A partition $\Pi$ of a set $S$ is a subdivision of $S$ into disjoint, nonempty subsets.

Example: $S = \{apple, bird, 1, 2, 3\}$

Example of partition:

$S = \{apple, 1\} \cup \{bird, 2\} \cup \{3, 8\}$

Example: $S = \mathbb{Z}$.

$\mathbb{Z} = \text{Even} \cup \text{Odd}$

Even = $\{2n: n \in \mathbb{Z}\}$

Odd = $\{2n + 1: n \in \mathbb{Z}\}$

Equivalence relation:

$S$ is a set. An equivalence relation on $S$ is a relation that holds between pairs of elements of $S$, written $a \sim b$, if $a, b \in S$, that's required to satisfy the following:

- **Transitivity**: if $a \sim b$ and $b \sim c$ then $a \sim c$
- **Symmetry**: if $a \sim b$ then $b \sim a$
- **Reflexivity**: $a \sim a$ for all $a, b, c \in S$
\( \forall a, b, c \in S. \)

\[
\text{Ex} \quad S = \mathbb{Z}. \text{ Define } a \sim b \text{ if the prime numbers that divide } a \text{ are the same as the prime numbers that divide } b. \\
\text{Ex} \quad p, q \text{ distinct primes} \\
p \sim p^2 \sim p^3 \\
p \sim p^q \sim p^{2q}.
\]

\[
\text{Ex} \quad \text{Conjugacy in a gp. } G \text{ a gp.} \\
\text{Define } a \sim b \text{ if } \exists g \in G \text{ s.t. } b = gag^{-1}. \\
\text{transitivity: } a \sim b \text{ and } b \sim c \\
\exists g_1 \in G \text{ s.t. } b = gag_1, \quad \exists g_2 \in G \text{ s.t. } c = g_2bg_2. \\
\Rightarrow c = g_2b g_2^{-1} = g_2(gag_1g_1^{-1})g_2^{-1} = (g_2g_1)(a)(g_2g_1)^{-1} \Rightarrow c \sim a. \\
\text{symmetry: } a \sim b \Rightarrow \exists g \in G \text{ s.t. } b = gag^{-1} \\
\Rightarrow a = g^{-1}bg = g'bg(g')^{-1} \Rightarrow b \sim a.
\]
reflexive \[ a \sim a \quad \forall a \in G \]

Ex. Cosets \[ G = aG, \quad H \leq G \quad \text{a subgp} \]
Define \[ a \sim b \iff \exists h \in H \quad \text{st.} \quad b = ah \]

This is an equiv rel

transitivity: \[ a \sim b \quad \text{and} \quad b \sim c \quad \text{then} \]
\[ b = ah_1 \quad \quad c = bh_2 \]
for some \( h_1 \in H \quad \quad \text{for some} \quad h_2 \in H \]
\[ \implies c = bh_2 = (ah_1)h_2 = a(h_1h_2) \]

\( H \), \( b/c \) \( H \) is a subgp.

symmetry \[ a \sim b \iff b = ah \quad \text{for some} \quad h \in H \]
\[ \implies a = bh^{-1} \quad \quad \text{But,} \quad h^{-1} \in H \quad b/c \quad H \text{ is closed under inverses} \]
\[ \implies b \sim a. \]

reflexivity \[ a \sim a \quad \text{b/c} \quad a = a.1, \quad 1 \in H. \]

Suppose \( S \) is a set w/ an equiv rel \( \sim \), and \( a \in S \).
Then the equivalence class of \( a \) in \( S \)
\[ S_a = \{ s \in S : a \sim s \} \]

Non-example \( S = \mathbb{R} \). Define \( a \sim b \) if \( a \leq b \).

transitive \[ a \leq b \quad \text{and} \quad b \leq c \quad \implies a \leq c \]
transitive: \( a \leq b \) and \( b \leq c \Rightarrow a \leq c \)

reflexive: \( a \leq a \).

not symmetric: \( a \leq b \not\Rightarrow b \leq a \).

Pf:\ An\ equivalence\ rels\ on\ a\ set\ \( S \)\ determines\ a partition\ of\ \( S \), via\ the\ equivalence\ classes.\ Conversely,\ a\ partition\ of\ \( S \)\ determines\ an\ equivalence\ rels\ \( \equiv \).

Pf: Suppose \( S \) is a disjoint union of nonempty subsets \( S_a \):
\[ S = \bigcup S_a \]

Define \( a \equiv b \) in \( S \), if \( a \) and \( b \) are both in \( S_a \) for some subset \( S_a \) of the partition.

This is an equivalence rels \( \equiv \) (check this).

Conversely, suppose given an equivalence rels \( \equiv \) on \( S \).
Let \( S_a = \{ s \in S : a \equiv s \} \) be the equiv class of \( a \).

Claim: The equiv classes partition \( S \):
\[ S = \bigcup S_a \]

i.e.
1) every elt of \( S \) is in some \( S_a \)
2) If \( S_a \cap S_b \not= \emptyset \), then \( S_a = S_b \)

\( S_a \subset S \).
1) is clear b/c if $a \in S$ then $a \in S_a$. This is because $a \sim a$ by reflexivity.

2) Suppose $c \in S_a \cap S_b$. Then $a \sim c$ and $b \sim c$.

   $\Rightarrow a \sim c$ and $c \sim b$ $\Rightarrow a \sim b$

   (by symmetry) (by transitivity)

WTP: $S_a = S_b$.

Suppose $x \in S_b$. Then $a \sim b$, $b \sim x \Rightarrow a \sim x$.

$\Rightarrow x \in S_a$.

$\Rightarrow S_b \subseteq S_a$.

$S_a \subseteq S_b$.

$\therefore S_a = S_b$.

Ex: Fix $n \in \mathbb{Z}$, $n \neq 0$. Define $a \sim b$ if $a - b \in \mathbb{Z}_n$. This is an equivalence reln b/c it is a special case of cosets for $G = \mathbb{Z}$, $H = \mathbb{Z}_n$ the subgroup.

Ex: Suppose $n = 10$. Then $a \sim b$ if $a$ and $b$ have the same final digit (if expressed in base 10).

Equivalence classes

- $1 \cdot 2$, $1 \cdot 2 + 1$, $1 \cdot 2 + 2$, ..., $10 \cdot 2 + 9$
Ex: \( S, T \) sets, \( f: S \to T \)

Then define and if \( f(a) = f(b) \).
This is an equivalence rel.

**Equiv. classes**  
If \( t \in T \), define

\[
\tilde{f}^{-1}(t) = \left\{ s \in S : f(s) = t \right\}
\]

Called "the fibers" of the map \( f \).

The equiv. classes are the non-empty fibers; they partition \( S \).