Problem 1. Let
\[ \sigma = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 3 & 6 & 1 & 2 & 7 & 5 & 8
\end{array} \right). \]

a) (3 pts) Write \( \sigma \) as a product of disjoint cycles.

b) (3 pts) Determine whether \( \sigma \) is even or odd.

Solution. We first consider the orbit
\[ O_{\sigma,1} : \{ \sigma^n(1) : n \in \mathbb{N} \} = \{1, 4\}. \]
Next, we consider the orbit
\[ O_{\sigma,2} : \{ \sigma^n(2) : n \in \mathbb{N} \} = \{2, 3, 6, 7, 5\}. \]
Using both of these and then also noting that 8 is a fixed point, we get that
\[ \sigma = (1, 4) \cdot (2, 3, 6, 7, 5). \]

b) We can also write \( \sigma \) as a product of transpositions and get that
\[ \sigma = (1, 4) \cdot (2, 5) \cdot (2, 7) \cdot (2, 6) \cdot (2, 3) \]
and as a result \( \sigma \) may be represented as a product of 5 transpositions and hence is odd.

Problem 2. (4 pts) Let \( \sigma = (2, 6, 1)(4, 3, 5) \) and \( \tau = (2, 3, 4)(6, 1, 5) \) be two elements in \( S_6 \). Find a permutation \( \mu \in S_6 \) so that \( \sigma = \mu \tau \mu^{-1} \).

Solution. As we did in a previous homework assignment, we want \( \mu \) to send the 3-cycles to 3-cycles. There are actually many different ways to do this, but one such way is to define \( \mu \in S_6 \) so that \( \mu(1) = 3, \mu(2) = 2, \mu(3) = 6, \mu(4) = 1, \mu(5) = 5, \) and \( \mu(6) = 4 \). Putting this in two line notation, we see that
\[ \sigma = \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 2 & 6 & 1 & 5 & 4
\end{array} \right). \]

Problem 3. (8 pts) List all the abelian groups of order 540 up to isomorphism.

Solution. Observe that we have the following prime factorization \( 540 = 2^2 \cdot 3^3 \cdot 5 \). As a result, by the fundamental theorem of finite abelian groups, we have the following list of abelian groups of order 540:

- \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5 \)
- \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \)
- \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \)
- \( \mathbb{Z}_4 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5 \)
- \( \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \)
- \( \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \)

Note that there are 6 = \( p(2)p(3)p(1) \) total nonisomorphic abelian groups of order 540 where \( p(n) \) is the number of integer partitions of \( n \).

Problem 4. a) (3 pts) Find the order of \((1, 3)\) in \( \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \).

b) (6 pts) Find a homomorphism from \( \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \) onto \( \mathbb{Z}_{30} \).
Solution. a) Observe that \( \text{ord}_{\mathbb{Z}_9}(1) = 6 \) and \( \text{ord}_{\mathbb{Z}_{10}}(3) = 10 \) and hence we have that

\[
\text{ord}((1,3)) = \text{lcm}(6,10) = 30.
\]

b) Observe that we have that \( \mathbb{Z}_{30} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{10} \) so by appropriately composing homomorphisms, it suffices to define a surjective homomorphism \( \phi : \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \to \mathbb{Z}_3 \oplus \mathbb{Z}_{10} \). Observe that the map \( \phi(i,j) \mapsto (i \pmod{3}, j) \) is a surjective homomorphism in each coordinate and hence is as desired.

Remark. There are a few other solutions that work here. One such surjective homomorphism is \( \phi \) given by \( \phi(i,j) = 5i-3j \). Another solution uses part a) and the fact that there are two cosets \( H, (3,5)+H \) where we send an element \((i,j) \mapsto k \) if \((i,j) = k(1,3) \) or if \((i,j) = (3,5) + k(1,2) \). One can then check that both of these maps work.

Problem 5. Let \( H = \langle (1,3) \rangle \) in \( \mathbb{Z}_9 \oplus \mathbb{Z}_6 \).

a) (6 pts) Find all of the left cosets of \( H \) in \( \mathbb{Z}_9 \oplus \mathbb{Z}_6 \).

b) (3 pts) Find the order of \((0,2) + H\) in \( \mathbb{Z}_9 \oplus \mathbb{Z}_6 \).

Solution. a) First, we observe that \( \text{ord}(1,1) = \text{lcm}(9,2) = 18 \) and hence we have that \(|H| = 18\). An application of Lagrange’s theorem therefore yields that there are \( \frac{54}{18} = 3 \) left cosets of \( H \) in \( \mathbb{Z}_9 \oplus \mathbb{Z}_6 \). These are \( H, (0,1) + H, (0,2) + H \).

b) We have that

\[
\begin{align*}
(0,2) + H &+ (0,2) + H = (0,4) + H = (0,2) + H
\end{align*}
\]

and hence we have that \( \text{ord}((0,2) + H) \mid 3 \) which since \((0,2) + H \neq H \) yields that \( \text{ord}((0,2) + H) = 3 \).

Problem 6. Let \( G \) be an abelian group of order 309. Show that \( G \) is isomorphic to \( \mathbb{Z}_{309} \).

Solution. Observe that we have the following prime factorization \( 309 = 3^1 \cdot 103 \), and hence by the fundamental theorem of finite abelian groups we necessarily have that \( G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{103} \). Next, we note that \((1,1) \in \mathbb{Z}_3 \oplus \mathbb{Z}_{103} \) is so that \( \text{ord}((1,1)) = \text{lcm}(3,103) = 309 \) and hence \((1,1) \) generates the group \( \mathbb{Z}_3 \oplus \mathbb{Z}_{103} \). Hence, we have that \( G \) is cyclic and thus since \(|G| = 309\), we necessarily have that \( G \cong \mathbb{Z}_{309} \) as desired.

Problem 7. Let \( H = \langle (2,3) \rangle \) in \( \mathbb{Z} \oplus \mathbb{Z} \).

a) (5 pts) Find all the left cosets of \( H \) in \( \mathbb{Z} \oplus \mathbb{Z} \).

b) (5 pts) Show that \( \frac{\mathbb{Z} \oplus \mathbb{Z}}{H} \) is a cyclic group.

Solution. a) Observe that we have that

\[
\mathbb{Z} \oplus \mathbb{Z} \cong \langle (2,3), (1,1) \rangle
\]

and that every \((i,j) \in \mathbb{Z} \oplus \mathbb{Z}\) can be written uniquely in the form \((i,j) = a(1,1) + b(2,3)\) for some \(a,b \in \mathbb{Z}\) since we have that

\[
\det \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = 2 - 3 = -1.
\]

As a result, we have that we may write all of the cosets of \( H \) in \( \mathbb{Z} \oplus \mathbb{Z} \) in the form \((a,a) + H\) for \( a \in \mathbb{Z} \).

b) Since we may write cosets of \( H \) in \( \mathbb{Z} \oplus \mathbb{Z} \) in the form \((a,a) + H = a(1,1) + H\), we have that

\[
\langle (1,1) + H \rangle \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{H}
\]

and hence the quotient group is cyclic.
Remark. One could have also noted that the cosets are of the form \((0, n) + H\) and \((1, n) + H\) for \(n \in \mathbb{Z}\), but this took a bit more work to show that these are sufficient and then the resulting quotient group is cyclic.

Remark. The most common error here was to say the quotient group was isomorphic to \(\mathbb{Z}_2 \oplus \mathbb{Z}_3\) and that the cosets are of the form \(H, (0, 1) + H, (0, 2) + H, (1, 0) + H, (1, 1) + H, (1, 2) + H\). The issue here is that the element \((0, 100) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3\) is not in any of the above cosets; and hence this list of cosets is not complete.

Problem 8. (5 pts) Let \(G\) be a finite abelian group. Assume that \(G\) has at least three elements of order 3. Show that \(|G|\) is divisible by 9.

Solution. Let \(g_1\) be one such element in \(G\) of order 3, then consider \(H = \langle g_1 \rangle\). Since \(H\) is a normal subgroup of \(G\) as \(G\) is an abelian group, we can make sense of the quotient group \(G/H\). We then have of the other two elements of order 3, which we’ll denote as \(g_2, g_3\), that at most one of them is in \(H\). Without loss of generality, suppose that \(g_2 \not\in H\) and hence we can consider the coset \(g_2H\) and note that

\[(g_2H)^3 = g_2^3H = eH = H\]

and hence we have that ord\((g_2H)\) = 3 as \(g_2H \neq H\). Thus, we have that \(3||G/H|| = 3\cdot||G/H||\) and hence since \(|G| = 3 \cdot |G/H|\) we necessarily have that \(9||G|\) which is what we wanted to show.

Remark. As \(G\) is a finite abelian group, we have that for some \(k \in \mathbb{N}\)

\[G \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{p_i^{r_i}}.\]

Seeking a contradiction, suppose that \(|G|\) is not divisible by 9. Then, since \(G\) has an element of order 3, \(|G| = 3n\) for some \(n\) which is not divisible by 3. We therefore have that

\[G \cong \mathbb{Z}_3 \oplus \bigoplus_{i=2}^{k} \mathbb{Z}_{p_i^{r_i}}\]

where we let \(H = \bigoplus_{i=2}^{k} \mathbb{Z}_{p_i^{r_i}}\). Since \(|H| = n\) and \(n\) is not divisible by 3, we have that \(H\) does not have any elements of order 3. As a result, the only elements in \(G\) of order 3 in \(G\) are of the form \((1,0,\ldots,0)\) and \((2,0,\ldots,0)\). Hence, \(G\) only has two elements of order 3. This is a contradiction. Thus, we have that \(n\) is necessarily divisible by 3, and hence \(|G|\) is divisible by 9.

Remark. One could also find a subgroup \(H \leq G\) so that \(|H| = 9\), but this requires some careful casework.

Problem 9. (5 pts) Let \(G\) be a group and let \(H\) be a normal subgroup of \(G\). Assume that \((G : H) = n\). Show that \(g^n \in H\) for all \(g \in G\).

Solution. Observe that since \(H\) is a normal subgroup of \(G\), we may make sense of the quotient group \(G/H\) and also that \(|G| = (G : H) = n\). Let \(g \in G\) and consider \(gH \in G/H\). By a problem from the third quiz, we have that any element to the size of group is the identity, which in this case yields that \((gH)^n = H\). However, we then have that

\[(gH)^n = H \iff g^nH = H \iff g^n \in H\]

which is what we wanted to show.

Remark. This problem is actually a generalization of the problem from the third quiz. This is a stronger statement which can be seen by taking \(H = \{e\}\) which is necessarily normal.