^{\delta_Y} e^{-(\delta_Y - s) t(c, s)/4}}{(\delta_Y - s)} = c.$$ 

where $b_0$ and $b_1$ are given in Lemma 5.13.

2. For $0 < c < 1$ and $t > 0$, define $\lambda(t, c, s) > 0$ by

$$\lambda(t, c, s) := (2c_0 D_1 p_{Y}^6 \tau_Z) \frac{e^{2ts}}{c}.$$ 

where $c_0$ is given by (8.12).
As it is evident from the above, the definition of \( t(c,s) \) is motivated by the linear algebra lemma 5.13. Indeed, for any vector \( v \in e_1 G \) and \( t \geq t(c,s) \), we have we have

\[
\sup_{1 \leq \rho \leq 2} \frac{1}{\mu_y[-\rho, \rho]} \int_{-\rho}^{\rho} \frac{1}{\|vu_r a_t\|^s} d\mu_y(r) \leq c\|v\|^{-s}.
\]

The choice of \( \lambda(t, c, s) \) is to control the additive difference between \( f_s(yu_r a_t) \) and \( \sum_{v \in I} \|vu_r a_t\|^{-s} \) uniformly over all \( r \in [-1, 1] \) such that \( yu_r \in Y_0 \), so that we would get:

\[
A_t f_s(y) \leq c \cdot f_s(y) + \frac{\lambda(t,c,s)}{2} \omega(y)^s
\]

(see Lemma 9.11, (9.15) and (9.16)).

**Markov operator for the height function.** In this subsection, we use notation from section 6.

It will be convenient to introduce the following notation:

**Notation 9.6.** Let \( Q \subset G \) be a compact subset.

1. Let \( d_Q \geq 1 \) be the infimum of all \( d \geq 1 \) such that for all \( g \in Q \) and \( v \in \mathbb{R}^4 \),

\[
d^{-1}\|v\| \leq \|vg\| \leq d\|v\|.
\]

Note that \( d_Q \asymp \max_{g \in Q} \|g\| \), up to an absolute multiplicative constant.

2. We also define \( c_Q \geq 1 \) to be the infimum of all \( c \geq 1 \) such that for any \( x \in X_0 \), \( g \in Q \) with \( xg \in X_0 \), and for all \( 1 \leq i \leq \ell \)

\[
c^{-1}\omega_i(x) \leq \omega_i(xg) \leq c\omega_i(x).
\]

We note that \( c_Q \asymp \max_{g \in Q} \|g\| \) up to an absolute multiplicative constant.

**Lemma 9.9.** For any \( 0 < c \leq 1/2 \) and \( t \geq t(c,s) \), there exists \( D_2 \asymp e^{2t} \) so that for all \( y \in Y_0 \) and \( 1 \leq \rho \leq 2 \),

\[
A_{t,\rho} \omega(y)^s \leq c \cdot \omega(y)^s + D_2.
\]

**Proof.** Let \( t \geq t(c,s) \). We compare \( \omega(yu_r a_t) \) and \( \omega(y) \) for \( r \in [-2,2] \).

Setting

\[
Q := \{ u_r a_r u_s : |r|, |s| \leq 2, |\tau| \leq t \},
\]

we have \( c_Q \asymp e^t \).

Recall that the constant \( \varepsilon_X \) satisfies

\[
\varepsilon_X^{-1} \gg \sup_{y \in X_0 \cap Y_0} \omega(y).
\]

We consider two cases.

**Case 1:** \( \omega(y) \leq 2\varepsilon_Q / \varepsilon_X \). In this case, for \( h \in Q \) with \( yh \in Y_0 \),

\[
\omega(yh) \leq 2\varepsilon_Q^2 / \varepsilon_X.
\]

Hence, the claim in this case follows if we choose \( D_2 = 2\varepsilon_Q^2 / \varepsilon_X \asymp e^{2t} \).
Case 2: $\omega(y) > 2c_Q/\varepsilon_X$. There exists $1 \leq i \leq \ell$ such that

$$\omega_i(y) > 2c_Q/\varepsilon_X,$$

and hence $y \in h_i$.

By the definition of $c_Q$, we have

$$\omega_i(yh) > 2/\varepsilon_X,$$

and hence $yh \in h_i$ for all $h \in Q$ with $yh \in Y_0$. Choose $g_0 \in G$ so that $y = [g_0]$. In view of Lemma 6.5, see in particular (6.6), there exists $\gamma \in \Gamma$ such that simultaneously for all $h \in Q$ with $yh \in Y_0$,

$$\omega(yh) = \|v_i\gamma g_0h\|^{-1}.$$

Since $v_i = e_1g_i^{-1} \in e_1G$ (see (6.1)), we may apply Lemma 5.13 (linear algebra lemma II) and deduce:

$$A(t, \rho) \omega(y)^s = \frac{1}{\mu_y([-\rho, \rho])} \int_{-\rho}^{\rho} \frac{1}{\|v_i\gamma u + at\|^s} d\mu_y(r) \leq b_0b_1p_\gamma^s e^{-(\delta_Y-s)t/4} \|v_i\gamma\|^{-s} \leq c \cdot \omega(y)^s;$$

in the last inequality we used the fact that $t \geq t(c, s)$. The proof is now complete. □

**Log-continuity of $F_{s, \lambda}$.** The following log-continuity lemma with a control on the multiplicative constant $\sigma$ is the first hypothesis in Proposition 7.5.

**Lemma 9.10 (Log-continuity lemma).** There exists $1 \leq \sigma \ll p_\gamma^8$ so that the following holds: for every $\lambda \geq \tau_Z$, we have

$$\sigma^{-1}F_{s, \lambda}(y) \leq F_{s, \lambda}(yh) \leq \sigma F_{s, \lambda}(y)$$

for all $y \in Y_0$ and all $h \in B_H(2)$ so that $yh \in Y_0$.

We first obtain the following estimate for $f$ on nearby points:

**Lemma 9.11.** Let $Q \subset H$ be a compact subset. For any $y \in Y_0$ and $h \in Q$ such that $yh \in Y_0$, we have

$$f_s(yh) \leq \sum_{v \in I_Z(y)} \|vh\|^{-s} + (c_0c_Qd_QD_1^6p_{Y,\tau_Z}) \omega(y)^s$$

where $c_0$ is as in Lemma 8.12, and the sum is understood as 0 in the case when $I_Z(y) = \emptyset$.

**Proof.** Let $y \in Y_0$ and $h \in Q$ with $yh \in Y_0$. If $I_Z(yh) = \emptyset$, then by (9.8), we have

$$f_s(yh) = \omega(yh)^s \leq c_0^6 \omega(y)^s$$

proving the claim.

Now suppose that $I_Z(yh) \neq \emptyset$. Setting

$$\varepsilon := (d_QD_1\omega(y))^{-1},$$

we have
we write
\begin{equation}
(9.12) \quad f_s(yh) = \sum_{v \in I_Z(yh), \|v\| < \varepsilon} \|v\|^{-s} + \sum_{v \in I_Z(yh), \|v\| \geq \varepsilon} \|v\|^{-s}.
\end{equation}

Since \#I_Z(yh) \leq c_0 p_Y^6 \tau_Z by Lemma 8.12, we have
\begin{equation}
(9.13) \quad \sum_{v \in I_Z(yh), \|v\| \geq \varepsilon} \|v\|^{-s} \leq (c_0 p_Y^6 \tau_Z \varepsilon)^{-s} \leq (c_0 d_Q D_1 p_Y^6 \tau_Z \omega(y)^s).
\end{equation}

If there is no \( v \in I_Z(yh) \) with \( \|v\| \leq \varepsilon \), then this proves the claim by (9.12). If \( v \in I_Z(yh) \) satisfies \( \|v\| < \varepsilon \), then
\[ \|vh^{-1}\| \leq d_Q \varepsilon = D_1^{-1} \omega(y)^{-1}; \]
in particular, \( vh^{-1} \in I_Z(y) \). Therefore, by setting \( v' = vh^{-1} \),
\[ \sum_{v \in I_Z(yh), \|v\| < \varepsilon} \|v\|^{-s} \leq \sum_{v' \in I_Z(y)} \|v'\|^{-s}. \]
Together with (9.13), this finishes the proof. \( \square \)

**Proof of Lemma 9.10.** Since \( B_H(2)^{-1} = B_H(2) \), it suffices to show the inequality \( \leq \). By Lemma 9.11, applied with \( c := c_{B_H(2)} \) and \( d := d_{B_H(2)} \), we have that for all \( h \in B_H(1) \) with \( yh \in Y_0 \),
\[ f_s(yh) \leq d \sum_{v \in I_Z(y)} \|v\|^{-s} + c_0 d_Q D_1 p_Y^6 \tau_Z \omega(y)^s. \]

Recall that \( \varepsilon_X^3 \leq \tau_Z \leq \lambda \) and that \( D_1 \ll p_Y^2 \).

If \( I_Z(y) = \emptyset \), then
\[ F_{s,\lambda}(yh) \ll p_Y^8 \tau_Z \omega(y)^s + \lambda \omega(y)^s \ll p_Y^8 \lambda \omega(y)^s \ll p_Y^8 (f_s(y) + \lambda \omega(y)^s) \ll p_Y^8 F_{s,\lambda}(y). \]

If \( I_Z(y) \neq \emptyset \), then
\[ F_{s,\lambda}(yh) \leq d \cdot f_s(y) + c_0 d_Q D_1 p_Y^6 \tau_Z \omega(y)^s + \lambda \omega(yh)^s \ll f_s(y) + p_Y^8 \lambda \omega(y)^s \ll p_Y^8 F_{s,\lambda}(y). \]

This finishes the upper bound. The lower bound can be obtained similarly.

**Main inequality.** We will apply the following lemma to obtain the second hypothesis of Proposition 7.5 for \( c := (8 \sigma p_Y^{6y})^{-1} < 1/2. \)

**Lemma 9.14 (Main inequality).** Let \( 0 < c \leq 1/2 \). For \( t \geq t(c/2, s) \) and \( \lambda = \lambda(t, c, s) \), we have the following: for any \( y \in Y_0 \) and \( 1 \leq \rho \leq 2 \), we have
\[ \mathcal{A}_{t,\rho} F_{s,\lambda}(y) \leq c F_{s,\lambda}(y) + \lambda D_2 \]
where \( D_2 \ll e^{2t} \) is as in Lemma 9.9.
Proof. The following argument is based on comparing the values of $f_s(yu, a_t)$ and $f_s(y)$ for $r \in [-2, 2]$ such that $yu_r \in Y_0$.

Let $Q := \{u_r a_r u_s : |r|, |s| \leq 2, |r| \leq t\}$. Then

$$c_Q \asymp e^t$$ and $$d_Q \asymp e^t$$

where $c_Q$ and $d_Q$ are as in (9.6). Hence, by Lemma 9.11, we have that for any $|r| \leq 2$ such that $yu_r \in Y_0$,

$$f_s(yu_r a_t) \leq \sum_{v \in I_Z(y)} \|yu_r a_t\|^{-s} + c_0 D_1 p_Y^6 \tau Z \omega(y)^s e^{2ts}$$

where $c_0$ is as in Lemma 9.11.

By averaging (9.15) over $[-\rho, \rho]$ with respect to $\mu_y$, and applying (9.5), we get

$$A_{t, \rho} f_s(y) \leq c \cdot f_s(y) + c_0 D_1 p_Y^6 \tau Z \omega(y)^s e^{2ts}$$

$$\leq c \cdot f_s(y) + \frac{c_0}{2} \omega(y)^s.$$  

Then by Lemma 9.9 and (9.16), we have

$$A_{t, \rho} F_s, \lambda(y) = A_{t, \rho} f_s(y) + A_{t, \rho} \lambda \omega(y)^s$$

$$\leq c \cdot f_s(y) + \frac{c_0 \lambda}{2} \omega(y)^s + \frac{c_0 \lambda}{2} \omega(y)^s + \lambda D_2$$

$$= c \cdot F_s, \lambda(y) + \lambda D_2.$$  

By Theorem 4.8, we have $s_Y \asymp p_Y$. For the sake of simplicity of notation, we put

$$\alpha_{Y, s} := \left(\frac{s_Y}{s_Y - s}\right)^{1/(s_Y - s)} \asymp \left(\frac{p_Y}{s_Y - s}\right)^{1/(s_Y - s)}.$$  

We are now in a position to apply Proposition 7.5 to get the following estimate:

**Theorem 9.18 (Margulis function on average).** There exists $\lambda_s > 1$ such that

$$m_Y (F_{s, \lambda_s}) \ll \alpha_{Y, s} \tau Z.$$  

Proof. Let $1 \leq \sigma < p_Y^\delta$ be given by Lemma 9.10. Let $c := (8\sigma p_Y^\delta)^{-1} < 1/2$, $t_s := t(c, s)$ and $\lambda_s := \lambda(t_s, c, s)$ be given by (9.4). Then in view of Lemmas 9.10 and 9.14, $F_{s, \lambda_s}$ satisfies the conditions of Proposition 7.5 with $t = t_s$ and $D_0 = \lambda_s D_2$, where $D_2 \ll e^{2t_s}$ is given in Lemma 9.9. Therefore

$$m_Y (F_{s, \lambda_s}) \leq 64 \lambda_s p_Y^\delta D_2.$$  

Since

$$e^{(s_Y - s) t_s} = \left(\frac{8\sigma b_0 p_Y^\delta}{(s_Y - s)^4}\right)^4 \ll \left(\frac{p_Y}{s_Y - s}\right)^*$$ and $$\lambda_s = (2c_0 D_1 p_Y^6 \tau Z) \frac{e^{2t_s}}{c},$$

we get

$$\lambda_s p_Y^\delta D_2 \ll p_Y^* e^{4t_s} \tau Z \ll \alpha_{Y, s} \tau Z.$$  


Combining this with (9.19) finishes the proof. □

10. Quantitative isolation of a closed orbit

In this section, we deduce Theorem 1.5 from Theorem 9.18. Let $Y, Z$ be non-elementary closed $H$-orbits in $X$. We allow the case $Y = Z$ as well. Let $\frac{\delta_Y}{2} \leq s < \delta_Y$.

Recall the definitions of $f_{s,Y,Z}$ and $F_{s,\lambda} = F_{s,\lambda,Y,Z}$ from Def 9.1. Let $\lambda_s$ be given by Theorem 9.18. Using the log-continuity lemma for $F_{s,\lambda_s}$ (Lemma 9.10), we first deduce the following estimate:

**Proposition 10.1.** For any $0 < \varepsilon < \varepsilon_X$ and $y \in Y_0 \cap X_\varepsilon$, we have

$$f_{s,Y,Z}(y) \leq F_{s,\lambda_s}(y) \ll \frac{\alpha_{Y,s}^{*}\tau_Z}{m_Y(B(y, \varepsilon))}.$$

**Proof.** Let $y \in Y_0 \cap X_\varepsilon$. Then $\text{inj}(y) \geq \varepsilon$ and hence $y B_H(\varepsilon) = B(y, \varepsilon)$. For all $h \in B_H(\varepsilon_X)$, $F_{s,\lambda_s}(y) \leq \sigma F_{s,\lambda_s}(yh)$ for some constant $\sigma \ll \rho_Y^0$ by Lemma 9.10. By applying Theorem 9.18, we get

$$F_{s,\lambda_s}(y) \leq \frac{\sigma \int_{x \in y B_H(\varepsilon)} F_{s,\lambda_s}(x) dm_Y(x)}{m_Y(B(y, \varepsilon))} \leq \frac{\sigma \cdot m_Y(F_{s,\lambda_s})}{m_Y(B(y, \varepsilon))} \ll \frac{\alpha_{Y,s}^{*}\tau_Z}{m_Y(B(y, \varepsilon))}.$$

Recall from (6.8) that for all $x \in X_0$,

$$(10.2) \quad \frac{1}{\alpha \tau} \cdot \text{inj}(x) \leq \omega(x)^{-1} \leq \frac{\alpha}{\tau} \cdot \text{inj}(x).$$

Using the next lemma, we will be able to use the estimate for $f_{s,Y,Z}$ obtained in Proposition 10.1 to deduce a lower bound for $d(y, Z)$.

**Lemma 10.3.** (1) Let $y \in Y_0$ and $z \in Z - B_Y(y, \text{inj}(y))$. If $d(y, z) \leq \frac{1}{2\alpha c_1 D_1} \text{inj}(y)$, then

$$d(y, z)^{-s} \leq c_1 f_{s,Y,Z}(y)$$

where $c_1 \geq 1$ is as in (8.1).

(2) If $Y \neq Z$, then for any $y \in Y_0$,

$$d(y, Z)^{-s} \ll \rho_Y^2 f_{s,Y,Z}(y).$$

**Proof.** As $Z$ is closed and $d(y, z) \leq \frac{1}{2\alpha c_1 D_1} \text{inj}(y) < \text{inj}(y)/2$, the hypothesis $z \in Z - B_Y(y, \text{inj}(y))$ implies that $z$ is of the form $y \exp(v + v')$ with $v \in is\mathfrak{sl}_2(\mathbb{R}) - \{0\}$ and $v' \in is\mathfrak{sl}_2(\mathbb{R})$.

By (8.1),

$$\|v\| \leq \|v + v'\| \leq c_1 d(y, z) \leq D_1^{-1} \text{inj}(y)/(2\alpha) \leq (D_1 \omega(y))^{-1}.$$

It follows that $v \in I_Z(y, D_1)$. Therefore

$$(10.4) \quad d(y, z)^{-s} \leq c_1^s \|v\|^{-s} \leq c_1 \|v\|^{-s} \leq c_1 f_s(y),$$

proving (1).
We now turn to the proof of (2); suppose thus that $Y \neq Z$. Then there exists $z \in Z$ such that $d(y, Z) = d(y, z)$. In view of (1), it suffices to consider the case when $d(y, z) > \frac{1}{2\alpha c_1 D_1} \operatorname{inj}(y)$.

Since $s \leq 1$, $\omega(y)^s \leq f_s(y)$, and $D_1 \ll p_Y^2$, we get

$$d(y, z)^{-s} \leq 2\alpha c_1 D_1 \operatorname{inj}(y)^{-s} \leq 2\alpha^2 c_1 D_1 \omega(y)^s \ll p_Y^2 f_s, Y, Z(y)$$

where we also used (10.2). The proof is complete. \qed

Theorem 1.5(1) is a special case of the following theorem when $Y \neq Z$:

**Theorem 10.5** (Isolation in distance). For any $0 < \varepsilon < \varepsilon_X$, $y \in Y_0 \cap X_\varepsilon$, and $z \in Z$, at least one of the following holds:

1. $z \in B_Y(y, \varepsilon) = y B_H(e, \varepsilon)$, or
2. $d(y, z) \gg \alpha_{Y, s}^{-1/s} m_Y(B(y, \varepsilon))^{1/s} \tau_Z^{-1/s}$, where $\alpha_{Y, s}$ is as given in (9.17).

**Proof.** As $y \in X_\varepsilon$, $\operatorname{inj}(y) \geq \varepsilon$. Suppose that $z \notin B_Y(y, \varepsilon)$. We first observe that since $\tau_Z \geq \varepsilon_X^2$, $m_Y(B(y, \varepsilon))^{1/s} \leq \varepsilon^{1/s} \leq \varepsilon$, and $p_Y^{-2} \gg \alpha_{Y, s}^{-1/s}$, we have

$$\frac{\varepsilon}{2\alpha c_1 D_1} \gg p_Y^{-2} \varepsilon \gg \alpha_{Y, s}^{-1/s} m_Y(B(y, \varepsilon))^{1/s} \tau_Z^{-1/s}.$$

Therefore if $d(y, z) \geq \frac{1}{2\alpha c_1 D_1} \varepsilon$, (2) holds.

If $d(y, z) \leq \frac{1}{2\alpha c_1 D_1} \varepsilon \leq \frac{1}{2\alpha c_1 D_1} \operatorname{inj}(y)$, then by Lemma 10.3, $d(y, z)^{-s} \leq c_1 f_s(y)$. Hence it remains to apply Proposition 10.1 to finish the proof. \qed

The following theorem is Theorem 1.5(2):

**Theorem 10.6** (Isolation in measure). Let $0 < \varepsilon \leq \varepsilon_X$. Let $Y \neq Z$. We have

$$m_Y\{y \in Y : d(y, Z) \leq \varepsilon\} \ll \alpha_{Y, s}^* \tau_Z \varepsilon^s.$$

**Proof.** Let $\lambda_s$ be given by Theorem 9.18. By Lemma 10.3(2),

$$d(y, Z)^{-s} \leq c f_{s, Y, Z}(y) \leq C : F_{s, \lambda_s}(y)$$

for some $1 < C \ll p_Y^2$.

For $0 < \varepsilon < \varepsilon_X$, if we set

$$\Omega_\varepsilon := \{y \in Y_0 : F_{s, \lambda_s}(y) \geq C^{-1} \varepsilon^{-s}\},$$

then $\{y \in Y_0 : d(y, Z) \leq \varepsilon\} \subset \Omega_\varepsilon$. On the other hand, we have

$$C^{-1} \varepsilon^{-s} m_Y(\Omega_\varepsilon) \leq \int_{\Omega_\varepsilon} F_{s, \lambda_s} d m_Y \leq m_Y(F_{s, \lambda_s}).$$

Since $m_Y(F_{s, \lambda_s}) \ll \alpha_{Y, s}^* \tau_Z$ by Theorem 9.18, we get that

$$m_Y\{y \in Y_0 : d(y, Z) \leq \varepsilon\} \leq m_Y(\Omega_\varepsilon) \ll \alpha_{Y, s}^* \tau_Z \varepsilon^s.$$

\qed
Proof of Proposition 1.16. Let $F_s = F_{s,\lambda_s}$ be as in Theorem 9.18. Then $F_s$ satisfies (1) in the proposition by Lemma 10.3. It satisfies (3) by Lemma 9.10.

Moreover, in view of Lemmas 9.10 and 9.14, $F_s$ satisfies the conditions of Proposition 7.5. Hence, by Proposition 7.6 it also satisfies (2) in the proposition. □

Number of properly immersed geodesic planes. When $\text{Vol}(M) < \infty$, we record the following corollary of Theorem 10.5. Let $\mathcal{N}(T)$ denote the number of properly immersed totally geodesic planes $P$ in $M$ of area at most $T$.

We deduce the following upper bound from Theorem 10.5 using the pigeonhole principle:

**Corollary 10.7.** Let $\text{Vol}(M) < \infty$. For any $1/2 < s < 1$, we have

$$\mathcal{N}(T) \ll_s T^{\frac{s}{2}-1}$$

where the implied constant depends only on $s$.

**Proof.** We obtain an upper bound for the number of closed $H$-orbits in $X$ which yields the above result. The proof is based on applying Theorem 10.5.

If $X$ is compact, let $\rho = 0.1e_X$. If $X$ is not compact, then the quantitative non-divergence of the action of $U$ on $X$ implies that there exists $\rho > 0$ so that

$$m_Y(X - X_\rho) < 0.01$$

for every closed orbit $Y = xH$, e.g., see [8].

For every $S > 0$ put

$$\mathcal{Y}(S) := \{xH : xH \text{ is closed and } S/2 < \text{Vol}(xH) \leq S\}.$$ 

In view of the above choice of $\rho$, we have $\text{Vol}(xH) \geq \rho^3 \gg 1$ for every closed orbit $xH$. Let $n_0 = \lfloor 3 \log_2(\rho) \rfloor$ and for every $T > 1$, let $n_T = \lceil \log_2 T \rceil$. Then we have

$$\{xH : xH \text{ is closed and } \text{vol}(xH) \leq T\} \subset \bigcup_{n_0}^{n_T} \mathcal{Y}(2^k).$$

Let $\eta \asymp \rho$ be so that the map $g \mapsto xg$ is injective for all $x \in X_\rho$ and all $g \in \text{Box}(\eta) := \exp(B_{isl_2(\mathbb{R})}(0, \eta)) \exp(B_{sl_2(\mathbb{R})}(0, \eta))$.

Fix some $1/2 < s < 1$ and some $z \in X$. We claim that

$$\#(\text{connected components of } \mathcal{Y}(2^k) \cap z.\text{Box}(\eta)) \ll \alpha_s^{12/s} \eta^{6k/s}$$

where the implied constant depends on $\rho$.

For any connected component $C$ of $\mathcal{Y}(2^k) \cap z.\text{Box}(\eta)$, there exists some $v \in isl_2(\mathbb{R})$ so that

$$C = z \exp(v) \exp(B_{sl_2(\mathbb{R})}(0, \eta)).$$
Let us write $C = C_v$. Now in view of Theorem 10.5, for every two connected components $C_v \neq C_{v'}$, we have

$$\|v - v'\| \gg \rho \alpha_s^{-4/s} 2^{-2k/s}. \tag{10.10}$$

Because $\dim(\mathfrak{t}) = 3$, the cardinality of an $\alpha_s^{-4/s} 2^{-2k/s}$-separated set in $B_{\mathfrak{sl}_2(\mathbb{R})}(0, \eta)$ is $\ll \alpha_s^{12/s} 2^{6k/s}$. The claim in (10.9) thus follows from (10.10).

Let $\{z_j.\text{Box}(\eta) : 1 \leq j \leq M\}$ be a covering of $X_\rho$ with sets of the form $z.\text{Box}(\eta)$; we may find such a covering with $M = O(1)$. Then we compute

$$\mathcal{N}(2^k) \leq 2^{-k+1} \sum_{\mathcal{Y}(2^k)} \text{vol}(xH)$$

by def. of $\mathcal{Y}(2^k)$

$$\ll 2^{-k} \sum_{j=1}^M \sum_{C_v \subset z_j.\text{Box}(\eta)} \text{vol}(C_v) \quad \text{by (10.8)}$$

$$\ll \alpha_s^{12/s} \sum_{j=1}^M 2^{\frac{6k}{s} - k} \ll \alpha_s^{12/s} 2^{\frac{6k}{s} - k} \quad \text{by (10.9) and } M = O(1)$$

in the above we also used the fact that $\text{vol}(C_v) \ll 1$.

We conclude that

$$\mathcal{N}(T) \ll \alpha_s^{12/s} \sum_{k=n_0}^{n_T} 2^{\frac{6k}{s} - k} \ll_s T^\frac{6}{s} - 1$$

which implies the claim. \hfill \Box

11. Appendix: Proof of Theorem 1.1 in the compact case

In this section we present the proof of Theorem 1.1 when $X$ is compact. As was mentioned in the introduction, this case is due to G. Margulis.

Let $Y \neq Z$ be two closed $H$-orbits in $X = \Gamma \backslash G$. Recall $\epsilon_X = \min_{x \in X} \text{inj}(x)$.

Fix $0 < s < 1$, and define $f_s : Y \to [2, \infty)$ as follows: for any $y \in Y$,

$$f_s(y) = \begin{cases} \sum_{v \in I_Z(y)} \|v\|^{-s} & \text{if } I_Z(y) \neq \emptyset \\ \epsilon_X^{-s} & \text{otherwise} \end{cases}$$

where

$$I_Z(y) = \{v \in i\mathfrak{sl}_2(\mathbb{R}) - \{0\} : 0 < \|v\| < \epsilon_X, \ y \exp(v) \in Z\}.$$

Note that in the case at hand, $f_s$ is a bounded Borel function on $Y$.

We use the following special case of Lemma 5.6: for any $v \in i\mathfrak{sl}_2(\mathbb{R})$ with $\|v\| = 1$, $1/3 \leq s < 1$ and $t > 0$, we have

$$\int_0^1 \frac{ds}{\|v u_s t\|^s} \leq b_0 \frac{e^{(s-1)t/4}}{1 - s} \tag{11.1}$$

where $v h = \text{Ad}(h)(v)$ for all $h \in H$. 


Remark 11.2. It is worth noting that the symmetric interval $[-1, 1]$ was used in Lemma 5.6. We remark that this is necessary in the infinite volume setting; indeed the half interval $[0, 1]$ may even be a null set for $\mu_y$ for some $y$, see (4.1) for the notation.

For $t > 0$ and $y \in Y$, define

$$A_t f_s(y) = \int_0^1 f_s(yu_r a_t) dr.$$  \tag{11.3}

**Proposition 11.4.** Let $1/3 \leq s < 1$. There exists $t = t(s) > 0$ such that for all $y \in Y$,

$$A_t f_s(y) \leq \frac{1}{2} f_s(y) + \alpha_s^4 \text{Vol}(Z)$$  \tag{11.5}

where $\alpha_s \asymp (1 - s)^{1/(1-s)}$.

**Proof.** Let $b_0$ be as in (11.1), and let $t = t(s)$ be given by the equation

$$b_0\frac{e^{(s-1)t/4}}{1-s} = 1/2.$$

We compare $f_s(yu_r a_t)$ and $f_s(y)$ for $r \in [0, 1]$. Let $C_1 \asymp e^t$ be large enough so that $\|vh\| \leq C_1 \|v\|$ for all $v \in \mathfrak{is} \mathfrak{t}_2(\mathbb{R})$ and all

$$h \in \{u_r a_{-r} u_r : 0 < r', r < 1, |\tau| \leq t\}.$$

Let $v \in I_Z(yu_r a_t)$ be so that $\|v\| < \varepsilon_X/C_1$. Then $\|va u_{-r}\| \leq \varepsilon_X$; in particular, $va u_{-r} \in I_Y(y)$.

In the following, if $I_Z(\cdot) = \emptyset$, the sum is interpreted as to equal to $\varepsilon_X^{-s}$. In view of the above observation and the definition of $f_s$, we have

$$f_s(yu_r a_t) = \sum_{v \in I_Z(yu_r a_t)} \|v\|^{-s}$$

$$= \sum_{v \in I_Z(yu_r a_t)} \|v\|^{-s} + \sum_{v \in I_Z(yu_r a_t), \|v\| \geq \varepsilon_X/C_1} \|v\|^{-s}$$

$$\leq \sum_{v \in I_Z(y)} \|vu_r a_t\|^{-s} + \sum_{v \in I_Z(y), \|v\| \geq \varepsilon_X/C_1} \|v\|^{-s}. \tag{11.6}$$

Moreover, note that $\#I_Z(y) \ll \text{Vol}(Z)$. Hence,

$$\sum_{\|v\| \geq \varepsilon_X/C_1} \|v\|^{-s} \ll C_1^s \varepsilon_X^{-s} \text{Vol}(Z) \ll e^{st} \varepsilon_X^{-s} \text{Vol}(Z). \tag{11.7}$$

We now average (11.6) over $[0, 1]$. Then using (11.7) and (11.1) we get

$$A_t f_s(y) \leq \frac{1}{2} f_s(y) + O(e^{st} \text{Vol}(Z)).$$

As $(1 - s)^{1/(1-s)} \asymp e^{st/4}$, this proves (11.5). \hfill \Box

Let $m_Y$ be the $H$-invariant probability measure on $Y$:

**Corollary 11.8.** We have

$$m_Y(f_s) \leq 2\alpha_s^4 \text{Vol}(Z).$$
Proof. Since $m_Y$ is an $H$-invariant probability measure, $m_Y(A_t f_s) = m_Y(f_s)$. Hence the claim follows by integrating (11.5) with respect to $m_Y$. □

Proof of Theorem 1.1. There exists $\sigma > 0$ such that for any $h \in B_H(\varepsilon_X)$ and $y \in Y$, $f_s(y) \leq \sigma f_s(yh)$.

Hence, using Corollary 11.8, we deduce

$$f_s(y) \leq \frac{\sigma \int_{B_H(\varepsilon_X)} f_s(yh)dm_Y(yh)}{m_Y(B(y, \varepsilon_X))} \leq \frac{\sigma \cdot m_Y(f_s)}{m_Y(B(y, \varepsilon_X))} \leq 2\sigma \alpha_4^4\varepsilon^{-3} \text{Vol}(Y) \text{Vol}(Z).$$

Since $d(y, Z)^{-s} \leq c_1 f_s(y)$ for an absolute constant $c_1 \geq 1$ (see (10.4)), we have

$$d(y, Z) \gg \alpha_4^{-4/s} \text{Vol}(Z)^{-1/s} \text{Vol}(Y)^{-1/s}.$$

This shows Theorem 1.1(1). Theorem 1.1(2) follows from Corollary 11.8 as

$$m_Y\{y \in Y : d(y, Z) \leq \varepsilon\} \leq m_Y\{y \in Y : f_s(y) \geq \varepsilon^{-s}\} \leq m_Y(f_s)\varepsilon^{s}.$$

by the Chebyshev inequality. □

References


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