Math 103B Quiz 1 Solutions

April 21, 2020

**Problem 1.** Find, with justification, the characteristic of the following rings.

(a) (5 pts) $\mathbb{Z} \times \mathbb{Z}_6$.

(b) (5 pts) $M_2(\mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{12})$. (Recall that $M_2(R)$ denotes $2 \times 2$ matrices with entries in $R$.)

**Solution.** a) We first note that the multiplicative identity in $\mathbb{Z} \times \mathbb{Z}_6$ is $(1, 1)$ as $1 \in \mathbb{Z}$ and $1 \in \mathbb{Z}_6$ are the multiplicative identities in their respective rings. Then, suppose there exists $n \geq 1$ with $n \cdot (1, 1) = (0, 0)$. In the first component, this implies $n \cdot 1 = 0$ in $\mathbb{Z}$; a contradiction. Thus the characteristic in $\mathbb{Z} \times \mathbb{Z}_6$ is zero.

b) We first note that the multiplicative identity in $M_2(\mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{12})$ is

$$I := \begin{pmatrix} (1, 1, 1) & (0, 0, 0) \\ (0, 0, 0) & (1, 1, 1) \end{pmatrix}. \quad (1)$$

Taking multiples of $I$, we see that $nI = 0$ in $M_2(\mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_{12})$ if and only if $n(1, 1, 1) = 0$ if and only if $4|n$ and $8|n$ and $12|n$. The minimum such $n \geq 1$ to satisfy these conditions is the least common multiple of $4, 8, 12$, which is seen to be 24.

**Problem 2.** (10 pts) Find, with justification, all of the solutions of the equation

$$x^2 - 4x - 5 = 0$$

in $\mathbb{Z}_7 \times \mathbb{Z}_9$.

**Solution.** We first note that $x = (x_1, x_2)$ is a solution if and only if $x_1$ is a solution to the equation in $\mathbb{Z}_7$ and $x_2$ is a solution to the equation in $\mathbb{Z}_9$. From here, one can simply plug in all values of $\mathbb{Z}_7$ and $\mathbb{Z}_9$ to get the desired solution. However, for completeness, here is the desired solution in each component.

In $\mathbb{Z}_7$, as $\mathbb{Z}_7$ is an integral domain, we see that $(x + 1)(x - 5) = 0$ implies that $x = -1 = 6$ or $x = 5$.

In $\mathbb{Z}_9$, we note that 3 and 6 are zero divisors as $3 \cdot 3 = 0$ and $6 \cdot 3 = 0$. We similarly have that $x = -1 = 8$ and $x = 5$ are solutions. For all other values of $x$, note that $9|(x + 1)(x - 5)$ implies that $x + 1 \in \{3, 6\}$ and $x - 5 \in \{3, 6\}$ or $x \in \{2, 5\} \cap \{8, 2\} = \{2\}$.
As a result, the solutions are of the form
\[(i, j) \subset \mathbb{Z}_7 \times \mathbb{Z}_9 : i \in \{5, 6\}, j \in \{2, 5, 8\}\].

**Problem 3.** Find, with justification, all the units in the ring \(\mathbb{Z}_6 \times \mathbb{Z} \times \mathbb{Z}_{13}\).

**Solution.** We first note that the units in a product ring are precisely the product of the units in each component. As such, it suffices to determine the units in each component. In \(\mathbb{Z}_6\), the units consist of the integers \(1 \leq i \leq 5\) which are relatively prime to 6; i.e. \(\{1, 5\}\). In \(\mathbb{Z}\), the only units are \(\{\pm 1\}\) and as \(\mathbb{Z}_{13}\) is a field, the units consist of all nonzero elements; namely \(\mathbb{Z}_{13} \setminus \{0\}\). As a result, the units in \(\mathbb{Z}_6 \times \mathbb{Z} \times \mathbb{Z}_{13}\) are as follows:

\[U(\mathbb{Z}_6 \times \mathbb{Z} \times \mathbb{Z}_{13}) = \{(i, j, k) : i \in \{1, 5\}, j \in \{\pm 1\}, k \in \mathbb{Z}_{13} \setminus \{0\}\}\].

**Problem 4.** Let \(R\) be a ring, and let \(a \in R\) be a fixed element. Let \(S = \{x \in R : ax = 0\}\). Show that \(S\) is a subring of \(R\).

**Solution.** Note that commutativity of addition, associativity of addition/multiplication, and distributivity all carry down from \(R\) being a ring. Thus, it suffices to check closure of the various operations.

- **0 \in S:** As proved in class, we have that \(a \cdot 0 = 0\), so \(0 \in S\).
- **For \(x \in S, -x \in S\):** Note that \(ax = 0\) implies that \(-ax = 0\) and thus \(a(-x) = 0\), i.e. \(x + y \in S\).
- **For \(x, y \in S, x + y \in S\):** By distributivity, \(0 = ax + ay = a(x + y)\), so \(x + y \in S\).
- **For \(x, y \in S, xy \in S\):** \(ax = 0\) implies that \(axy = (ax)y = (0)y = 0\).

Thus \(S\) is indeed a subring of \(R\).

**Problem 5.** Let \(R\) be a ring. Let \(0 \neq a \in R\), and define \(f : R \to R\) by

\[f(x) = ax\] for all \(x \in R\).

(a) (8 pts) Assume \(a\) is not a zero divisor. Prove that \(f\) is one-to-one.
(b) (7 pts) Prove or disprove by giving a counter example: if \(a\) is not a zero divisor, then \(f\) is onto.

**Solution.** a) Let \(x, y \in R\) be such that \(f(x) = f(y)\) or \(ax = ay\). Then, \(a(x - y) = ax - ay = 0\). Since \(a\) is nonzero and not a zero divisor, we must have that \(x = y\), thus showing injectivity.

b) We claim this is false. Consider \(R = \mathbb{Z}\) and \(a = 2\). Then \(a\) is not a zero divisor (there are no zero divisors in \(\mathbb{Z}\)) but \(f\) is not onto since there is no \(x \in \mathbb{Z}\) such that \(2x = 1\).