Problem 1. Let
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 & 8 \end{pmatrix}. \]

a) (6 pts) Write, with justification, \( \sigma \) as a product of disjoint cycles.
b) (4 pts) Determine with justification, whether \( \sigma \) is even or odd.

Solution. a) By considering the orbits \( O_{\sigma, i} := \{ \sigma^n(i) : i \in \mathbb{N} \} \), we may write \( \sigma \) as a product of disjoint cycles. First, we have that \( O_{\sigma, 1} := \{ \sigma^n(1) : i \in \mathbb{N} \} = \{ 1, 4, 2, 3 \} \).

Similar, we may consider \( O_{\sigma, 5} := \{ \sigma^n(5) : i \in \mathbb{N} \} = \{ 5, 6, 7 \} \).

Lastly, we note that \( \sigma(8) = 8 \) and hence we omit if from cycle notation of \( \sigma \) and get \( \sigma = (1, 4, 2, 3) \circ (5, 6, 7) \).

b) Let \( \tau = (i_1, i_2, \ldots, i_l) \) be a cycle of length \( l \). Then, we have that \( \tau = (i_1, i_l) \circ (i_1, i_{l-1}) \circ \cdots \circ (i_1, i_2) \) and hence we have that a cycle of odd length is written in terms of an even number of transposition and hence is even. Similarly, a cycle of even length is written in terms of an odd number of transposition and hence is odd. As \( \sigma \) is the product of an even permutation and a odd permutation, it is odd. One could also write \( \sigma = (1, 3) \circ (1, 2) \circ (1, 4) \circ (5, 7) \circ (5, 6) \)

and observe that \( \sigma \) is the product of 5 transposition and hence is odd.

Problem 2. Let \( \sigma = (1, 2) \circ (5, 1, 3) \circ (4, 2, 6, 7) \) be a permutation in \( S_8 \).

a) (5 pts) Write \( \sigma \) as a product of disjoint cycles.
b) (5 pts) Compute, with justification, \( \sigma^{100} \).

Solution. a) We have that \( \sigma = (1, 3, 5, 2, 6, 7, 4) \) is cycle notation; or in two line notation that
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 1 & 2 & 7 & 4 & 8 \end{pmatrix}. \]

b) Observe that \( \sigma \), when written as a product of disjoint cycles, is comprised of a 7-cycle and a 1-cycle and hence we have that \( \text{ord}(\sigma) = 7 \). This yields that \( \sigma^{100} = \sigma^{7 \cdot 14} \circ \sigma^2 = \sigma^2 \) and that \( \sigma^2 = (1, 5, 6, 4, 3, 2, 7) \) or in two line notation
\[ \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 2 & 3 & 6 & 4 & 1 & 8 \end{pmatrix}. \]

Problem 3. (5 pts) Let \( H \) be the subgroup of \((\mathbb{Z}_{20}, +_{20})\) which is generated by 4. Find all the left cosets of \( H \) in \( \mathbb{Z}_{20} \).

Solution. Observe that \( \text{ord}(4) = 5 \) and that \( H = \langle 4 \rangle = \{ 0, 4, 8, 12, 16 \} \) and hence an application of Lagrange’s theorem yields that \( [\mathbb{Z}_{20} : \langle 4 \rangle] = \frac{20}{5} = 4 \) and hence we necessarily have four left cosets. They are \( 0 + H = \{ 0, 4, 8, 12, 16 \}; \ 1 + H = \{ 1, 5, 9, 13, 17 \}; \ 2 + H = \{ 2, 6, 10, 14, 18 \}; \ 3 + H = \{ 3, 7, 11, 15, 19 \}. \)
Problem 4. (5 pts) Prove or disprove by an example. Suppose \([G : H] = 3\), then every right coset is a left coset.

**Solution.** First, we observe that the statement is true (independent of the index of the subgroup \(H\) in \(G\)) in the case where \(G\) is an abelian group. Thus, towards finding a counterexample, we want a group which is not abelian and hence consider \(S_3\). (Note that we showed on a previous homework that the center of \(S_n\), the elements which commute with all \(\sigma \in S_n\), consists of only the identity element. This tells us morally that \(S_n\) is very “not” abelian and hence seems like a good guess for a counterexample.) Next, we consider the subgroup \(H = \langle (1, 2) \rangle\) where \(\text{ord}((1, 2)) = 2\) and hence \([G : H] = 3\). Next, we observe that

\[
(1, 3)H = \{(1, 3), (1, 2, 3)\}; \quad H(1, 3) = \{(1, 3), (1, 3, 2)\}
\]

and hence we have that \((1, 3)H \neq H(1, 3)\) which yields that the left cosets are not the same as the right cosets and thus we have found a valid counterexample.

Problem 5. (5 pts) Let \(G\) be a finite group of order \(n\) and let \(e \in G\) be the identity element. Show that \(g^n = e\) for all \(g \in G\).

**Solution.** Fix \(g \in G\) and consider \(H = \langle g \rangle\). Then, we have that \(\text{ord}(g) = |H|\) and by Lagrange’s theorem, we have that \(\text{ord}(g)|n\) as \(|G| = n\) and \(\text{ord}(g) = |H|\). As a result, we have that there exists a \(k \in \mathbb{N}\) so that \(k \text{ord}(g) = n\). Now, we have that

\[
g^n = g^{k \text{ord}(g)} = (g^{\text{ord}(g)})^k = e^k = e.
\]