Midterm Exam 2

Instructions: Same as for Midterm 1.
1. For full credit, you need to solve the first three problems, and one of the remaining two.
2. At the end of the test, write the letter (A-F) of your version on the front of the Blue Book, insert the assignment sheet into it, and hand it in.

A.1. (10 pts) For each of the statements below, decide True or False (1 point) and give a short justification for your answer (1 point) (if the justification is wrong, no points will be given).

(a) The set of all \( n \times n \) matrices with zeros on the diagonal is a subspace of \( M_{n \times n} \).
(b) If a system \( \mathcal{Y} = \{v_1, \ldots, v_k\} \) in a finite-dimensional vector space \( V \) is linearly independent then it is a basis. Hint: The answer depends on \( \dim V \).
(c) To find the coordinates of an \( \mathbb{R}^n \) vector with respect to a new basis one needs to multiply with the matrix \( B \) formed from the basis vectors.
(d) The area of the triangle in \( \mathbb{R}^n \) with vertices at \((0,0), (1,2), \) and \((3,1)\) equals 5.
(e) If \( \det A = 2 \) and \( \det B^T = -3 \) then \( \det(B^{-1}A^T) = -2/3 \).

A.2. a) (6 pts) Find the dimension and a basis for the null-space \( \text{Nul}(A) \), where

\[
A = \begin{pmatrix} 2 & -5 & 8 & -7 \\ -1 & 3 & -4 & 4 \\ 0 & 2 & -7 & 9 \end{pmatrix}.
\]

b) (2 pts) Determine the rank of the matrix \( B = -2A^T \).

Turn over!
A.3. a) (6 pts.) Using the determinant criterion, decide if $B = A - 2I_4$ is invertible, where

$$A = \begin{pmatrix} 1 & 3 & -4 & 6 \\ 0 & 2 & -1 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & -3 & 4 & 0 \end{pmatrix}.$$ 

b) (2 pts) Find the determinants of $2B$ and $B^3$ (if you haven’t solved a), assume that $\det(B) = -1$.

c) (2 pts) Find the cofactor $C_{23}$ of $B$. Hint: If you have solved a) then you have it already computed.

A.4. (6 pts) Show that the three polynomials $b_1(t) = t^2, \ b_2(t) = t(1 - t), \ b_3(t) = (1 - t)^2$ form a basis in $\mathbb{P}_2$, and find the coordinate vector of $p(t) = t^2 + t + 1$ in this basis.

A.5. (6 pts) Find the range space and kernel of the linear map $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ given by the formula

$$T(A) = 2A + A^T.$$
A1 \( (a) \) Yes, \( \text{det}(A + B) \neq \text{det}(A) + \text{det}(B) \) for both \( A, B \) have same two diagonals.

(b) No, if \( k < \dim V \), yes if \( k = \dim V \) (by Thm 9 of Chapter 4)

(c) No, \( B^{-1} \) would have been the correct choice

(d) No, \( \text{area}(A) = \frac{1}{2} \text{area}(\tilde{A}) = \frac{1}{2} \left| \begin{array}{cc} 3 & 1 \\ 1 & 2 \end{array} \right| = \frac{5}{2} \)

(e) Yes, \( \text{det}(B^{-1}) = \frac{1}{\text{det}(B)} = \frac{1}{\text{det}(B^T)} = -\frac{1}{3} \), \( \text{det}(A^T) = \text{det}(A) = 2 \)

\[ \text{det}(B^{-1}A^T) = -\frac{2}{3} \]

A2 \( a) \) Nullspace = solution set of \( Ax = 0 \). Do REF, look for free variables.

\[ \begin{bmatrix} -1 & 3 & -4 & 4 \\ 2 & -5 & 8 & -7 \\ 0 & 2 & -7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

1. \( \text{dim} \text{Nul}(A) = 1 \) if \( \text{Span} B = \text{Nul}(A) \).

2. Basis any \( \neq 0 \) solution of \( Ax = 0 \). \( \{ x_4 = 1, x_3 = 1, x_2 = -1, x_1 = -3 \} \)

3. \[ B = \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\} \]

4. \( \text{rank}(2A^T) = \text{rank}(A^T) = \text{rank}(A) \) \( \text{# pivots in REF} \) \( = 3 \)

A3 \( a) \) Need to check \( \text{det}(B) \neq 0 \) for invertibility:

\[ \text{det}(B) = \begin{vmatrix} -1 & 3 & -4 & 6 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & 0 & 2 \\ 1 & -3 & 4 & -2 \end{vmatrix} = -(-1) \begin{vmatrix} -1 & 3 & 6 \\ 2 & 2 & 2 \\ 1 & -3 & -2 \end{vmatrix} = 2 \begin{vmatrix} 0 & 4 & 7 \\ 1 & 1 & -2 \\ 0 & -4 & -3 \end{vmatrix} = -2 \begin{vmatrix} 4 & 7 \\ -4 & -3 \end{vmatrix} = -32 \]

1. \( B \) invertible

2. \( \text{det}(2B) = 2^4 \cdot \text{det}(B) = 16 \cdot (-32) = -672 \), \( \text{det}(B^3) = (-32)^3 \)

3. \( C_{23} = 32 \) (see above)
A4. Basis: Since \( \dim(V) = 3 \), need to check for linear independence or spanning property.

Linear independence:

\[
\begin{align*}
&c_1 b_1(t) + c_2 b_2(t) + c_3 b_3(t) = (c_1 - c_2 + c_3) t^2 + (c_2 - 2c_3) t + c_3 = 0 \\
\Rightarrow &
\begin{cases}
    c_3 = 0 \\
    c_2 = 2c_3 = 0 \\
    c_1 = c_2 - c_3 = 0
\end{cases} \Rightarrow \text{linear independence}
\end{align*}
\]

Spanning property: Same ansatz, if \( p(t) = a_0 + a_1 t + a_2 t^2 \) then:

\[
\begin{align*}
    a_0 &= c_3 \\
    a_1 &= c_2 - 2c_3 \\
    a_2 &= c_1 - c_2 + c_3
\end{align*}
\]

i.e.,

\[
\begin{align*}
    c_3 &= a_0 \\
    c_2 &= a_1 - 2a_0 \\
    c_1 &= a_2 - a_1 + 3a_0
\end{align*}
\]

Thus, each \( p(t) \) is a linear combination of the \( b_i(t) \).

In particular, if \( a_0 = a_1 = a_2 = 1 \) then:

\[
1 + t + t^2 = 3 b_1(t) - b_2(t) + b_3(t), \quad [1+t+t^2]_{B} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}
\]

A5. If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( T(A) = \begin{pmatrix} 3a & 2b+c \\ 2c+b & 3d \end{pmatrix} \).

\( T(A) = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) means \( 3a = 3d = 0 \Rightarrow a = d = 0 \\
2b + c = 2c + b = 0 \Rightarrow b = c = 0 \)

Thus, \( \ker(T) = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \) trivial (dim = 0)

Since \( T \) maps space \( M_{2 \times 2} \) into itself, and by the above \( T \) is one-to-one, it is also onto (and invertible) thus:

\( \text{Ran}(T) = M_{2 \times 2} \).

Alternative: For any \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \), find \( A \) such that \( T \) maps to it!
Midterm Exam 2

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2. At the end of the test, write the letter (A-F) of your version on the front of the Blue Book, insert the assignment sheet into it, and hand it in.

B.1. (10 pts) For each of the statements below, decide True or False (1 point) and give a short justification for your answer (1 point) (if the justification is wrong, no points will be given).
   (a) If $W_1, W_2$ are subspaces of $V$ then $W_1 \cup W_2$ is also a subspace of $V$.
   (b) Any system of fewer than dim $V$ vectors in $V$ can be complemented to a basis by including some more elements of $V$.
   (c) To find the coordinates of an $\mathbb{R}^n$ vector with respect to a new basis one needs to multiply with the inverse of the matrix $B$ formed from the basis vectors.
   (d) The area of the triangle in $\mathbb{R}^n$ with vertices at $(0,0)$, $(2,1)$, and $(1,-3)$ equals $7/2$.
   (e) If $\det A = 2$ and $\det B = -3$ then $\det(A - B) = 5$.

B.2. a) (6 pts) Find the dimension and a basis for the null-space $\text{Nul}(A)$, where

$$A = \begin{pmatrix} -2 & -6 & 8 & -2 \\ 1 & 4 & -3 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$ 

b) (2 pts) Determine the rank of the matrix $B = 2A^T$.

Turn over!
B.3. a) (6 pts.) Using the determinant criterion, decide if $B = A + 2I_4$ is invertible, where

$$A = \begin{pmatrix}
-1 & 3 & -5 & 3 \\
3 & 0 & -1 & 2 \\
0 & 1 & -2 & 0 \\
1 & 2 & 7 & -2
\end{pmatrix}. $$

b) (2 pts) Find the determinants of $-2B$ and $(2B)^{-1}$ (if you haven't solved a), assume that $\det(B) = -1$.

c) (2 pts) Find the cofactor $C_{32}$ of $B$. Hint: If you have solved a) then you have it already computed.

B.4. (6 pts) Show that the three polynomials

$$b_1(t) = t, \quad b_2(t) = t(1 - t), \quad b_3(t) = 1 - t$$

form a basis in $\mathbb{P}_2$, and find the coordinate vector of $p(t) = t^2 + t + 1$ in this basis.

B.5. (6 pts) Find the range space and kernel of the linear map $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ given by the formula

$$T(A) = A - 2A^T.$$
B1. (a) No, e.g. union of two lines through origin (with different directions) in $\mathbb{R}^2$ is not a subspace.
   (b) No, the system must also be linearly independent.
   (c) Yes, since $x = c_1b_1 + \cdots + c_m b_m \iff x = B[x]_B$ we get $[x]_B = B^{-1}x$.
   (d) $\frac{1}{2} \left| \frac{1}{2} \right| = \text{area}(A) = \frac{7}{2}$, Yes.
   (e) No, there is no rule how to compute $\det(A \pm B)$ from $\det(A)$, $\det(B)$.

B2. Nullspace = Solution set of $AX = 0$. Do REF, look for free variables:

\[ A \sim \begin{bmatrix} 1 & 4 & -3 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & -1 \end{bmatrix} \]

b) Rank $A = \#$ pivots.
Rank $(2A^T) = 4 - \dim \text{Null}(A)$

Thus, $\dim \text{Null}(A) = 1$; span $B = \text{Null}(A)$, where
Basis: Any $x = 0$ solution of $Ax = 0$ \( \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \} \)

B3. a) Need to check $\det(B) \neq 0$ for invertibility:
\[
\det(B) = \begin{vmatrix} 1 & 3 & -5 & 3 \\ 3 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot C_{32} = -3 \cdot \begin{vmatrix} 1 & -5 & 3 \\ -1 & 2 & 3 \\ 1 & 7 & 1 \end{vmatrix} = -3 \cdot \frac{1}{16} \cdot \begin{vmatrix} 1 & -5 \\ -1 & 2 \end{vmatrix} = -66 + 24 = -42
\]

b) $\det(-2B) = (-2)^4 \det(B) = -16 \cdot 42 = \ldots$, $\det(2B)^{-1} = \frac{1}{16 \det(B)} = \frac{1}{16 \cdot 42}$

C) $C_{32} = 42$, see above.
B4 Since \( \dim P_2 = 3 \), for basis we have to check either for linear independence:

\[
\begin{align*}
& c_1 b_1(t) + c_2 b_2(t) + c_3 b_3(t) = c_3 + (c_1 + c_2 - c_3)t + c_2 t^2 = 0 \\
\implies & c_2 = 0, c_3 = 0, c_1 = c_1 - c_2 = 0
\end{align*}
\]

or for the spanning property: For any \( \phi(t) = a_0 + a_1 t + a_2 t^2 \)
there is a (unique) set of \( c_1, c_2, c_3 \) such that

\[
\begin{align*}
& c_1 b_1(t) + c_2 b_2(t) + c_3 b_3(t) = a_0 + a_1 t + a_2 t^2 = \phi(t)
\end{align*}
\]

This gives \( a_0 = c_3, a_1 = c_1 + c_2 - c_3, a_2 = -c_2 \) thus

\[
\begin{align*}
& c_1 = a_1 + a_2 + a_0, c_2 = -a_2, c_3 = a_0 \quad \text{clearly if}
\end{align*}
\]

In particular,

\[
\begin{align*}
1 + t + t^2 = 3b_1(t) + 6b_2(t) + b_3(t) \quad \begin{pmatrix} 1 + t + t^2 \end{pmatrix}_P = \begin{pmatrix} 3 \\ -1 \end{pmatrix}
\end{align*}
\]

B5 If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( T(A) = \begin{pmatrix} -a & b - 2c \\ c - 2b & -d \end{pmatrix} \). Thus

\[
\begin{align*}
T(A) = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{mean} -a = -d = 0, \quad \text{b} - 2c = c - 2b = 0 \quad \implies \quad \begin{cases} a = b = c = d = 0 \end{cases}
\end{align*}
\]

\[\text{Ker}(T) = \{ 0 \} = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \quad \text{trivial subspace}\]

Since \( T \) maps \( M_{2 \times 2} \) into itself (\( V = V \)) and \( \text{Ker}(T) = \{ 0 \} \) implies one-to-one, \( T \) is also onto, i.e.

\[\text{Ran}(T) = M_{2 \times 2}\].

\[\text{Alternative: For any} \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \text{find} A \text{ that maps to if} \]
Midterm Exam 2

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2. At the end of the test, write the letter (A-F) of your version on the front of the Blue Book, insert the assignment sheet into it, and hand it in.

C.1. (10 pts) For each of the statements below, decide True or False (1 point) and give a short justification for your answer (1 point) (if the justification is wrong, no points will be given).
   (a) If \( W_1, W_2 \) are subspaces of \( V \) then \( W_1 \cap W_2 \) is also a subspace of \( V \).
   (b) The rank of a triangular \( n \times n \) matrix equals the number of its non-zero entries on the diagonal.
   (c) To find the coordinates of an \( \mathbb{R}^n \) vector with respect to a new basis one needs to multiply with the matrix \( B \) formed from the basis vectors.
   (d) The area of the triangle in \( \mathbb{R}^n \) with vertices at (1,2), (3,1), and (0,0) equals 5.
   (e) If \( \det A = 3 \) and \( \det B^T = -2 \) then \( \det(B^{-1}A^T) = 3/2 \).

C.2. a) (6 pts) Find the dimension and a basis for the null-space \( \text{Nul}(A) \), where

\[
A = \begin{pmatrix}
3 & 3 & -1 & 4 \\
1 & 4 & 2 & 3 \\
0 & 9 & 7 & 3
\end{pmatrix}
\]

b) (2 pts) Determine the rank of the matrix \( B = 3A^T \).

Turn over!
C.3. a) (6 pts.) Using the determinant criterion, decide if \( B = A - 3I_4 \) is invertible, where

\[
A = \begin{pmatrix}
3 & -1 & 5 & 4 \\
2 & 6 & 1 & -2 \\
0 & 0 & 3 & 2 \\
1 & 6 & -3 & 2 \\
\end{pmatrix}.
\]

b) (2 pts) Find the determinants of \(-\frac{1}{2}B\) and \(B^2\) (if you haven’t solved a), assume that \(\det(B) = 8\).

c) (2 pts) Find the cofactor \(C_{34}\) of \(B\). Hint: If you have solved a) then you have it already computed.

C.4. (6 pts) Show that the three polynomials

\[
b_1(t) = t^2, \quad b_2(t) = 1, \quad b_3(t) = (1 - t)^2
\]

form a basis in \(\mathbb{P}_2\), and find the coordinate vector of \(p(t) = t^2 + t + 1\) in this basis.

C.5. (6 pts) Find the range space and kernel of the linear map \(T : M_{2\times 2} \rightarrow M_{2\times 2}\) given by the formula

\[
T(A) = 3A + A^T.
\]
Version C  Midterm 2

C1 a) Yes, Theorem in book. Need to deduce closeness W_1 and W_2 which is obvious. E.g.

\[ u, v \in W_1, W_2 \Rightarrow \begin{cases} u + v \in W_1 \Rightarrow u + v \in W_1 \\ u, v \in W_2 \Rightarrow u + v \in W_2 \end{cases} \Rightarrow u + v \in W_1 \cap W_2 \]

Subspace property of \( W_1 \) and \( W_2 \) \( \Rightarrow \) subspace property of \( W_1 \cap W_2 \)

b) No, e.g. \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) has rank 1 but no \( \neq 0 \) diagonal elements in general, (# nonzero diagonal entries) # pivots = rank 1

c) No, \( B^{-1} \) would be correct

d) \( \det(A) = \frac{1}{2} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = \frac{5}{2} \) \( \Rightarrow \) No

e) \( \det(B^{-1}A^T) = \frac{\det(A^T)}{\det(B)} = \frac{\det(A)}{\det(B)} = -\frac{3}{2} \) \( \Rightarrow \) Yes

C2 a) Null-space is solution set of \( Ax = 0 \). REF \( \rightarrow \) free variables

\[ A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -9 & -7 & -5 \\ 0 & 9 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -9 & -7 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\( \dim \text{Null}(A) = 1 \)

out-of-variable \( x_3 \)

\( \text{Basis: Any 4 to solution vector of } Ax = 0 \)

\( x_4 = 0, x_3 = 1, x_2 = -\frac{2}{9}, x_1 = \frac{28}{9} - 2 = \frac{10}{9} \)

\( \text{Null}(A) = \text{Span} \{ \begin{bmatrix} 10 \\ -7 \\ 9 \\ 0 \end{bmatrix} \} \)

b) \( \text{rank}(3A^T) = \text{rank}(A^T) = \text{rank}(A) = 4 - \dim(\text{Null}(A)) = \# \text{pivots} = 3 \)

C3 a) Need to decide for \( \det(B) \neq 0 \):

\[ \det(B) = \begin{vmatrix} 0 & -1 & 5 & 4 \\ 2 & 4 & 3 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & 6 & 3 & -1 \end{vmatrix} = 2 \cdot C_{34} = -2 \begin{vmatrix} 0 & -1 & 5 \\ 2 & 3 & 1 \\ 1 & 6 & -3 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & 5 \\ 6 & -3 & -1 \\ 1 & 6 & 3 \end{vmatrix} = -4 \cdot 2.7 + 2 \cdot 16 = -76 \]
b) \[ \det \left( -\frac{1}{2} B \right) = \left( \frac{1}{2} \right)^4 \det(B) = -\frac{19}{4}, \quad \det(B^2) = (-76)^2 \]

C) \[ C_{34} = -38 \]

C4. Since \( \dim(P_2) = 3 \), we have to check for linear independence:

\[ c_1 b_1(t) + c_2 b_2(t) + c_3 b_3(t) = (c_2 + c_3) - 2c_3 t + (c_1 + c_3) t^2 = 0 \]

\[ \Rightarrow c_3 = 0 \Rightarrow c_2 = -c_3 = 0 \Rightarrow c_1 = -c_3 = 0 \]

or the spanning property

\[ c_1 b_1(t) + c_2 b_2(t) + c_3 b_3(t) = a_0 + a_1 t + a_2 t^2 \]

\[ \Rightarrow a_0 = c_2 + c_3, \quad a_1 = -2c_3, \quad a_2 = (c_1 + c_3) \]

\[ \Rightarrow c_3 = -\frac{1}{2} a_1, \quad c_2 = a_0 + \frac{1}{2} a_1, \quad c_1 = a_1 + \frac{1}{2} a_1 \]

In particular,

\[ 1 + t + t^2 = \frac{3}{2} b_1(t) + \frac{3}{2} b_2(t) - \frac{1}{2} b_3(t) \quad \Rightarrow \begin{bmatrix} 1 + t + t^2 \end{bmatrix}_B = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \]

C5. If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( T(A) = \begin{pmatrix} 4a & 3b+c \\ 3c+b & 4d \end{pmatrix} \).

Thus

\[ T(A) = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ 4a + 3c + d = 0 \quad 3c + b + 3c + d = 0 \Rightarrow a = b = c = d = 0 \]

Thus \( \ker(T) = \{ 0 \} \) is trivial subspace.

Since this is the one-to-one property of \( T \) and \( T \) maps \( M_{2 \times 2} \) into itself, we must have

\[ \text{Range}(T) = M_{2 \times 2} \] (onto property! )