1. Either provide an example to support your answer or prove your claim: (5 points each)
   (1) What is a unit in a unital ring?
   (2) Give a ring \( R \) and a non-principal ideal \( I \).
   (3) Is there a non-commutative ring of order 4?
   (4) Let \( R \) be a ring and assume that it has a subring isomorphic to \( \mathbb{Q} \). Does \( R \) have a unity?

2. Let \( S = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Q} \right\} \subseteq \mathbb{M}_2(\mathbb{Q}) \).
   (1) (10 points) Prove that \( S \) is a commutative unital subring of \( \mathbb{M}_2(\mathbb{Q}) \).
   (2) (5 points) Prove that \( S \) is a field.
   (3) (10 points) Prove that \( f : S \rightarrow \mathbb{Q}[\sqrt{2}] \) given by
       \[ f(\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}) = a + \sqrt{2}b \]
       is an isomorphism.
   (4) (5 points) Is \( R = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \) a field? Explain your answer.
   (5) (Bonus) Prove that \( R \) is isomorphic to \( \mathbb{R} \oplus \mathbb{R} \).

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