1. Let $G$ be a finite group and $X$ be a finite set. Assume $G \trianglelefteq X$. For any $g \in G$, let $x^g$ be the set of fixed points of $g$, i.e. 
\[ x^g := \{ x \in X \mid g \cdot x = x^g \}. \]
Prove that 
\[ |G^X| = \mathbb{E}(|X^g|). \]
($\mathbb{E}(|X^g|)$ is the average of $|X^g|$, i.e. $\frac{1}{|G|} \sum_{g \in G} |X^g|$.)

*Hint:* Consider 
\[ S = \bigcup_{(g, x) \in G \times X} \{ g \cdot x = x^g \}. \]
\[ \Rightarrow |S| = \sum_{g \in G} |X^g| = \sum_{x \in X} |G_{x^g}|. \]

* $G_{g \cdot x} = g \cdot G_x \cdot g^{-1}.$

2. Let $G$ be a finite group and $X$ be a finite set. Assume $G \trianglelefteq X$ transitively and $|X| \neq 1$. Prove that there is $g \in G$ with no fixed points.

*Hint:* Use $\mathbb{E}$ and notice that $x^{id} = x.$
3(a). Let $G$ be a finite group and $H \leq G$. Prove that
\[ \bigcup_{g \in G} Hg^{-1} \neq G. \]

(b) Does part (a) hold for an infinite group?

Hint. (a) There are various ways to solve this problem. Here is one method: assume the contrary and get a contradiction using Problem 2.

(b) Think about $G = \text{GL}_2(\mathbb{C})$ and $H = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | ac \neq 0 \}$.

4. For any permutation $\sigma \in S_n$, let $m_\sigma$ be the number of fixed points of $\sigma$, i.e.
\[ m_\sigma := \left| \{ 1 \leq i \leq n \mid \sigma(i) = i \} \right|. \]
Prove that $\sum_{\sigma \in S_n} m_\sigma = n!$.

5. $M, N \leq G$ and $M \cap N = \{ e \} \Rightarrow \forall m \in M, n \in N, mn = nm$.

Prove that $H \triangleleft G$.

Hint. Let $N$ be the core of $H$. Then as we showed in class we have $G/N \hookrightarrow S_p$. Now use the assumption to deduce $|G/N| = p$.

7. Let $G$ be a group, $H \leq G$, $N \leq G$.
   
   (a) Assume $|H| < \infty$, $[G:N] < \infty$ and $(|H|, [G:N]) = 1$. 
   Prove that $H \triangleleft N$.
   
   (b) Assume $|N| < \infty$, $[G:H] < \infty$ and $(|N|, [G:H]) = 1$. 
   Prove that $N \triangleleft H$.

8. Let $G$ be a finite non-abelian simple group.
   
   Assume $\exists \ H \unlhd G$, $[G:H] = n$. Then 
   $$G \hookrightarrow A_n.$$