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Some Expansions of the Dual Basis of Z_λ

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Abstract.

A zigzag or ribbon is a connected skew diagram that contains no 2×2 boxes. Given a composition $\beta = (\beta_1, \dots \beta_k)$, we let Z_β denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \dots \beta_k$ reading from top to bottom. For each n, the set $\{Z_\lambda\}_{\lambda \vdash n}$ is a basis for Λ_n, the space of homogeneous symmetric functions of degree n. In this paper, we investigate some characteristics of the dual basis of $\{Z_\lambda\}_{\lambda \vdash n}$ relative to the Hall inner product which we denote by $\{DZ_\lambda\}_{\lambda \vdash n}$. We give a combinatorial interpretation for the coefficients in the expansion of DZ_λ in terms of the monomial symmetric functions $\{m_\mu\}_{\mu \vdash n}$ as a certain signed sum of paths in partition lattice under refinement. We shall show that in many cases, we can give an explicit formulas for the coefficients $a_{\mu,\lambda} = DZ_\lambda \mid_{m_\mu}$. In addition, we give explicit formulas for the coefficients that arise in the expansion of DZ_λ in terms of Schur functions for several special cases.

1. Introduction

Zigzag (or ribbon) Schur functions are the skew Schur functions with a ribbon shape and indexed by compositions. A composition $\beta = (\beta_1, \dots, \beta_n)$ of n, denoted $\beta \models n$, is a sequence of positive integers such that $\beta_1 + \beta_2 + \ldots + \beta_k = n$. We define a zigzag shape to be a connected skew shape that contains no 2 x 2 array of boxes. Given a composition $\beta = (\beta_1, \dots \beta_k)$, we let Z_β denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \dots \beta_k$ reading from top to bottom. For example Figure 1 shows the zigzag shape corresponding to the composition $(2, 3, 1, 4)$. As pointed out in [**2**], zigzag Schur functions

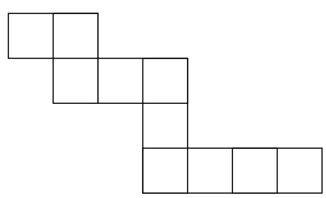

Figure 1. The ribbon shape corresponding to the composition $(2, 3, 1, 4)$, so that $s_{(2,4,4,7)/(1,3,3)} = Z_{(2,3,1,4)}$.

arise in many contexts. For example, the scalar product of any two zigzags gives the number of permutations σ such that σ and σ^{-1} have the associated pair of descent sets [**8**]. Zigzags can also be used to compute the number of permutations with a given descent set and cycle structure [**4**]. MacMahon [**7**] showed their coefficients in terms of the monomial symmetric functions count descents in permutations with repeated elements. They also show up as sl_n-characters of the irreducible components of the Yangian representation in level 1 modules of $\hat{sl}_n$[**5**].

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The zigzag Schur functions corresponding to partitions of n form a basis of Λ_n, the space of homogeneous symmetric functions of degree n, and therefore they have a dual basis relative to the Hall inner product which we denote by $\{DZ_\lambda\}_{\lambda\vdash n}$. We shall call DZ_λ the dual zigzag symmetric function corresponding to λ. The basis $\{DZ_\lambda\}_{\lambda\vdash n}$ has not been extensively studied. Let $\{m_\lambda\}_{\lambda\vdash n}$ denote the set of monomial symmetric functions, $\{h_\lambda\}_{\lambda\vdash n}$ denote the set of homogeneous symmetric functions, and $\{s_\lambda\}_{\lambda\vdash n}$ denote the set of Schur functions. The main result of this paper is to give a combinatorial interpretation to coefficients that arise in the expansion of DZ_λ in terms of the monomial symmetric functions. That is, we shall give a combinatorial interpretation to $a_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu a_{\mu,\lambda} m_\mu$$

Our main result will show that $a_{\mu,\lambda}$ is a signed sum over the weights of certain paths in the lattice of partitions under refinement. In general such a signed sum is complicated, but we will show that in many special cases, we can explicitly evaluate this sum. For example, we will show that $a_{\mu,(n)} = 1$ for all μ so that

$$DZ_{(n)} = \sum_\mu m_\mu = s_{(n)}$$

where $s_{(n)}$ is the Schur function associated to the partition with only one part.

Once we have found our combinatorial interpretation for $a_{\mu,\lambda}$, we can obtain combinatorial interpretations for the expansion of DZ_λ in terms of any other basis by using the combinatorial interpretations of the transition matrices between bases of symmetric functions found in [**1**]. In particular, we shall use this method to find explicit values for $b_{\mu,\lambda}$ where $DZ_\lambda = \sum_\mu b_{\mu,\lambda} s_\mu$ for certain special cases.

In Section 2, we shall review the necessary background for symmetric functions and the combinatorial interpretation of the entries of the transition matrices between various bases of symmetric functions that we shall need. In particular, we shall use the Jacobi-Trudi identity to give a combinatorial interpretation of the coefficients $Z_\lambda \mid_{h_\mu}$. In Section 3, we outline the proof of our main theorem and give some examples of the computations involved in computing the coefficients $a_{\mu,\lambda}$. In Section 4, we give closed forms for several of the coefficients, independent of the size of the composition. In Section 5, we give the expansion of several dual zigzags in terms of Schur functions which are independent of the size of the partition.

2. Background Information

For $n \in \mathbb{N}$, $\lambda \vdash n$ denotes that λ is a partition of n, meaning that $\lambda = (\lambda_1, \ldots, \lambda_k)$, where the λ_i's are in weakly decreasing order, and $\lambda_1 + \ldots + \lambda_k = n = |\lambda|$. We will notate the number of parts of λ, k, as $l(\lambda)$.

We can construct a pictorial representation of λ using squares, or cells, called the Ferrer's diagram, F_λ. F_λ is the diagram with λ_i left-justified squares in the i^{th} row, (with row 1 as the bottom row). We can also construct the difference of the Ferrer's diagrams for two partitions λ and μ, as long as F_λ contains F_μ. More formally, we say that $\mu \subseteq \lambda$ if and only if $l(\mu) \leq l(\lambda)$ and $\mu_{l(\mu)-i} \leq \lambda_{l(\lambda)-i}$ for every $0 \leq i \leq l(\mu)$. Then we denote $F_\lambda - F_\mu$ as $F_{\lambda/\mu}$, for the skew shape λ/μ. Any F_λ can be expressed as a skew Ferrer's diagram by letting $\mu = \emptyset$ so that $F_{\lambda/\mu} = F_{\lambda/\emptyset} = F_\lambda$. For λ a partition of n, a *column-strict tableau* T of shape λ

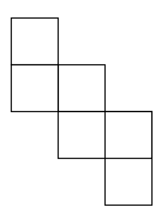

FIGURE 2. The skew Ferrer's diagram of $(1,2,3,3)/(1,2)$

is any filling of F_λ with natural numbers such that entries in each row are weakly increasing from left to right, and entries in each column are strictly increasing from bottom to top. We define the content of T to be $c(T) = (\alpha_1, \alpha_2, \ldots,)$ where α_i is the number of times that i occurs in T. If λ is a partition denoted by $\lambda = (\lambda_1, \ldots, \lambda_l) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, where m_i is the number of parts of λ equal to i, then we define $z_\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \ldots m_n!$.

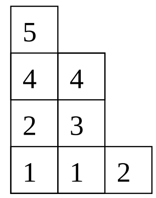

FIGURE 3. A column strict tableau of shape $(1,2,2,3)$ and content $(2,2,1,2,1)$

There are six standard bases of the space of homogeneous symmetric functions of degree n, $\Lambda_n(x)$, which are generally notated as: $\{m_\lambda\}_{\lambda\vdash n}$ (the monomial symmetric functions), $\{h_\lambda\}_{\lambda\vdash n}$ (the complete homogeneous symmetric function), $\{e_\lambda\}_{\lambda\vdash n}$ (the elementary symmetric functions), $\{p_\lambda\}_{\lambda\dashv n}$ (the power sum symmetric functions), $\{f_\lambda\}_{\lambda\dashv n}$ (the forgotten symmetric functions) and $\{s_\lambda\}_{\lambda\vdash n}$ (the Schur functions), where λ is a partition of n [**1**].

The Hall inner product is a standard scalar product on the space of homogeneous symmetric functions $\Lambda_n(x)$, which is defined by:

$$\langle m_\lambda, h_\mu\rangle = \delta_{\lambda,\mu}$$

where

$$\delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

Under this scalar product, $\{s_\lambda\}_{\lambda\vdash n}$ and $\{p_\lambda/\sqrt{z_\lambda}\}_{\lambda\vdash n}$ are known to be self-dual, and $\{e_\lambda\}_{\lambda\vdash n}$ and $\{f_\lambda\}_{\lambda\dashv n}$ are dual.

When given two bases of $\Lambda_n(x)$, $\{a_\lambda\}_{\lambda\vdash n}$ and $\{b_\lambda\}_{\lambda\vdash n}$, we first fix some ordering of the partitions of n, e.g. the lexicographic order, and then we may think of the bases as row vectors, $\langle a_\lambda\rangle_{\lambda\vdash n}$ and $\langle b_\lambda\rangle_{\lambda\vdash n}$. We can define the transition matrix $M(a,b)$ that transforms the basis $\langle a_\lambda\rangle_{\lambda\vdash n}$ into the basis $\langle b_\lambda\rangle_{\lambda\vdash n}$ by:

$$\langle b_\lambda\rangle_{\lambda\vdash n} = \langle a_\lambda\rangle_{\lambda\vdash n} M(a,b)$$

The $(\lambda,\mu)th$ entry of $M(a,b)$ is given by the equation

$$b_\lambda = \sum_{\mu\vdash n} a_\mu M(a,b)_{\mu,\lambda}.$$

The main goal of this paper is to find a combinatorial interpretation of the entries of $M(m,DZ)$. That is, we want find a combinatorial interpretation for the $a_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_{\mu} a_{\mu,\lambda} m_\mu$$

In addition, we shall also be interested in finding a combinatorial interpretation for the entries of $M(s,DZ)$. That is, we want to find a combinatorial interpretation for $b_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_{\mu} b_{\mu,\lambda} s_\mu$$

We now give examples of the expansion of $\{DZ_\lambda\}_{\lambda\vdash n}$ when $n=6$. We first give the expansion of DZ_λ in terms of the monomial symmetric functions, when $\lambda\vdash 6$.

$$\begin{aligned}
DZ_{(6)} &= m_6+m_{5,1}+m_{4,2}+m_{4,1,1}+m_{3,3}+m_{3,2,1}+m_{3,1,1,1}\\
&\quad +m_{2,2,2}+m_{2,2,1,1}+m_{2,1,1,1,1}+m_{1,1,1,1,1,1}\\
DZ_{(5,1)} &= m_{5,1}+m_{4,1,1}+m_{3,2,1}+2m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}-2m_{1,1,1,1,1,1}\\
DZ_{(4,2)} &= m_{4,2}+m_{4,1,1}+2m_{2,2,2}+m_{2,2,1,1}+2m_{2,1,1,1,1}+7m_{1,1,1,1,1,1}\\
DZ_{(4,1,1)} &= m_{4,1,1}+m_{3,1,1,1}+m_{2,2,1,1}+3m_{2,1,1,1,1}+8m_{1,1,1,1,1,1}\\
DZ_{(3,3)} &= m_{3,3}+m_{3,2,1}+m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}\\
DZ_{(3,2,1)} &= m_{3,2,1}+2m_{3,1,1,1}+m_{2,2,1,1}+m_{2,1,1,1,1}-3m_{1,1,1,1,1,1}\\
DZ_{(3,1,1,1)} &= m_{3,1,1,1}+m_{2,1,1,1,1}+m_{1,1,1,1,1,1}\\
DZ_{(2,2,2)} &= m_{2,2,2}+m_{2,2,1,1}+2m_{2,1,1,1,1}+5m_{1,1,1,1,1,1}\\
DZ_{(2,2,1,1)} &= m_{2,2,1,1}+3m_{2,1,1,1,1}+9m_{1,1,1,1,1,1}\\
DZ_{(2,1,1,1,1)} &= m_{2,1,1,1,1}+5m_{1,1,1,1,1,1}\\
DZ_{(1,1,1,1,1,1)} &= m_{1,1,1,1,1,1}
\end{aligned}$$

The case where $n=6$ is the last case that can be comfortably handled by hand, for cases $n=7$ through $n=11$, we used a computer to assist with the calculations. Because the combinatorics of the transition matrices between standard bases is known, our method can also be used to find the expansion in terms of the other symmetric function bases. Given two partitions λ and μ of n, we say that λ is a refinement of μ, written $\lambda\le_r\mu$, if λ can be created from μ by splitting some of the parts of μ into pieces. We give a reference for expansion in terms of the Schur functions. We have that:

$$DZ_\lambda=\sum_{\mu\le_r\lambda}a_{\mu,\lambda}m_\mu.$$

But the transition matrix between the monomial symmetric functions and the Schur functions is:

$$M(s,m)_{\lambda\mu}=K^{-1}_{\mu,\lambda},$$

where $||K^{-1}_{\mu,\lambda}||$ is the inverse Kostka matrix. So

$$\begin{aligned}
DZ_\lambda &= \sum_{\mu\le_r\lambda}a_{\mu,\lambda}\sum_\gamma s_\gamma K^{-1}_{\mu,\gamma}\\
&= \sum_\gamma s_\gamma\sum_{\mu\le_r\lambda}a_{\mu,\lambda}K^{-1}_{\mu,\gamma}.
\end{aligned}\qquad(2.1)$$

The expansion of DZ_λ in terms of the Schur functions, when $\lambda\vdash 6$, is given below.

$$\begin{aligned}
DZ_{(6)} &= s_6\\
DZ_{(5,1)} &= s_{5,1}-s_{4,2}+s_{3,2,1}-s_{2,2,2}-s_{2,2,1,1}\\
DZ_{(4,2)} &= s_{4,2}-s_{3,3}-s_{3,2,1}+2s_{2,2,2}+s_{2,2,1,1}\\
DZ_{(4,1,1)} &= s_{4,1,1}-s_{3,2,1}+s_{2,2,2}+s_{2,2,1,1}\\
DZ_{(3,3)} &= s_{3,3}-s_{2,2,2}\\
DZ_{(3,2,1)} &= s_{3,2,1}-2s_{2,2,2}-s_{2,2,1,1}\\
DZ_{(3,1,1,1)} &= s_{3,1,1,1}-s_{2,2,1,1}\\
DZ_{(2,2,2)} &= s_{2,2,2}\\
DZ_{(2,2,1,1)} &= s_{2,2,1,1}\\
DZ_{(2,1,1,1,1)} &= s_{2,1,1,1,1}\\
DZ_{(1,1,1,1,1,1)} &= s_{1,1,1,1,1,1}
\end{aligned}$$

We now examine some of the combinatorics behind the transitions matrices for the standard bases, which will be useful to us later. In particular, we will need to use the expansion of skew-Schur functions as h_λ [**3**]

To do so, we introduce rim hooks, special rim hooks and special rim hook tabloids. More detail is given in [**3**] where they are used to give a combinatorial interpretation of the inverse Kostka matrix.

For a partition λ, consider the Ferrer's diagram F_λ. A rim hook of λ is a sequence of cells, h, along the northeast boundary of F_λ such that any two consecutive cells in h share an edge and if we remove h from F_λ, we are left with the Ferrer's diagram of another partition. More generally, h is a rim hook of a skew shape λ/μ if h is a rim hook of λ which does not intersect μ.

A rim hook tableau of shape λ/ν and type μ, T, is a sequence of partitions

$$T = (\nu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda),$$

such that for each $1 \leq i \leq k, \lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ of size μ_i.

We define the sign of a rim hook $h_i = \lambda^{(i)}/\lambda^{(i-1)}$ to be

$$sgn(h_i) = (-1)^{r(h_i)-1}$$

where $r(h_i)$ is the number of rows that h_i occupies. The sign of a rim hook tableau T is

$$sgn(T) = \Pi_{i=1}^{k} sgn(h_i).$$

Given two partitions $\lambda^{(i-1)} \subset \lambda^{(i)}$, we say that $\lambda^{(i-1)}/\lambda^{(i)}$ is a special rim hook if $\lambda^{(i-1)}/\lambda^{(i)}$ is a rim hook of $\lambda^{(i)}$ and $\lambda^{(i-1)}/\lambda^{(i)}$ contains a cell from the first column of λ.

A special rim hook tabloid (SRHT) T of shape λ/μ is a sequence of partitions

$$T = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda),$$

such that for each $1 \leq i \leq k, \lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook of $\lambda^{(i)}$. We have a partition determined by the integers $|\lambda^{(i)}/\lambda^{(i-1)}|$ which is the type of the special rim hook tabloid T. Notice that we have used used the word *tabloid* instead of *tableau* in order to highlight there is no implicit order in the size of each successive special rim hook, unlike rim hook tableau.

The sign of a special rim hook, $h_i = \lambda^{(i)}/\lambda^{(i-1)}$, and the sign of a special rim hook tabloid T, are defined as we did for rim hooks and rim hook tableaux. We show an example of a special rim hook tabloid of type $(6, 5, 4, 2)$ and shape $(5, 4, 4, 3, 1)$ in Fig 4. For λ and μ partitions of n, Eğecioğlu and Remmel [**3**] show that

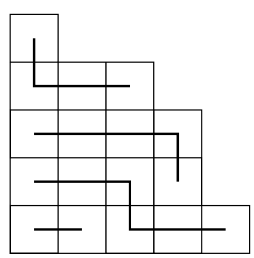

FIGURE 4. A special rim hook tabloid of shape (5,4,4,3,1) and type (6,5,4,2)

$$s_{\lambda/\mu} = \sum_{\mu} K^{-1}_{\mu,\lambda/\nu} h_\mu$$

where

$$K^{-1}_{\mu,\lambda/\nu} = \sum_{T \text{ is a SRHT of shape } \lambda \text{ and type } \mu} sgn(T).$$

We note that is also the case that

$$M(s,m)_{\lambda\mu} = K^{-1}_{\mu,\lambda}.$$

Recall that we defined a composition β on n, denoted $\beta \models n$, as a list of positive integers $(\beta_1, \beta_2, \ldots, \beta_k)$ such that $\beta_1 + \beta_2 + \ldots + \beta_k = n$. We call each of the β_i a component of β, and we say that β has length $l(\beta) = k$ and size $|\beta| = n$. From this definition, we can see that β is a partition if each of its components are weakly decreasing. For any composition β, we define the partition determined by β, $\lambda(\beta)$, which we obtain by reordering the components of β in weakly decreasing order, e.g. $\lambda(2, 8, 9, 4) = (9, 8, 4, 2)$. Notice that two compositions β, γ can determine the same partition, e.g. if $\beta = (2,8,9,4)$ and $\gamma = (2,9,8,4)$, then $\lambda(2,8,9,4) = (9,8,4,2) = \lambda(2,9,8,4)$.

There is a natural correspondence between a composition $\beta \models n$ and a subset $Set(\beta) \subseteq [n-1] = \{1,2,\ldots,n-1\}$ where

$$Set(\beta) = \{\beta_1, \beta_1+\beta_2, \beta_1+\beta_2+\beta_3, \ldots, \beta_1+\beta_2+\ldots+\beta_{k-1}\}.$$

We can also reverse this process so that for any subset $S = \{j_1, j_2, \ldots, j_{k-1}\} \subseteq [n-1]$, we can find the composition $\beta_n(S) \models n$ where

$$\beta_n(S) = (j_1, j_2 - j_1, \ldots, n - j_{k-1}).$$

For example, the composition $\beta = (2,9,8,4)$ has $Set(\beta) = \{2,11,19\} \subseteq [22]$. We also define $shape_n(S) = \lambda(\beta_n(S))$. For example if $S = \{2,5,6,10\}$ and $n = 11$, then $\beta_{11}(S) = (2,3,1,4,1)$, and $shape_{11}(S) = (4,3,2,1,1)$.

Recall that if we are given two partitions λ and μ of n, we say that λ is a refinement of μ, written $\lambda \leq_r \mu$, if λ can be created from μ by splitting some of the parts of μ into pieces. For example, $(4,2,1,1,1,1) \leq_r (5,3,2)$ since we can split 5 into $4+1$ and 3 into $1+1+1$ to obtain λ. The cover relations in the lattice of partitions of n under refinement arise by starting with a partition λ and combining two of the parts of λ to get μ. Similarly, given two compositions β and γ, we say that β is a refinement of γ, denoted $\beta \leq_r \gamma$, if by adding together adjacent components of β, we can obtain γ. For example, $421131 \leq_r 4314$, meaning $\gamma = 421131$ is a refinement of $\beta = 4314$. If we only add together a single pair of adjacent components of a partition β to get γ, then we will say that γ cover β.

We now create some notation to bridge our transition from partitions to compositions and subsets of $[n-1]$.

$$[\mu \to \lambda] = |\{S \subseteq Set(\mu) : shape_n(S) = \lambda\}|$$

As an example, let's calculate $[(2,1^4) \to (4,2)]$. Note that $Set(2,1^4) = \{2,3,4,5\}$. We want to find $|\{S \subseteq \{2,3,4,5\} : shape_6(S) = (4,2)|$. The only two subsets of $\{2,3,4,5\}$ that have the appropriate shape are $\{2\}$ and $\{4\}$, so $[(2,1^4) \to (4,2)] = 2$.

3. Main result

$\{Z_\lambda : \lambda \vdash n\}$ has dual basis $\{DZ_\lambda : \lambda \vdash n\}$. We give an interpretation for $a_{\mu,\lambda}$, where

$$DZ_\lambda = \sum_{\mu \leq_r \lambda} a_{\mu,\lambda} m_\mu.$$

THEOREM 3.1.

$$a_{\mu,\lambda} = (-1)^{l(\mu)-l(\lambda)} \sum_{P \in Path(\mu,\lambda)} [P](-1)^{l(P)}$$

where $P = (\mu_0, \mu_1, \ldots, \mu_k)$,$\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda$ *and* $[P] = [\mu_0 \to \mu_1][\mu_1 \to \mu_2]\ldots[\mu_{k-1} \to \mu_k]$.

We will now demonstrate how this theorem can be used to calculate $a_{\mu,\lambda}$ in the case where $\mu = (1^6)$ and $\lambda = (3,2,1)$. Since our theorem says we sum over all paths in the lattice of partitions under refinement, (which from here on we will refer to as the refinement lattice), we give the relevant portion of the refinement lattice in Fig. 5. First we give several examples of how to calculate $[\alpha \to \beta]$. Recall that $Set(\lambda) = \{\lambda_1, \lambda_1+\lambda_2, \ldots, \lambda_1+\ldots+\lambda_{k-1}\}$. We first calculate $[(1^6) \to (2,1^4)]$, which is equal to $|\{S \subset Set(1^6) : shape_6(S) = (2,1^4)\}|$. $Set(1^6) = \{1,2,3,4,5\}$, and the subsets $\{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}$, and $\{1,2,3,4\}$ all have shape equal to $(2,1^4)$. Therefore $[(1^6) \to (2,1^4)] = 5$. Similarly $[(1^6) \to (3,2,1)] = 6$ since $\{3,4\}, \{3,5\}, \{2,5\}, \{2,3\}, \{1,3\}, \{1,4\}$ are the only subsets T of $Set(1^6) = \{1,2,3,4,5\}$ such that $shape_6(T) = (3,2,1)$. Finally we calculate $[(2,1^4) \to (3,1^3)]$. In this case, $Set(2,1^4) = \{2,3,4,5\}$ and the only subset T of $Set(2,1^4)$such that $shape_6(T) = (3,1^3)$ is $\{3,4,5\}$. Thus $[(2,1^4) \to (3,1^3)] = 1$.

From these three examples we see that a considerable amount of work goes into calculating $[\alpha \to \beta]$ for every possibility in our refinement lattice. In Table 1, we give the values needed to calculate $[\alpha \to \beta]$ for all pairs in the refinement lattice from (1^6) to $(3,2,1)$.

$[1^6 \to 2,1^4] = 5$	$[2,1^4 \to 3,1^3] = 1$	$[3,1^3 \to 3,2,1] = 2$
$[1^6 \to 3,1^3] = 4$	$[2,1^4 \to 2^2,1^1] = 3$	$[2^2,1^2 \to 3,2,1] = 1$
$[1^6 \to 2^2,1^2] = 6$	$[2,1^4 \to 3,2,1] = 4$	
$[1^6 \to 3,2,1] = 6$		

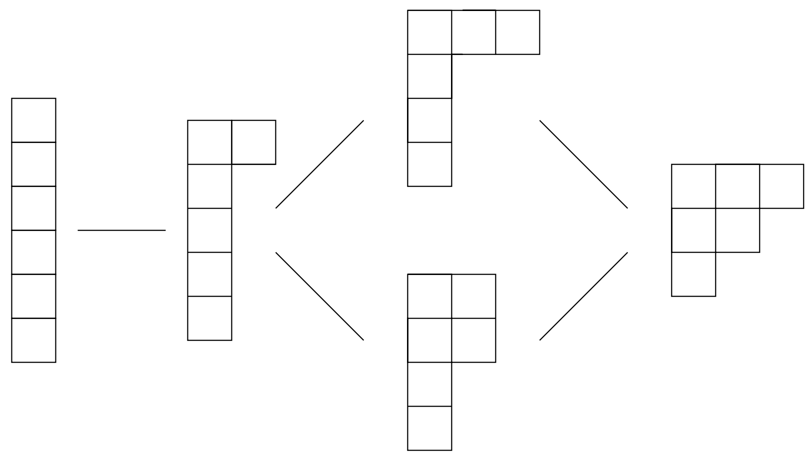

FIGURE 5. The refinement lattice from (1,1,1,1,1,1) to (3,2,1).

Now that we have calculated those values, we can calculate the weights of each possible path in our refinement lattice. These paths and weights are listed in Table 2. The length of the path will be used in our calculation of $a_{\mu,\lambda}$.

Possible Paths	Length of Path	Weight of Path
$[(1^6) \to (3,2,1)]$	1	6
$[(1^6) \to (3,1^3)][(3,1^3) \to (3,2,1)]$	2	8
$[(1^6) \to (2^2,1^2)][(2^2,1^2) \to (3,2,1)]$	2	6
$[(1^6) \to (2,1^4)][(2,1^4) \to (3,2,1)]$	2	20
$[(1^6) \to (2,1^4)][(2,1^4) \to (3,1^3)][(3,1^3 \to (3,2,1)]$	3	10
$[(1^6) \to (2,1^4)][(2,1^4) \to (2^2,1^2)][(2^2,1^2 \to (3,2,1)]$	3	15

Finally, we combine this information:

$$\begin{aligned} a_{(1^6),(3,2,1)} &= (-1)^{6-3} \sum_{P \in Path((1^6),(3,2,1))} -1^{l(P)}[P] \\ &= -1^3(-1^1(6) + -1^2(8+6+20) + -1^3(10+15)) \\ &= -(-6+34-25) \\ &= -3. \end{aligned}$$

We should note that although this first example took many calculations, we have now done almost all of the work for several other coefficients for $n = 6$, since our refinement lattice contains several smaller refinement lattices corresponding to other pairs of partitions. In addition, we will see later that we have now already calculated an infinite number of coefficients for $n > 6$.

Outline of proof:

We start by expanding the zigzag Schur functions in terms of the homogeneous symmetric functions $\{h_\lambda\}_{\lambda \vdash n}$ derived from the Jacobi-Trudi by Egecioglu and Remel [**3**],

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j}) = \sum_\nu K^{-1}_{\nu,\lambda/\mu} h_\mu$$

where $h_0 = 1$ and $h_k = 1$ if $k < 0$. Applying it specifically to zigzag Schur functions and using compositions as subscripts, we can show that for any $\alpha \models n$,

$$Z_\alpha = (-1)^{l(\alpha)} \sum_{\beta \leq_r \alpha} (-1)^{l(\beta)} h_{\lambda(\beta)}.$$

Alternatively,

$$Z_\lambda = h_\lambda + \sum_{T\subset Set(\lambda)} (-1)^{|Set(\lambda)-T|} h_{\lambda(\beta(T))}.$$

Recall that $[\mu \to \lambda] = |\{S \subseteq Set(\mu) : shape_n(S) = \lambda\}|$. So

$$Z_\lambda = h_\lambda + \sum_{\lambda \leq_r \alpha} (-1)^{l(\lambda)-l(\alpha)} [\lambda \to \alpha] h_\alpha.$$

To prove that $\{DZ_\lambda\}_{\lambda\vdash n}$ is the dual basis of $\{Z_\lambda\}_{\lambda\vdash n}$, we need to show that

$$\sum_\gamma Z_\gamma(x) DZ_\gamma(y) = \sum_\gamma h_\gamma(x) m_\gamma(y)$$

or, equivalently,

$$\sum_\gamma Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x)m_\mu(y)} = \delta_{\lambda,\mu}.$$

Given our expansion of $Z_\lambda(x)$ in terms of $h_\lambda(x)$'s and the fact that $\langle h_\lambda(x), m_\mu(x)\rangle = \delta_{\lambda,\mu}$, we can then show that

$$\sum_\gamma Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x)} = \sum_{\alpha\leq_r \lambda} (-1)^{l(\alpha)-l(\lambda)} [\alpha \to \lambda] m_\alpha(y)$$

and

$$\begin{aligned} \sum_\gamma Z_\gamma(x) DZ_\gamma(y)|_{h_\lambda(x)m_\mu(y)} &= \sum_{\mu\leq_r\alpha\leq_r\lambda} (-1)^{l(\alpha)-l(\lambda)} [\alpha\to\lambda] a_{\mu,\alpha} \\ &= \sum_{\mu\leq_r\alpha\leq_r\lambda} \sum_{P\in Paths(\mu,\alpha)} [P][\alpha\to\lambda] \\ &= \sum_{Q\in Paths(\mu,\lambda)} sgn(Q)[Q] \end{aligned}$$

Thus we need only show that $\sum_{Q\in Paths(\mu,\lambda)} sgn(Q)[Q] = \delta_{\lambda,\mu}$. This we do by defining a weight preserving involution on the set of paths in the lattice of partitions under refinement.

4. Coefficients of the expansion of Dual Zigzags as monomials

We saw in our example calculating $a_{(1^6),(3,2,1)}$ how difficult and time-consuming it can be to find these coefficients. However, in a number of special cases, this sum of paths leads to a closed form.

In particular, if the indices of the coefficient we are looking for are only one step apart in the refinement lattice, then it is no longer a sum over paths, since there is only one path, and the formula for the coefficient simplifies.

(1) If λ and μ are a cover relation in the refinement lattice, then $a_{\mu,\lambda} = [\mu \to \lambda]$.

(2) Similarly, we can show that $a_{\mu,\mu} = 1$ for all μ.

(3) For any μ such that $\mu \vdash n$, $a_{\mu,(n)} = 1$, so that we find $DZ_{(n)} = \sum_\mu m_\mu = s_{(n)}$.

We outline a proof of (3) by induction on the length of the refinement.

$$\begin{aligned} a_{\mu,(n)} &= (-1)^{l(\mu)-1} \sum_{P\in Path(\mu,(n))} (-1)^{l(P)}[P] \\ &= (-1)^{l(\mu)-1} \sum_{\mu<_r\alpha<_r(n)} (-1)[\mu\to\alpha] \sum_{P\in Path(\alpha,(n))} (-1)^{l(P)}[P] \\ &\quad +(-1)^{l(\mu)-1}(-1)[\mu\to(n)] \end{aligned}$$

Our inductive assumption that $a_{\alpha,(n)} = 1$ gives that $\sum_{P\in Path(\alpha,(n))} (-1)^{l(P)}[P] = (-1)^{l(\alpha)-1}$.

$$a_{\mu,(n)} = (-1)^{l(\mu)-1}\Big(\sum_{\mu<_r\alpha<_r(n)} (-1)[\mu\to\alpha](-1)^{l(\alpha)-1}\Big) + (-1)^{l(\mu)-1}(-1)[\mu\to(n)]$$

But if we think about the definition of $[\mu \to \alpha]$, now we are summing over all possibilities of ways to remove at least one element from $Set(\mu)$ so

$$\begin{aligned} a_{\mu,(n)} &= (-1)^{l(\mu)-1} \sum_{\emptyset \subsetneq S \subseteq Set(\mu)} (-1)^{|Set(\mu)|-|S|} \\ &= (-1)^{l(\mu)-1}((\sum_{\emptyset \subseteq S \subseteq Set(\mu)} (-1)^{|Set(\mu)|-|S|}) - (-1)^{|Set(\mu)|}) \end{aligned}$$

But $\sum_{S \subseteq Set(\mu)} (-1)^{|S|} = 0$. So

$$a_{\mu,(n)} = (-1)^{l(\mu)}(0 - (-1)^{|Set(\mu)|}) = (-1)^{l(\mu)}((-1)^{|Set(\mu)|+1})$$

But $|Set(\mu)| + 1 = l(\mu)$, so $a_{\mu,(n)} = 1$.

Other results can be found using careful examination of the lattice of refinement.The proofs of some of the below items are very straightforward; for example, the proof of item 4 is plain because the relevant portion of the refinement lattice contains only two shapes, and our composition $\{1, 2, \dots, k-1\}$ has shape $(2, 1^{k-2})$ when we remove any element, and there are $k-1$ ways to do so. The proofs of other items are more difficult, for example item (10) relies on opposite-signed binomial coefficients.

Results with $\mu = (1^k)$ and $\lambda = (b, 1^{k-b})$ for $b = 1, 2, \dots, 7$:

(4) $a_{(1^k),(2,1^{k-2})} = k - 1$

(5) $a_{(1^k),(3,1^{k-3})} = 1$

(6) $a_{(1^k),(4,1^{k-4})} = \binom{k-1}{2} - 2$

(7) $a_{(1^k),(5,1^{k-5})} = -\frac{1}{2}(k-1)(k-4) + 3$

(8) $a_{(1^k),(6,1^{k-6})} = \frac{1}{6}(k^3 - 3k^2 - 16k - 6)$

(9) $a_{(1^k),(7,1^{k-7})} = -\frac{1}{3}(k)(k+1)(k-7) + 1$

Other results useful for Schur function expansions:

(10) $a_{(1^k),(3^2,1^{k-6})} = 0$

(11) $a_{(1^k),(3,2,1^{k-5})} = -\frac{1}{2}k(k-5)$

(12) $a_{(2,1^{k-2}),(4,1^{k-4})} = k - 3$

(13) $a_{(2,1^{k-2}),(3,2,1^{k-5})} = 1$

Theorem 4.1. *If $d \neq 1$,*

$$a_{(2^c,1^b),(2^{c+d},1^{b-2d})} = \frac{b(b-1)\cdots(b-d+2)}{d!}(b-2d+1)$$

If $d = 1$, the product on the right is not defined, however the $d = 1$ case is covered further above when $c = 0$.

Finding the value of one coefficient also tells us the value of an infinite number of other coefficients. Let $\mu = (\mu_1, \dots, \mu_j)$. Define $k\mu$ to be the partition obtained when each part of μ is multiplied by k, so that $k\mu = (k\mu_1, \dots, k\mu_j)$.

Theorem 4.2. *For all $j \in \mathbb{N}$,*

$$a_{\mu,\lambda} = a_{k\mu,k\lambda}.$$

The proof of this theorem follows from an obvious bijection between paths in the refinement lattice of (μ, λ) to paths in the refinement lattice of $(k\mu, k\lambda)$. In particular, if we apply Theorem 3 to Theorem 2 we obtain another theorem, and an entire new list of results from above.

Theorem 4.3. *Given $\mu = (\mu_1, \dots, \mu_s)$ and $\lambda = (\lambda_1, \dots, \lambda_t)$. Then for any j such that $1 \leq j < min(\mu_s, \lambda_t)$,*

$$a_{\mu,\lambda} = a_{(\mu_1,\dots,\mu_s,j),(\lambda_1,\dots,\lambda_t,j)}.$$

The proof of this theorem follows from examining the compositions and noticing that we must always have the last element of the composition in our subsets S in order for $shape_n(S)$ to match $(\lambda_1, \dots, \lambda_t, k)$. This theorem works in "both directions", so to speak. Knowledge of a coefficient with $\mu \vdash n$ and $\lambda \vdash n$ both with smallest part larger than 1 gives another coefficient for larger n. Conversely, knowledge of a coefficient with μ and λ with identical unique smallest part gives another coefficient for smaller n by removing that smallest part from both μ and λ.

Combining Theorem 4.2 and Theorem 4.3 enables us to calculate many coefficients. Starting with $a_{\mu,\lambda}$, we can first multiply each part by k, then add smaller parts on the end, and so on.

5. The expansion of Dual Zig Zags as Schur functions

Our method of expansion in terms of Schur functions in section 2 is useful not only in calculating particular expansions, but can also be used to make general statements independent of the size of λ.

We can use the fact that $b_{\mu,\lambda}$ can be expressed as $a_{\mu,\lambda}$ to prove further results, in particular that:

(1)$DZ_{(1^n)} = s_{1^n}$

(2)$DZ_{(n)} = s_{(n)}$

(3)$DZ_{(2^k,1^{n-2k})} = s_{(2^k,1^{n-2k})}\ \forall\ k$

(4)$DZ_{(3^k,1^{n-3k})} = s_{(3,1^{n-3})} - s_{(2^2,1^{n-4})}\ \forall\ k$

(5)$DZ_{(3,2,1^{n-5})} = s_{(3,2,1^{n-5})} - 2s_{(2^3,1^{n-6})} - s_{(2^2,1^{n-4})}$

(6) $DZ_{(4,1^{n-4})} = s_{(4,1^{n-4})} - s_{(3,2,1^{n-5})} + s_{(2^2,1^{n-4})} + s_{(2^3,1^{n-6})}$

The proof of (1) was given above. The proofs of the others involve using the combinatorial interpretation of the coefficients that arise in (2.1) and defining some appropriate involutions to simplify the sum.

6. Conclusions and Further Research

In this paper we have demonstrated a combinatorial interpretation of the coefficients in the expansion of DZ_λ in terms of the monomial symmetric functions, as well as several formulas for calculating them. In addition, we found several formulas for the DZ_λ themselves in terms of the Schur functions. We have exemplified that in general these are hard to calculate; however, once we have calculated one coefficient, we have calculated an infinite number of coefficients.

There are many unanswered questions in this area. Of particular interest is what happens when we apply the omega transformation to DZ_λ. That is, recall the $\omega : \Lambda_n \to \Lambda_n$ is defined by the fact for all $\lambda \vdash n$, $\omega(h_\lambda) = e_\lambda$. Then the question is: can we write $\omega(DZ_\lambda)$ in terms of $\{Z_\lambda\}_{\lambda\vdash n}$ or $\{DZ_\lambda\}_{\lambda\vdash n}$? We can clearly write it in terms of $\{f_\lambda\}_{\lambda\vdash n}$, since we can already expand DZ in terms of $\{m_\lambda\}_{\lambda\vdash n}$, and $\omega(m_\lambda) = f_\lambda$.

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