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Research statement

My current research is focused on finding generating functions for statistics on the family of groups which are the wreath product of the symmetric group S_k with the symmetric group S_n , i.e. groups of the form $S_k \wr S_n$ where $k \geq 2$.

This statement is divided into 4 sections. In the first, I give a brief overview of the important definitions and some theorems relating to statistics on S_n . These definitions and results are then extended to $S_k \wr S_n$ in the second section, where I describe some new results. In the third and fourth sections I discuss other problems on S_n that I have extended to $S_k \wr S_n$, as well as ideas for future projects.

1 Permutations Statistics on S_n

A permutation statistic s on the symmetric group S_n can be viewed as a map S_n into the natural numbers \mathbb{N} . For example, the statistics for the number of descents, the major index, and the number of inversions are given by

$$\text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}, \text{des}(\sigma) = |\text{Des}(\sigma)|, \text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i, \text{ and } \text{inv}(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i > \sigma_j).$$

The indicator function $\chi(A)$ equals 1 if A is true and 0 if A is false. The statistics for the number of rises, the co-major index, and the number of co-inversions are given by

$$\text{Ris}(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}, \text{ris}(\sigma) = 1 + |\text{Ris}(\sigma)|, \text{comaj}(\sigma) = \sum_{i \in \text{Ris}(\sigma)} i, \text{ and } \text{coinv}(\sigma) = \sum_{1 \leq i < j \leq n} \chi(\sigma_i < \sigma_j)$$

We have found the analogues for $S_k \wr S_n$ of the trivariate generating function of descents, major index, and inversions for $S_k \wr S_n$. That is, Ira Gessel gave a generating function $\sum_{\sigma \in S_n} x^{\text{des}(\sigma)} u^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)}$ both in his thesis and in a paper coauthored with Adriano Garsia [6, 7]. Our approach is to extend the methods of Mendes and Remmel [8] who showed how the Garsia-Gessel formula could be proved by applying a certain ring homomorphism defined on the ring Λ of symmetric functions over infinitely many variables x_1, x_2, \dots . Define the homogeneous symmetric functions h_n and the elementary symmetric functions e_n by

$$\prod_{i \geq 1} \frac{1}{1 - x_i t} = \sum_{n \geq 0} h_n t^n \quad \text{and} \quad \prod_{i \geq 1} (1 + x_i t) = \sum_{n \geq 0} e_n t^n.$$

Then Mendes and Remmel showed that the Garsia-Gessel formula and the corresponding analogues for the hyperoctahedral group B_n and its subgroup D_n could be derived by applying a ring homomorphism to the simple identity

$$\sum_{n \geq 0} h_n t^n = \frac{1}{\sum_{n \geq 0} e_n (-t)^n}. \tag{1}$$

To state their results we shall use standard notation from hypergeometric function theory. For $n \geq 1$ and $\lambda \vdash n$, define the q -analogues of n , $n!$, and $\binom{n}{\lambda}$ by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad \text{and} \quad \begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}.$$

If we replace $[n]_q$ by $[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \dots + pq^{n-2} + q^{n-1}$ in these formulas, we obtain the corresponding p, q -analogues $[n]_{p,q}$, $[n]_{p,q}!$, and $\begin{bmatrix} n \\ \lambda \end{bmatrix}_{p,q}$. By convention, let $[0]_q = [0]_{p,q} = 0$ and $[0]_q! = [0]_{p,q}! = 1$. In addition, let $(x, y; p, q)_0 = 1$ and $(x, y; p, q)_n = (x - y)(xp - yq) \cdots (xp^{n-1} - yq^{n-1})$. A p, q -analog for the exponential function is defined by

$$\mathbf{e}_{p,q}^t = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}.$$

Then Mendes and Remmel [8] defined the ring homomorphism ξ by defining it on the elementary symmetric function e_n so that

$$\xi(e_n) = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}}.$$

Since products of elementary symmetric functions form a basis for the ring of symmetric functions, the definition of the homomorphism ξ uniquely extends to all other symmetric functions. They then applied this homomorphism to (1) to prove the following result which is a direct consequence of the Garsia-Gessel formula.

Theorem 1.1

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!(x, y; u, v)_{n+1}} \sum_{\sigma \in \mathcal{S}_n} x^{\text{des}(\sigma)} y^{\text{ris}(\sigma)} u^{\text{maj}(\sigma)} v^{\text{comaj}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ = \sum_{k \geq 0} \frac{x^k}{y^{k+1} \mathbf{e}_{p,q}^{-t(u/v)^0} \cdots \mathbf{e}_{p,q}^{-t(u/v)^k}}. \end{aligned}$$

2 Analogues of the trivariate generating function for des, maj, and inv

One of the problems in dealing with the groups $S_k \wr S_n$ is that there are several ways in which one can define a natural descent type statistic. We shall write an element of $\Gamma \in S_k \wr S_n$ in one-line notation as

$$\Gamma = (\alpha_1, \sigma(1)) (\alpha_2, \sigma(2)) \dots (\alpha_n, \sigma(n)),$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in (S_k)^n$ and $\sigma = \sigma(1) \cdots \sigma(n) \in S_n$. We define our descent statistics on $S_k \wr S_n$ in terms of coordinate-wise comparisons. In our research, we have explored 5 different ways to compare coordinates. Most of them rely in part on the standard **lexicographic** ordering on S_k , whereby $\alpha_r < \alpha_s$ if and only if there an j such that $\alpha_r(i) = \alpha_s(i)$ for $i \leq j$ and $\alpha_r(j) < \alpha_s(j)$. For example, $42135 < 42513$. This ordering is a *total* ordering on S_k , since any two elements of S_k must differ in at least one coordinate. Then we define the following 5 orderings on pairs of the form (α, k) where $\alpha \in S_k$ and $k \in \mathbb{N}$.

Ordering $<_1$: We define $(\alpha_r, \sigma(r)) <_1 (\alpha_s, \sigma(s))$ if and only if either (i) $\alpha_r < \alpha_s$ or (ii) $\alpha_r = \alpha_s$ and $\sigma(r) < \sigma(s)$.

This is an extension of **lexicographic** ordering. It compares the $\vec{\alpha}$ coordinate first, then uses the σ coordinate as a tiebreaker, if necessary. Like the lexicographic ordering on S_n , $<_1$ is a total ordering.

Ordering $<_2$: We define $(\alpha_r, \sigma(r)) <_2 (\alpha_s, \sigma(s))$ if and only if $\alpha_r = \alpha_s$ and $\sigma(r) < \sigma(s)$.

This is a **restricted lexicographic** ordering. It is a subordering of $<_1$, and is most certainly *not* a total ordering.

Ordering $<_3$: We define $(\alpha_r, \sigma(r)) <_3 (\alpha_s, \sigma(s))$ if and only if $\alpha_r < \alpha_s$ and $\sigma(r) < \sigma(s)$.

This is a **product** ordering; like $<_2$, $<_3$ is not a total ordering.

Ordering $<_4$: We define $(\alpha_r, \sigma(r)) <_4 (\alpha_s, \sigma(s))$ if and only if $\alpha_r \leq \alpha_s$ and $\sigma(r) < \sigma(s)$.

This ordering is a slight modification of $<_3$, and we will call it a **weak product** ordering.

Ordering $<_5$: We define $(\alpha_r, \sigma(r)) <_5 (\alpha_s, \sigma(s))$ if and only if $\sigma(r) < \sigma(s)$.

We can think of $<_5$ as a **reverse lexicographic** ordering, since this ordering compares the $\sigma(i)$ first. Since $<_5$ never needs the β_i to settle a tie, the descent set of $(\vec{\alpha}, \sigma)$ will just be the descent set of σ .

For the $S_3 \wr S_8$ element $\Gamma = (\alpha^3, 5)(\alpha^3, 4)(\alpha^1, 6)(\alpha^6, 1)(\alpha^5, 7)(\alpha^2, 8)(\alpha^1, 3)(\alpha^1, 2)$, (where the superscripts indicate $\alpha^1 < \alpha^2 < \dots < \alpha^6$) we get the following descent sets:

ORDERING TYPE	ORDERING #	Des(Γ)
LEXICOGRAPHIC	$<_1$	{1, 2, 4, 5, 6, 7}
RESTRICTED LEXICOGRAPHIC	$<_2$	{1, 7}
PRODUCT	$<_3$	{6}
WEAK PRODUCT	$<_4$	{1, 6, 7}
REVERSE LEXICOGRAPHIC	$<_5$	{1, 3, 6, 7}

For each of these orderings $<_o$, we can define analogues for $S_k \wr S_n$ of the statistics des , maj , and inv on S_n by letting

$$\begin{aligned} \text{Des}_o(\Gamma) &= \{i : \Gamma(i) >_o \Gamma(i+1)\}, \quad \text{des}_o(\Gamma) = |\text{Des}_o(\Gamma)|, \quad \text{maj}_o(\Gamma) = \sum_{i \in \text{Des}_o(\Gamma)} i, \\ \text{inv}_o(\Gamma) &= \sum_{1 \leq i < j \leq n} \chi(\Gamma_i >_o \Gamma_j), \quad \text{and} \quad \text{wrinv}(\Gamma) = \sum_{i=1}^n \text{inv}(\alpha_i) \end{aligned}$$

The last statistic is the **wreath inversion** number of Γ . Each definition except for $\text{wrinv}(\Gamma)$ generates five statistics (for $o = 1, 2, 3, 4, 5$). We have been able to find analogues of 1.1 for all five orderings. We again use carefully chosen homomorphisms θ_o on the ring of symmetric functions Λ , applied to an identity proven by Egecioglu and Remmel:

$$h_n = \sum_{\mu \vdash n} B_{\mu, n} (-1)^{n-\ell(\mu)} e_\mu \quad (2)$$

where $B_{\mu, n}$ equals the number of arrangements of the parts of μ , $\ell(\mu)$ equals the number of parts of μ , and $e_\mu = \prod_i e_{\mu_i}$.

We can interpret the righthand side of θ_o (equation 2) as a sum of weighted labelings of elements of $S_k \wr S_n$. A sign-reversing involution ι can then be used to show that the fixed points of ι are labeled in a way that allows us to measure $\text{des}(\Gamma)$, $\text{maj}(\Gamma)$, and $\text{inv}(\Gamma)$ for all $\Gamma \in S_k \wr S_n$.

For example, using ordering $<_1$, we prove that

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_q!(1, x; 1, u)_{n+1}} \sum_{\Gamma \in S_k \wr S_n} q^{\text{inv}_1(\Gamma)} p^{\text{wrinv}(\Gamma)} x^{\text{des}_1(\Gamma)} u^{\text{maj}_1(\Gamma)} \\ = \sum_{m \geq 0} x^m \left(\prod_{r=0}^m \prod_{\alpha \in S_k} \mathbf{e}_q(-t u^r p^{\text{inv}(\alpha)}) \right)^{-1} \end{aligned}$$

Using ordering $<_3$, we prove that

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_q!(1, x; 1, u)_{n+1}} \sum_{\Gamma \in S_k \wr S_n} q^{\text{inv}_3(\Gamma)} p^{\text{wrinv}(\Gamma)} x^{\text{des}_3(\Gamma)} u^{\text{maj}_3(\Gamma)} \\ = \sum_{m \geq 0} x^m \left(\prod_{j=1}^m \sum_{r \geq 0} \frac{(-t u^j)^r q^{\binom{2}{j}} e_r[[k]_p!]}{[r]_q!} \right)^{-1} \end{aligned}$$

Here $e_r[[k]_p!]$ is the result of replacing the variables of $e_n(x_1, \dots, x_k)$ by the powers of p that appear in $[k]_p!$, the so-called plethysmic substitution of $[k]_p!$ into elementary symmetric function e_n for the expert. The results of this work were presented at the 2008 Permutation Patterns conference in Dunedin, New Zealand. A paper detailing the results will appear in the conference proceedings.

3 Alternating elements

For S_n , a permutation $\sigma = \sigma_1 \dots \sigma_n$ is said to an **alternating** (or an **up-down**) permutation if $\sigma_i < \sigma_{i+1}$ if i is odd and $\sigma_i > \sigma_{i+1}$ if i is even. The generating function for alternating permutations was found by André in 19th century. If A_n is the number of alternating permutations in S_n , then André proved that

$$\sum_{n \geq 0} A_n \frac{x^n}{n!} = \sec x + \tan x.$$

To find the analogue of André's result for $S_k \wr S_n$, one first has to define what we mean by alternating permutations. Clearly, there are analogues of up-down permutations for all of the five orderings on $S_k \wr S_n$. We have studied three possible extensions of "alternating":

1. Γ is called **up-down** (or **u-d**, for short) if $(\alpha_1, \sigma(1)) <_o (\alpha_2, \sigma(2)) >_o (\alpha_3, \sigma(3)) <_o \dots$.
2. Γ is called **not down-not up (nd-nu)** if $(\alpha_1, \sigma(1)) \not>_o (\alpha_2, \sigma(2)) \not<_o (\alpha_3, \sigma(3)) \not>_o \dots$.
3. Γ is called **up-not up (u-nu)** if $(\alpha_1, \sigma(1)) <_o (\alpha_2, \sigma(2)) \not<_o (\alpha_3, \sigma(3)) <_o \dots$.

The third extension is a cross between the first two, which we can think of as strict and weak extensions of the concept of alternation.

Even for small cases of k we find surprising results. For $k = 2$ and the product ordering $<_3$ on $S_2 \wr S_n = B_n$, we find that that the number of u-d alternating permutations of B_n is $F_{n+2} A_n$ where F_n is the n -th Fibonacci number and A_n is the number of alternating permutations of S_n . However, the generating function for the number of u-nu alternating permutations $A_{2,n}^{<_3}$ of $S_2 \wr S_n$ with respect to the product ordering $<_3$ is given by

$$\sum_{n \geq 0} A_{2,n}^3 \frac{x^n}{n!} = \frac{1 + 2 \sin x + x \cos x}{\cos x - x \sin x}, \quad (3)$$

where $A_{k,n}^o$ equals the number of u-nu alternating elements in $S_k \wr S_n$ with respect to $<_o$. Again, the main idea behind proving (3) is to apply properly chosen ring homomorphisms to various symmetric function identities. In this case, it requires separate identities for groups of form $S_k \wr S_{2n}$ and for groups of the form $S_k \wr S_{2n+1}$.

We also are able to create similar generating functions for the Weyl group D_n , and more generally for any group $G \wr S_n$ where a total ordering – like the lexicographic ordering on S_k – is placed on G . In particular, we can extend the methods of Carlitz and Scoville [4] to find the generating functions for alternating elements with respect to the weak product ordering $<_4$.

For example, let $AD_{n,k}$ equal the number of $\sigma \in G \wr S_n$ for which $\text{Ris}_4(\sigma) = \{1, 3, 5, \dots\}$. Then if $k = |G|$, we prove that

$$\sum_{n \geq 0} \frac{x^{2n}}{(2n)!} AD_{2n,k} = \frac{(k-1)!}{\frac{d^{k-1}}{dx^{k-1}} (x^{k-1} \cos x)} \quad \text{and} \quad \sum_{n \geq 0} \frac{x^{2n+1}}{(2n+1)!} AD_{2n+1,k} = \frac{\frac{d^{k-1}}{dx^{k-1}} (x^{k-1} \sin x)}{\frac{d^{k-1}}{dx^{k-1}} (x^{k-1} \cos x)}.$$

4 Other research and future work

Alternating elements of $S_k \wr S_n$ fall into the larger category of studying elements with controlled descent sets. In addition to studying alternating elements of $S_k \wr S_n$, I will also be working on the following related problems:

1. It is natural to ask how many $\Gamma \in S_k \wr S_n$ have a given $S \subseteq [n-1]$ for its set of descents. Using identities on ribbon Schur functions, we can extend the work of Remmel and Riehl, who were able to compute various generating functions for permutation statistics on S_n over the set of permutations that contain a given descent set S .

2. For a given $m \geq 2$, one can ask for the number of **common descents** of an m -tuple of elements of $S_k \wr S_n$:

$$\text{comdes}_{\vec{o}}(\Gamma^{(1)}, \dots, \Gamma^{(m)}) = \left| \bigcap_{i=1}^m \text{Des}_{o_i}(\Gamma^{(i)}) \right|$$

We are able to give a generating function for common descents for *any* combination of orderings $\vec{o} = (o_1, o_2, \dots, o_m)$.

3. One way to interpret the statistic $\text{des}_o(\Gamma)$ is that it measures deviation from an “all rises” element of $S_k \wr S_n$. We can define the statistic $\text{altdes}_o(\Gamma)$ to measure deviation from a u-d alternating element:

$$\text{altdes}_o(\Gamma) = |\{i : \Gamma(2i+1) \not\prec_o \Gamma(2i+2)\} \cup \{j : \Gamma(2j) \not\prec_o \Gamma(2j+1)\}|$$

We can use homomorphisms similar to the ones we use for alternating elements to create a generating function for $\text{altdes}(\Gamma)$. Similarly we can measure deviation from nd-nu and u-nu alternating elements.

4. A well-known result from pattern avoidance theory states that the number of $\sigma \in S_n$ that avoid the pattern 132 (consecutive elements a, b, c with $a < c < b$) is the n th Catalan number C_n . This begs for a set of $S_k \wr S_n$ Catalan-analogue numbers. More generally, we can look at pattern avoidance and ask which results can be extended to $S_k \wr S_n$. Like with our three extensions of alternating elements, we may often have multiple ways of lifting a definition from S_n to $S_k \wr S_n$ when the ordering $<_o$ is not a total ordering.

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