Math Finance, Homework 4, Winter 2006
Due Wednesday, February 1, 2006

1. Consider a Multi-period Binomial Model with $T = 2$, $r = 1/4$, $S_0 = 3$, $d = 1$, $u = 2$, and $p = 3/4$.

   a. Compute $B = (B_0, B_1, B_2)$.

   b. Compute the four possible values of (S_0, S_1, S_2) and the probability of each outcome.

   c. Draw a tree diagram representing the possible outcomes of (S_0, S_1, S_2) and the respective probability associated with each branch.

   d. Give an example of a self-financing trading strategy ϕ such that ϕ_2 is not constant.

   e. For the strategy named in 1.d., compute all possible values of $(V_0(\phi), V_1(\phi), V_2(\phi))$.

   f. Let X be a European put option with strike price \$5.00 and expiration time 2.

      i. Find $X(\omega_i)$ for $i = 1, 2, 3, 4$.

      ii. Find the replicating strategy ϕ^*

      iii. Find the manufacturing cost of X.

2. Modify exercise 2 by letting $T = 3$ and X be a European put option with strike price \$13 and expiration $T = 3$.

   a. Find $X(\omega_i)$ for $i = 1, 2, 3, 4$.

   b. Find the replicating strategy ϕ^*

   c. Find the manufacturing cost of X.

3. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathbf{P}(i) = 1/6$ for $1 \leq i \leq 6$, and $Y(i) = i^2$ for $1 \leq i \leq 6$. Also, let $\mathcal{P} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, $\mathcal{P}' = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$, $\mathcal{F} = \sigma(\mathcal{P})$, and $\mathcal{F}' = \sigma(\mathcal{P}')$.

   a. Find $\mathbf{E}[Y \mid \mathcal{F}]$

   b. Find $\mathbf{E}[Y \mid \mathcal{F}']$

   c. Evaluate
   $$\mathbf{E}[\mathbf{E}[Y \mid \mathcal{F}'] \mid \mathcal{F}].$$

   d. Evaluate
   $$\mathbf{E}[\mathbf{E}[Y \mid \mathcal{F}] \mid \mathcal{F}'].$$

   e. What property of conditional expectation does your calculation demonstrate?

4. Suppose that you play the same game three successive times, each game is independent of the others, and that you win each game with probability two-thirds. Let $\omega_1 = (L, L, L)$, $\omega_2 = (L, L, W)$, $\omega_3 = (L, W, L)$, $\omega_4 = (L, W, W)$, $\omega_5 = (W, L, L)$, $\omega_6 = (W, L, W)$, $\omega_7 = (W, W, L)$, and $\omega_8 = (W, W, W)$, where W stands for win, L stands for loss, and the ith coordinate stands for the outcome of the ith game. Suppose that you receive \$5 if you win the first game, \$2 if you win the second game, and \$4 if you win the last game. Suppose that you lose \$3 on any game lost. Let G be the net winnings after the three games have been played. Compute the following:

   a, the expected value of G;

   b. the conditional expected value of G given the outcome of the first game;

   c. the conditional expected value of G given the outcome of the first two games;

   d. the conditional expected value of G given the outcome of the first three games.